# Math 951 Lecture Notes
# Chapter 3 – Energy Methods in Parabolic PDE Theory

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## Contents



## 1 Introduction

Previously, we have studied the existence and regularity of solutions of uniformly elliptic PDE. Such equations are generalizations of the famous Poisson equation

$$\Delta u = f, \quad x \in \Omega$$

posed on some domain $\Omega \subset \mathbb{R}^n$, equipped with an appropriate set of boundary conditions. We now turn our eye to study a class of "evolution equations" which generalize the well-known heat equation

$$u_t = \Delta u, \quad x \in \Omega, \ t > 0. \tag{1}$$

Usually the heat equation is posed with some initial condition $u(x, 0)$ prescribing the state of the system at time $t = 0$, along with appropriate boundary conditions on $\partial\Omega$. Note that (1) describes the evolution (in time) of a function of $x$. That is, for each time $t$ the solution of (1) will be a function of $x$, and (1) describes how that function of $x$ will change over time. Consequently, instead of thinking of a solution of (1) as being a function

$$u : \Omega \times [0, \infty) \to \mathbb{R},$$

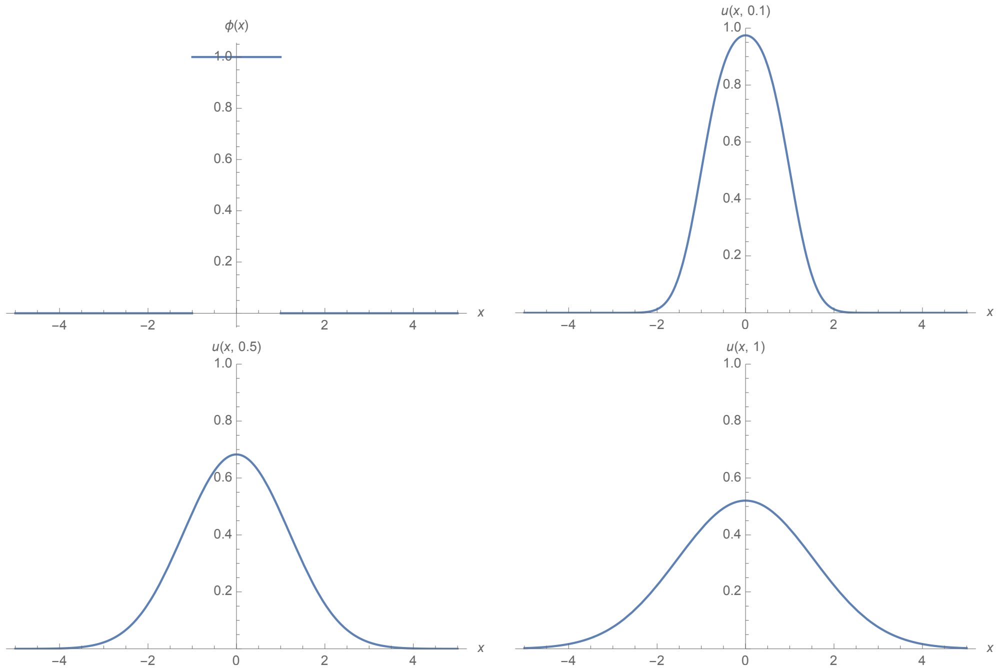


Figure 1: Time evolution of the solution of the heat equation posed on $\mathbb{R}$ with initial data $u(x,0) = \phi(x)$, where $\phi(x) = 1$ if $|x| < 1$ and $\phi(x) = 0$ for $x > 1$. Thus, for each time $t > 0$ we get a function $u(t) : \mathbb{R} \to \mathbb{R}$.

it is more natural to think of solutions of such evolution equations as being maps

$$u : [0, \infty) \to X,$$

where here $X$ is some appropriate function space on $\Omega$, possibly encoding the boundary conditions: for example, if (1) is equipped with homogeneous Dirichlet boundary conditions, then $X = H_0^1(\Omega)$. Note this is also consistent with how one would think of numerically simulating the solutions: for each time, you would plot the solution as a function of $x$. See Figure 1.

With the above change in perspective, we now have two scales of regularity to track: the regularity of the map $u : [0, \infty) \to X$ and the regularity of each $u(t) : \Omega \to \mathbb{R}$. For example, if the map $u : [0, \infty) \to X$ is continuous, we write

$$u \in C([0, \infty); X)$$

noting that the space $C([0, \infty); X)$ equipped with the norm

$$\|u\| := \sup_{t \in [0,\infty)} \|u(t)\|_X$$

is a normed linear vector space space. Note above we are essentially equipping $C([0,\infty);X)$ with the $L^\infty([0,\infty);X)$ norm. The next result, which we will use several times, implies that $C([0,\infty);X)$ is in fact a Banach space.

