Hints and Solutions: Which Number is Larger?

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March 6, 2009

First, let me restate the problem in a readable form: Which number is larger?

$$e^{\pi} \quad \text{or} \quad \pi^{\mathrm{e}}?$$

1 Hint # 1

Consider the function $f(x) = x^{1/x}$. Convince yourself that the above problem is really the same as asking which of the numbers $f(e)$ and $f(\pi)$ is largest. Remember, you don't really have the problem solved until you can justify any hints that I give you!

2 Hint # 2

If the first hint didn't help, maybe this will help shed some light. Consider the function $f(x) = \frac{\ln x}{x}$. Can you classify its critical points? How does this essentially solve the problem?

3 Hint # 3

If neither of the above hints helped, try considering the function $f(x) = e^x - x^e$. How does this help?

4 Solutions

4.1 Solution #1

This solutions is based off of the first hint above. Notice that if we take the πe^{th} root of the statement $e^\pi \; O \; \pi^e$, it yields $e^{1/e} \; O \; \pi^{1/pi}$ where O stands for either $>$ or $<$ or $=$. So, if we define $f(x) = x^{1/x}$, we just need to see which value is larger: $f(e)$ or $f(\pi)$. Well, taking logs we have $\ln(f(x)) = \frac{lnx}{x}$ so differentiating gives

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2} - \frac{\ln x}{x^2}$$
$$= \frac{1 - \ln x}{x^2}.$$

and hence

$$f'(x) = x^{1/x}\left(\frac{1-\ln x}{x^2}\right).$$

So, $f'(x) = 0$ exactly when $x = e$. Now, using the first derivative test, we see that e is a (global) maximum for this function and hence $f(e) > f(\pi)$, i.e. $e^\pi > \pi^e$.

4.2 Solution #2

Notice that in the above solution, all the work was showing that the function $\ln(f(x)) = \frac{\ln x}{x}$ has a (global) maximum at $x = e$. To see another way why this solves the problem, notice that

$$\begin{aligned} e^\pi &\quad O \quad \pi^e \\ \Rightarrow \pi\ln(e) &\quad O \quad e\ln(\pi) \\ \Rightarrow \frac{\ln(e)}{e} &\quad O \quad \frac{\ln(\pi)}{\pi} \end{aligned}$$

where again O stands for either $>$ or $<$ or $=$. So, another way of restating this problem is to see which of the above quantities is larger.

4.3 Solution #3

Thanks to Kabe Moen from the University of Kansas for pointing out this solution!!

In this solution, we follow hint #3 and consider the function $f(x) = e^x - x^e$ for $x > 0$. To solve the problem, we must determine whether $f(\pi)$ is positive or negative. To this end, we begin by locating the critical points of this function f. Taking the derivative we get

$$f'(x) = e^x - ex^{e-1} = e\left(e^{x-1} - x^{e-1}\right).$$

Clearly then $f'(e) = 0$ and $f'(1) = 0$. I claim these are in fact the only critical points of f. To see this, notice that $f'(x) = 0$ is equivalent with the equation

$$x - 1 = (e-1)\ln(x).$$

Since the left hand side is always concave down and increasing, this equation has at most two solutions. Moreover, it is not hard to find these solutions: one is at $x = 1$ and the other at $x = e$ thus justifying our claim.

Now, recall we are trying to determine the sign of the number $f(\pi)$. Up to this point, we have shown $f(e) = 0$, $f'(e) = 0$, and $f'(x) \neq 0$ for all $e < x < \infty$. We now want to show that $f'(x)$ is actually positive for $x > e$. I see at least two ways of doing this. For the first, notice that $1 < \ln(2) < 2$ implies that

$$(2e) - 1 > \ln(2)e - \ln(2).$$

Thus,

$$f'(2e) = e\left(e^{2e-1} - (2e)^{e-1}\right) > 0.$$

Another way you could do this is to notice

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(e^x - ex^{e-1}\right) = +\infty.$$

Since $f'(x) \neq 0$ for any $x > e$, this implies that $f'(x) > 0$ for all $x > e$.

The rest of the solution is now straightforward. Since $f(e) = 0$ and $f'(e) > 0$ for all $x > e$, we have that $f(x) > 0$ for all $x > e$. In particular, $f(\pi) > 0$ and hence

$$e^\pi > \pi^e.$$