Obviously False Proofs

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There are many "proofs" out there where clearly wrong statements are supposedly shown to be true. Some of the most common ones are proofs that $1 = 0$ or $2 = 1$ or some other absurdity. Most of these proofs have the same fatal flaw, which we now demonstrate in the following WRONG proof that $2 = 1$: Let $A = B$. Then elementary algebra yields

$$\begin{aligned} A^{2=}AB \\ \Rightarrow A^2 - B^2 = AB - B^2 \\ \Rightarrow (A-B)(A+B) = (A-B)B \\ \Rightarrow A + B = B \\ \Rightarrow 2B = B \quad \text{since} \quad A = B \quad \text{by} \quad \text{assumption} \\ \Rightarrow 2 = 1. \end{aligned}$$

Did you catch the problem? The problem is going from the third line to the fourth: in order to do so, one must divide by the quantity $A - B$, which was assumed to be zero! This gives a good illustration of why your high school teacher always told you that you can't divide by zero. On a more technical note, assuming $A = B$ and reworking the proof *backwards* illustrates how a false statement can imply a true statement: Setting, say, $A = B = 3$ and assuming $2 = 1$ you can deduce from the above that $A = B$ which is true! All you have to do is multiply both sides of the equation by zero (in this case, $A - B$) and you can get about what ever you want!!

OK, so thats how most false proofs along these lines go. Now for some more interesting ones which DO NOT rely on somehow secretly dividing by zero. Consider the following false calculus proof that $2 = 1$: from elementary school, we are taught that $2^2 = 2 + 2$, $3^2 = 3 + 3 + 3$, $4^2 = 4 + 4 + 4 + 4$ and so on. Thus, given any x it follows that

$$x^2 = \underbrace{x + x + \ldots + x}_{x-\text{times}}.$$

Taking derivatives of both sides yields

$$2x = \underbrace{1 + 1 + \ldots + 1}_{x-\text{times}} = x.$$

Dividing by x then gives $2 = 1$ as claimed.
Question: Is there a problem in the above argument? If so, what is it? If not, how do you explain the conclusion?

If you figure out the problem in the above proof, then you may enjoy this one[1]. Here, we will use elementary integration by parts to prove that $1 = 0$. Consider the antiderivative

$$\int \frac{dx}{x\ \ln(x)}.$$

We can integrate by parts by setting

$$u = \ln(x) \text{ and } dv = x^{-1}dx$$

from which it follows

$$du = \frac{-1}{x\ \ln(x)^2}dx \text{ and } v = \ln(x).$$

Thus, the integration by parts formula yields

$$\begin{aligned}\int \frac{dx}{x\ \ln(x)} &= \left(\ln(x) \cdot \frac{1}{\ln(x)}\right) - \int \frac{-\ln(x)\ dx}{x\ \ln(x)^2} \\ &= 1 + \int \frac{dx}{x\ \ln(x)}.\end{aligned}$$

Subtracting the integral on the right to the other side we are left with

$$1 = \int \frac{dx}{x\ \ln(x)} - \int \frac{dx}{x\ \ln(x)} = 0$$

as claimed.

Question: Is there a problem in the above argument? If so, what is it? If not, how do you explain the conclusion?

[1]Thanks to Johann Thiel for pointing out this problem. The following argument was given by one of Johann's Calculus 2 students during the Spring of 2009 at the University of Illinois at Urbana-Champaign. It is not hard to evaluate the integral by substitution, but a student tried to evaluate it using integration by parts and got the following weird result. It took the students some time to figure out what was going on..... can you?!