**Lemma 1.** *If $X$ is a Banach space, then the space $C([0,\infty);X)$ equipped with the above norm is complete and hence itself is a Banach space. That is, if $\{u_k\}_{k=1}^\infty$ is a Cauchy sequence in $C([0,\infty);X)$, then there exists a $u \in C([0,\infty);X)$ such that*

$$\lim_{k\to\infty} \sup_{t\geq 0} \|u_k(t) - u(t)\|_X = 0.$$

*Proof.* If $\{u_k\}_{k=1}^\infty$ is Cauchy then given any $\epsilon > 0$ there exists a $K > 0$ such that

$$\|u_j(t) - u_k(t)\|_X \leq \epsilon \tag{2}$$

for all $j,k \geq N$ and all $t \in [0,\infty)$. In particular, for each fixed $t \geq 0$ the sequence $\{u_k(t)\}$ is Cauchy in $X$ and hence, since $X$ is a Banach space, there exists a function $u(t) \in X$ such that $u_k(t) \to u(t)$ in $X$ for each $t \geq 0$.

To see that the function $u : [0,\infty) \to X$ is continuous, we use the fact that the estimate (2) is independent of time and apply a standard $\epsilon/3$-argument. Specifically, note by the triangle inequality that for all $j \in \mathbb{N}$ we have

$$\|u(t_1) - u(t_2)\|_X \leq \|u(t_1) - u_j(t_1)\|_X + \|u_j(t_1) - u_j(t_2)\|_X + \|u_j(t_2) - u(t_2)\|_X.$$

Thus, if $j > K$, then for all $t_1, t_2 \geq 0$ we have

$$\|u(t_1) - u(t_2)\|_X \leq \|u_j(t_1) - u_j(t_2)\|_X + 2\epsilon.$$

Since $u_j \in C([0,\infty);X)$ by assumption, we know there exists a $\delta > 0$ such that

$$\|u_j(t) - u_j(s)\|_X < \epsilon \quad \forall\ \ t,s \geq 0, \quad |t-s| < \delta.$$

It follows that if $|t_1 - t_2| < \delta$ then

$$\|u(t_1) - u(t_2)\|_X \leq 3\epsilon,$$

which implies that the function $u : [0,\infty) \to X$ is in fact uniformly continuous. □

# 2 Autonomous, Symmetric Equations

Given an open, bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, lets consider the following evolution problem: given $u_0 \in L^2(\Omega)$, find a function $u : \overline{\Omega} \times [0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} u_t + Lu = 0, & x \in \Omega,\ t > 0 \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{3}$$

where here

$$Lu := -\sum_{i,j=1}^{n} \left(a^{i,j} u_{x_i}\right)_{x_j}$$

is a time-independent, uniformly elliptic differential operator on $\Omega$. Specifically, here we are making the important assumption that the evolution equation (3) is autonomous, i.e. its coefficients do not depend on time[2] More precisely, we assume that $a^{i,j} = a^{j,i}$ and that $a^{i,j} \in L^\infty(\Omega)$.

The goal of this section is to essentially show the problem (3) can be solved (weakly) by classical separation of variables. To derive the appropriate weak formulation, note that if $v \in C_c^\infty(\Omega)$ then

$$\int_\Omega u_t(x,t)v(x)dx = -\int_\Omega \sum_{i,j=1}^{n} a^{i,j} u_{x_i}(x,t) v_{x_j}(x)dx.$$

Further, under reasonable assumptions on the regularity of $u(t)$ in time[3], the above could be written as

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx = -B[u(t),v],$$

where here $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the standard bilinear form associated to $L$. This motivates the following.

**Weak Formulation:** Find $u : [0,\infty) \to L^2(\Omega)$ such that

(a) $u \in C([0,\infty); L^2(\Omega))$ and $u(0) = u_0$.

(b) $u(t) \in H_0^1(\Omega)$ for all $t > 0$.

(c) for all $t > 0$ and $v \in H_0^1(\Omega)$ we have

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx = -B[u(t),v],$$

**Remark 1.** *Note that condition (a) ensures that the initial condition makes sense, while condition (b) ensures that $B[u(t),v]$ makes sense for each $v \in H_0^1(\Omega)$. Further, note that since the initial data is only $L^2(\Omega)$ it is only required that the solution approach the initial data as $t \to 0$ in $L^2(\Omega)$.*

[2]Recall a finite dimensional dynamical system is said to be autonomous if its of the form $x' = f(x)$, while it is non-autonomous if the vector field $f$ depends on $t$, i.e. is of the form $x' = f(x,t)$.

[3]Technically, I believe $H^1([0,\infty); H_0^1(\Omega))$ is sufficient. Note weak differentiation of the map $u : [0,\infty) \to H_0^1(\Omega)$ is defined as you might expect.

To state the main result for this section recall the First Existence Theorem for Uniformly Elliptic PDE guarantees that for each $f \in L^2(\Omega)$ there exists a unique $u = S(f) \in H_0^1(\Omega)$ that weakly satisfies the BVP

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Specifically, there exists a weak solution operator $S : L^2(\Omega) \to H_0^1(\Omega)$ such that

$$B[S(f), v] = \int_\Omega fv \; dx \quad \forall v \in H_0^1(\Omega).$$

Further, recall that since $\Omega$ is bounded that the operator $S : L^2(\Omega) \to L^2(\Omega)$ is compact. We may thus let $\{(\lambda_k, \phi_k)\}_{k=1}^\infty$ denote eigenvalue and eigenfunction pairs of $S$, where the eigenfunctions $\{\phi_k\}$ are chosen to form an O.N.B. of $L^2(\Omega)$ as well as an orthogonal basis of $H_0^1(\Omega)$ with respect to the inner product $B[\cdot, \cdot]$. It follows that $\{(\mu_k, \phi_k)\}_{k=1}^\infty$ are the eigenvalue and eigenfunction pairs associated to $L$, where here $\mu_k = \frac{1}{\lambda_k}$ is an increasing sequence satisfying

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \nearrow \infty.$$

With this setup, we now state the main result for this section.

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, and consider the linear diffusion problem* (3) *with $a^{ij} = a^{ji} \in L^\infty(\Omega)$ and $u_0 \in L^2(\Omega)$. Then the unique weak solution of* (3) *is given by*

$$u(t) = \sum_{k=1}^\infty a_k e^{-\mu_k t} \phi_k \tag{4}$$

*where here $a_k := \langle u_0, \phi_k \rangle_{L^2(\Omega)}$.*

Before proving this result, I want to emphasize this is EXACTLY what you should expect by performing a formal separation of variables argument to the IVBVP (3). Indeed, in an undergraduate PDE class one might attempt to solve this problem by seeking solutions of the form

$$u(x,t) = X(x)T(t)$$

which, when substituted into the PDE in (3) yields the equation (assume here $X, T \neq 0$)

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

Clearly, the only way a function of $x$ can be identically equal to a function of $t$ is if both functions are constant, say $-\lambda \in \mathbb{R}$. Coupled with the boundary condition in (3) this implies that $X$ must satisfy the uniformly elliptic BVP

$$\begin{cases} LX = \lambda X & \text{in } \Omega \\ X = 0 & \text{on } \partial\Omega. \end{cases}$$

while $T$ satisfies, for each eigenvalue $\lambda$ of above BVP, the ODE $T' + \lambda T = 0$. Using the notation above, it follows by elementary calculations that any function of the form

$$u(t) = \sum_{k=1}^{\infty} c_k e^{-\mu_k t} \phi_k, \quad c_k \in \mathbb{R}$$

will (formally)satisfy both the PDE and boundary condition in (3). Taking $t = 0$ and using that the eigenfunctions $\{\phi_k\}$ are chosen to form an O.N.B. of $L^2(\mathbb{R})$ gives us precisely the stated solution formula. Of course, the above is all formal and one has to deal with the convergence of the infinite series to make the above rigorous.

**Proof Of Main Theorem Above.** First, we show weak solutions of (3) are unique. To see this, suppose $u$ is such a weak solution and note since $\{\phi_k\}$ is complete in $L^2(\Omega)$ we have for all $t \geq 0$

$$u(t) = \sum_{k=1}^{\infty} b_k(t) \phi_k,$$

where here $b_k(t) := \langle u(t), \phi_k \rangle_{L^2(\Omega)}$. Since $u \in C([0, \infty); L^2(\Omega))$ we clearly see that each coefficient function $b_k(t)$ is continuous on $[0, \infty)$ with $b_k(0) = a_k$. Furthermore, for each $k$ we have

$$\frac{d}{dt} b_k(t) = \frac{d}{dt} \langle u(t), \phi_k \rangle_{L^2(\Omega)} = -B[u(t), \phi_k]$$

Using the weak solution operator $S$ and the fact that $B$ is symmetric, we see that

$$B[u(t), \phi_k] = B[\phi_k, u(t)] = \frac{1}{\lambda_k} B[S[\phi_k], u(t)] = \mu_k b_k(t)$$

so that, in the end, the $b_k$ satisfy the IVP

$$\begin{cases} b_k'(t) = -\mu_k b_k(t), & t > 0 \\ b_k(0) = a_k \end{cases}$$

which implies $b_k(t) = a_k e^{-\mu_k t}$, as claimed. This establishes uniqueness.

Now, we must prove the existence of a weak solution. We do this by proving that the function $u(t)$ defined in (4) satisfies (a)-(c) in the definition of a weak solution given above. To this end, for each $K \in \mathbb{N}$ set

$$u_K(t) := \sum_{j=1}^{K} a_j e^{-\mu_j t} \phi_j; \tag{5}$$

that is, $u_K$ is the $K$-th partial sum of the proposed solution representation. Then for all $t \geq 0$ and $K, J \in \mathbb{N}$ with $K > J$ we have

$$\|u_K(t) - u_J(t)\|_{L^2(\Omega)}^2 = \left\| \sum_{j=J+1}^{K} a_j e^{-\mu_j t} \phi_j \right\|_{L^2(\Omega)}^2$$

Observe that since the $\{\phi_k\}$ are orthonormal in $L^2(\Omega)$ the above norm can be expressed as

$$\begin{aligned}\left\|\sum_{j=J+1}^{K} a_j e^{-\mu_j t}\phi_j\right\|_{L^2(\Omega)}^2 &= \left\langle \sum_{j=J+1}^{K} a_j e^{-\mu_j t}\phi_j, \sum_{j=J+1}^{K} a_j e^{-\mu_j t}\phi_j \right\rangle_{L^2(\Omega)}\\ &= \sum_{j,\ell=J+1}^{K} a_j a_\ell e^{-(\mu_j+\mu_\ell)t}\left\langle\phi_j,\phi_\ell\right\rangle_{L^2(\Omega)}\\ &= \sum_{j=J+1}^{K} a_j^2 e^{-2\mu_j t}.\end{aligned}$$

Using that $\mu_j > 0$ for all $j$ it follows that for all $t \geq 0$ we have

$$\|u_K(t) - u_J(t)\|_{L^2(\Omega)}^2 \leq \sum_{j=J+1}^{K} a_j^2.$$

Since similar considerations to above imply that $\|u_0\|_{L^2(\Omega)}^2 = \sum_{j=1}^{\infty} a_j^2$, which is finite, it follows that the sequence $\{u_K\}_{K=1}^{\infty}$ is Cauchy in $C([0,\infty);L^2(\Omega))$. It follows that the function $u(t)$ defined in (4) belongs to $C([0,\infty);L^2(\mathbb{R}))$ with $u(0) = u_0$, proving that $u$ satisfies (a) in the definition of a weak solution.

Next, we prove by similar means that $u(t) \in H_0^1(\Omega)$ for all $t > 0$. To this end, note that since $\phi_k \in H_0^1(\Omega)$ for each $k$ we clearly have $u_K(t) \in H_0^1(\Omega)$ for each $t \geq 0$ and $K \in \mathbb{N}$. We now show that, for each $t > 0$, the sequence $u_K(t)$ is a Cauchy sequence in $H_0^1(\Omega)$. To this end, observe that using the defining relation $\phi_j = \mu_j S(\phi_j)$ we have for all $K, J \in \mathbb{N}$ with $K > J$ we have

$$\|u_K(t) - u_J(t)\|_{H^1(\Omega)}^2 = \left\|\sum_{j=J+1}^{K} a_j e^{-\mu_j t}\phi_j\right\|_{H^1(\Omega)}^2 = \left\|S\left(\sum_{j=J+1}^{K} a_j \mu_j e^{-\mu_j t}\phi_j\right)\right\|_{H^1(\Omega)}^2.$$

which, since $S : L^2(\Omega) \to H_0^1(\Omega)$ is bounded, implies that

$$\|u_K(t) - u_J(t)\|_{H^1(\Omega)}^2 \leq C\left\|\sum_{j=J+1}^{K} a_j \mu_j e^{-\mu_j t}\phi_j\right\|_{L^2(\Omega)}^2 \leq \sum_{j=J+1}^{K} a_j^2\left(\mu_j e^{-\mu_j t}\right)^2,$$

where the last equality follows again using the fact that the $\{\phi_k\}$ are orthonormal in $L^2(\Omega)$. Now, observe that for all $\mu, t > 0$ we have

$$\left|\mu e^{-\mu t}\right| = \frac{1}{t}\left|(\mu t)e^{-\mu t}\right| \leq \frac{1}{t}\left(\sup_{a>0} a e^{-a}\right) \leq \frac{C}{t}.$$

Thus, if we fix $\tau > 0$ then for all $t \geq \tau$ we have[4]

$$\|u_K(t) - u_J(t)\|^2_{H^1(\Omega)} \leq \frac{C}{\tau^2} \sum_{j=J+1}^{K} a_j^2.$$

It follows that for each fixed $\tau > 0$ the sequence $\{u_K\}$ is Cauchy in $C([\tau,\infty); H^1_0(\Omega))$. By our previous Lemma, it follows that $u \in C([\tau,\infty); H^1_0(\Omega))$ for each $\tau > 0$, proving that $u$ satisfies (c) in the definition of a weak solution.

Finally, it remains to show the function $u$ defined in (4) satisfies condition (c) in our definition of a weak solution of the diffusion equation (3). To this end, note for all $t_1, t_2 > 0$ we have

$$\begin{aligned}
\langle u(t_2), v\rangle_{L^2(\Omega)} - \langle u(t_1), v\rangle_{L^2(\Omega)} &= \sum_{j=1}^{\infty} a_j \left(e^{-\mu_j t_2} - e^{-\mu_j t_1}\right) \langle \phi_j, v\rangle_{L^2(\Omega)} \\
&= -\sum_{j=1}^{\infty} a_j \left(\int_{t_1}^{t_2} \mu_j e^{-\mu_j t} dt\right) \langle \phi_j, v\rangle_{L^2(\Omega)} \\
&= -\int_{t_1}^{t_2} \left\langle \sum_{j=1}^{\infty} a_j \mu_j e^{-\mu_j t} \phi_j, v \right\rangle_{L^2(\Omega)} \\
&= -\int_{t_1}^{t_2} B[u(t), v] dt.
\end{aligned}$$

To justify the last equality above, observe the above work implies the series $\sum_{j=1}^{\infty} a_j e^{-\mu_j t}\phi_j$ converges in $H^1_0(\Omega)$ for each $t > 0$ and hence, by the continuity of $B : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ in each slot, we have for all $v \in H^1_0(\Omega)$ that

$$B[u(t), v] = \sum_{j=1}^{\infty} a_j e^{-\mu_j t} B[\phi_j, v] = \sum_{j=1}^{\infty} a_j \mu_j e^{-\mu_j t} \langle \phi_j, v\rangle_{L^2(\Omega)}, \tag{6}$$

where again we have used $\phi_j = \mu_j S(\phi_j)$ and the definition of the weak solution operator $S$. As in our above work, we can show that for each $\tau > 0$ the sequence of partial sums

$$\sum_{j=1}^{K} a_j \mu_j e^{-\mu_j t} \langle \phi_j, v\rangle_{L^2(\Omega)}$$

is Cauchy in $C([\tau,\infty); \mathbb{R}) = C([\tau,\infty))$, and hence that the function $B[u(\cdot), v] \in C([\tau,\infty))$ for each $\tau > 0$. Alternatively, one can argue that since $u \in C([\tau,\infty); H^1_0(\Omega))$ and since $B[\cdot, v] : H^1_0(\Omega) \to \mathbb{R}$ is continuous for each fixed $v \in H^1_0(\Omega)$, that the composition map

$$t \mapsto B[u(t), v]$$

[4] The singular behavior at $t = 0$ is a reflection of the fact that the initial data is not required to be in $H^1_0(\Omega)$, and hence we should not in general expect the given series to converge in $H^1(\Omega)$ when $t = 0$. Note if we had $u_0 \in H^1_0(\Omega)$ this blow up could be avoided... can you see how?

is a continuous function on $[\tau, \infty)$. By the Fundamental Theorem of Calculus, it follows that for each $v \in H_0^1(\Omega)$ the mapping

$$(0, \infty) \ni t \mapsto \langle u(t), v \rangle_{L^2(\Omega)} \in \mathbb{R}$$

is differentiable with

$$\frac{d}{dt} \langle u(t), v \rangle_{L^2(\Omega)} = -B[u(t), v],$$

as desired. □

It is important to note the above procedure applies to many other autonomous, symmetric evolution equations. Naturally, this will be explored in the homework. For now, we end this section with a number of important observations.

**Observation 1:** If we write the weak solution of (3) as $u(t) = T(t)u_0$, this defines for each $t > 0$ a continuous linear operator $T(t) : L^2(\Omega) \to H_0^1(\Omega)$. It is not hard to show in this symmetric case that $T(t)$ is compact and self adjoint as an operator from $L^2(\Omega)$ into itself, and hence that the Spectral Theorem for Compact Operators applies to $T(t)$ for each $t$. Furthermore, since

$$T(t)\phi_k = e^{-\mu_k t}\phi_k$$

it follows that $\{(e^{-\mu_k t}, \phi_k)\}_{k=1}^{\infty}$ comprise all the eigenvalues and eigenvalues of $T(t)$. This one-parameter family of operators $\{T(t)\}_{t \geq 0}$ is an example of a semigroup and, in this case, it is usually denoted as $T(t) = e^{-Lt}$. We will talk more about semigroups of operators and their application to linear and nonlinear PDE later.

**Observation 2:** The solution $u(t)$ can be formally be rewritten as

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{\infty} e^{-\mu_j t}\phi_j(x) \left( \int_{\Omega} u_0(y)\phi_j(y)dy \right) \\ &= \int_{\Omega} \left( \sum_{j=1}^{\infty} e^{-\mu_j t}\phi_j(x)\phi_j(y) \right) u_0(y)dy \\ &= \int_{\Omega} G(x,y,t)u_0(y)dy. \end{aligned}$$

The function $G(x, y, t)$ is known as the "Greens function" for the parabolic operator $\partial_t + L$ on $\Omega$. Note this plays the same role as the heat kernel

$$K(x,y,t) = (4\pi t)^{-n/2}\, e^{-|x-y|^2} 4t$$

plays for the classical heat operator $\partial_t - \Delta$ on $\mathbb{R}^n$. This shows how our current work connects to the more "classical" ideas from Math 950.

**Observation 3:** Since $\mu_1 > 0$ the solution representation (4) implies all weak solutions of (3) exhibit exponential decay as $t \to \infty$. Another way of seeing this is to use a so-called energy estimate. This proceeds by using the solution $u(t)$ as the test function in (c) for the definition of a weak solution. Doing so, we find

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + B[u(t), u(t)] = 0.$$

From the calculations used in the previous result, we have

$$B[u(t), u(t)] = \sum_{j=1}^{\infty} a_j^2 \mu_j e^{-2\mu_j t} \geq \mu_1 \|u(t)\|^2_{L^2(\Omega)}$$

which immediately gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = -B[u(t), u(t)] \leq -\mu_1 \|u(t)\|^2_{L^2(\Omega)}.$$

By Gronwall's inequality it follows that $\|u(t)\|_{L^2(\Omega)} \leq e^{-\mu_1 t}\|u_0\|_{L^2(\Omega)}$, giving the expected exponential rate of decay. Note, in particular, one can prove uniqueness of weak solutions using the above estimate. Indeed, if $u_1$ and $u_2$ are both weak solutions of (3) with the same initial condition $u_0 \in L^2(\Omega)$, then by linearity the difference $v = u_1 - u_2$ is a weak solution of (3) with initial data $v(0) = 0$. Using the above exponential decay estimate it follows immediately that $v(t) = 0$ in $\Omega$ for every $t \geq 0$, yielding the desired uniqueness of solutions.

**Observation 4:** Continuing the above observation, we note that the Principle Eigenvalue Theorem (which we will prove later) implies that the lowest eigenvalue $\mu_1 > 0$ of $L$ is simple and that its corresponding eigenfunction $\phi_1$ may be chosen so that $\phi_1 > 0$ on $\Omega$. Using this, it follows that the solution $u(t)$ of (3) can be decomposed as

$$u(t) = a_1 e^{-\mu_1 t}\phi_1 + \mathcal{O}_{L^2(\Omega)}(e^{-\mu_2 t}).$$

In particular, so long as the initial condition has a non-trivial $L^2(\Omega)$-projection onto $\phi_1$, which is a pretty general condition, it follows that for $t \gg 1$ we have $u(t) \approx a_1 e^{-\mu_1 t}\phi_1$. This is, in some sense, a form of the Central Limit Theorem!

# 3 Review of the Method: Galerkin Approximations

Before we proceed, I want to review the methodology used to prove the existence of solutions of the linear diffusion equation (3). Fundamentally, the idea was that since the eigenfunctions of $L$ form a basis for $H_0^1(\Omega)$, **IF** a weak solution exists then for each $t > 0$ we must have

$$u(t) = \sum_{j=1}^{\infty} b_j(t)\phi_j$$

for some sequence of coefficient functions $b_k(t)$. To deal with issues of convergence in the above series, and to prove that such a solution actually exists, the rigorous proof started with considering finite sums of the form

$$u_K(t) = \sum_{j=1}^{K} b_j(t)\phi_j.$$

The specific point here is that with the choices $b_j(t) = a_j e^{-\mu_j t}$ made in the previous section, the finite sum $u_K(t)$ satisfies both

(i) $u_K(0) =$ the $L^2(\Omega)$ projection of the initial data $u_0$ onto the finite finite dimensional subspace

$$\Sigma_K := \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_K\},$$

and

(ii) For every $j \in \{1, 2, \ldots, K\}$ we have

$$\frac{d}{dt}\langle u_K(t), \phi_j\rangle_{L^2(\Omega)} = -\left\langle \sum_{\ell=1}^{K} a_\ell \mu_\ell e^{-\mu_\ell t}\phi_\ell, \phi_j \right\rangle_{L^2(\Omega)} = -B[u_K(t), \phi_j],$$

where the last identity follows from (6). Thus, by linearity it follows that

$$\frac{d}{dt}\langle u_K(t), v\rangle_{L^2(\Omega)} = -B[u(t), v]$$

for every $v \in \Sigma_K$.

Together, (i)-(ii) above imply that for each $K \in \mathbb{N}$ the partial sum $u_K(t)$ satisfies both the initial condition and the PDE (weakly) **EXACTLY** on the finite dimensional subspace $\Sigma_K$!! Since, formally, $\Sigma_K$ converges as $K \to \infty$ to a basis for $H_0^1(\Omega)$, we should expect our approximations to get better and better as $K \to \infty$.

Viewed in this way, the *big idea* of the above proof was to build a sequence of "approximate solutions" $\{u_K\}_{K=1}^{\infty}$ that satisfy the weak formulation **EXACTLY** on the finite dimensional subspace $\Sigma_K$, then pass to the limit $K \to \infty$. This idea is known as *Galerkin's method*, and the approximate solutions $\{u_K\}$ are known as the *Galerkin approximations* of the solution. Galerkin's method is widely used as a theoretical tool when studying linear and nonlinear PDE theory, and is equally powerful as a device for doing numerical calculations.

# 4 Extension to Non-Autonomous and Non-Symmetric Diffusion

We end this chapter with describing how the analysis in Section 2 applies to linear diffusion equations that are not necessarialy symmetric and autonomous. To this end, suppose $\Omega \subset \mathbb{R}^n$ is open and bounded and let $T > 0$ be fixed. Define the *parabolic cylinder*

$$\Omega_T := \Omega \times (0, T]$$

and consider the following linear diffusion system:

$$\begin{cases} u_t + Lu = f, \quad x \in \Omega_T, \\ \ u(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T]. \\ \ u(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{7}$$

where here $f : \Omega_T \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are given and

$$Lu = -\sum_{i,j=1}^{n} \left(a^{i,j}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u$$

where $a^{i,j} = a^{j,i}, b^i, c \in L^\infty(\Omega_T)$. Further, we assume that $f \in L^2(\Omega_T)$, $g \in L^2(\Omega)$, and that the operator $\partial_t + L$ is "uniformly parabolic" on $\Omega_T$, i.e. there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{i,j}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x,t) \in \Omega_T$ and $\xi \in \mathbb{R}^n$.

**Important Note:** The requirement that the operator $\partial_t + L$ is uniformly parabolic implies, in particular, that for each fixed $t > 0$ the operator $L$ is uniformly elliptic with ellipticity constant $\theta(t) > 0$. The "uniformity" requirement of being uniformly parabolic implies further that

$$\inf_{t \in (0,T]} \theta(t) > 0$$

i.e. we can choose a uniform lower bound for the ellipticity constants $\theta(t)$.

The proof of existence for solutions of the linear diffusion equation (7) follows in much the same spirit as that followed for the symmetric, autonomous example in Section 2. In particular, we use Galerkin's method to construct a sequence of approximate solutions which hold *exactly* on finite dimensional subspaces, and then pass to limits. One key difference here, however, is the choice of the basis for the subspaces. In Section 2, the coefficients of the operator $L$ were independent of time, and hence its eigenvalues and eigenfunctions were independent of time. it hence made sense to use the eigenfunctions of the uniformly elliptic operator $L$ as our basis. Here, however, we run into several issues: (i) the operator $L$ may not be symmetric, and hence it may not have eigenfunctions, and (ii) the coefficients of $L$ depend on time and hence any eigenfunctions would depend on $t$, which would clearly interfere the "separation of variables" approach used to construct the solution.

To avoid these obstacles, we note that all Gelerkin's method needs is *some* basis for $H_0^1(\Omega)$, and hence we can use for our basis the eigenfunctions associated to *any uniformly elliptic differential operator on* $\Omega$. To make our lives simple[5], let $\{\phi_k\}_{k=1}^\infty$ be the Dirichlet

[5] Of course, if $L$ were independent of $t$ then we should choose the eigenfunctions of $L$ since this basis is especially well suited to work with the operator $L$ and its associated bilinear form.

eigenfunctions associated to $-\Delta$ on $\Omega$, normalized as to form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H_0^1(\Omega)$ with respect to the inner product

$$H_0^1(\Omega) \times H_0^1(\Omega) \ni (u, v) \mapsto \int_\Omega Du \cdot Dv \, dx \in \mathbb{R}.$$

Following Galerkin's method, for each $K \in \mathbb{N}$ we attempt to construct an approximate solution $u_K$ of the form

$$u_K(t) = \sum_{j=1}^{K} b_j^K(t)\phi_j(x) \tag{8}$$

where we hope to choose the coefficient functions $b_j^K$ so that the $u_K$ satisfies the weak projected PDE[6]

$$\langle (u_K)_t \,, \phi_j \rangle_{L^2(\Omega)} + B[u_K, \phi_j; t] = \langle f, \phi_j \rangle_{L^2(\Omega)} \,, \quad t \in (0, T], \; j = 1, 2, \ldots, K \tag{9}$$

along with the projected initial condition

$$\begin{cases} u_K(0) = \text{the } L^2(\Omega) \text{ projection of } u_0 \text{ onto } \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_K\} \\ \qquad\quad = \displaystyle\sum_{j=1}^{K} \langle u_0, \phi_j \rangle_{L^2(\Omega)} \phi_j. \end{cases} \tag{10}$$

Note that since $\phi_j \in H_0^1(\Omega)$ we clearly have $u_K(t) \in H_0^1(\Omega)$ for each fixed $t \in (0, T]$.

Substituting the form (8) into (9)-(10) yields a system of ODE's for the coefficient functions. Indeed, by the $L^2$-normalization of the $\{\phi_j\}$ we have

$$\langle (u_K)_t \,, \phi_j \rangle_{L^2(\Omega)} = \left(b_j^K\right)'(t)$$

and

$$B[u_K, \phi_j; t] = \sum_{\ell=1}^{K} B[\phi_\ell, \phi_j; t] b_\ell^K(t).$$

Consequently, the projected system (9)-(10) is equivalent to the $K \times K$ system of ODE's

$$\left(b_j^K\right)'(t) + \sum_{\ell=1}^{K} B[\phi_\ell, \phi_j; t] b_\ell^K(t) = \langle f, \phi_j \rangle_{L^2(\Omega)} \,, \quad j = 1, 2, \ldots, K.$$

for the coefficient functions $\left(b_1^K(t), b_2^K(t), \ldots, b_K^K(t)\right)$, supplemented with the initial condition $b_j^K(0) = \langle u_0, \phi_j \rangle_{L^2(\Omega)}$. By standard existence and uniqueness theory for ODE's (take Math 850!), there exists unique absolutely continuous set of solutions $\{b_j^K(t)\}_{j=1}^K$ tot he above linear ODE system.

[6]Here, $B[\cdot, \cdot; t]$ denotes the natural $t$-dependent bilinear form associated with the operator $L = L(t)$.

**Remark 2.** *Note if the coeffiicents of $L$ are independent of $t$, and if $\{(\mu_j, \phi_j)\}_{j=1}^{\infty}$ are the eigenvalue/eigenfunction pairs of $L$ (normalized to be orthonormal in $L^2(\Omega)$), the above ODE system becomes the diagonal system*

$$\left(b_j^K\right)'(t) + \mu_j b_j^K(t) = \langle f, \phi_j \rangle_{L^2(\Omega)} .$$

*which, along with the initial condition, becomes*

$$b_j^K(t) = e^{-\mu_j t} + \int_0^t e^{-\mu_j (t-s)} \langle f(\cdot, s), \phi_j \rangle_{L^2(\Omega)} \, ds.$$

*In particular, note in this case the coefficient functions are independent of $K$.*

To complete the existence proof, one aims at proving the sequence of Galerkin approximations $\{u_K(t)\}$ converges, in an appropriate sense, to a weak solution of the IVBVP (7). In Section 2, this argument proceeded by directly proving the sequence $\{u_K\}$ was Cauchy in appropriate topologies. There, however, we heavily used the diagonal structure of $L$ when acting on its eigenbasis: see the above remark, for example. Consequently, in the general case we need a new idea.

To pass to limits, we essentially begin by proving a "parabolic energy estimate" which gives control (in particular, boundedness) over the sequence in appropriate topologies.

**Theorem 2** (Parabolic Energy Estimates)**.** *There exist a constant $C = C(T, \Omega, L) > 0$ such that for all $K \in \mathbb{N}$ we have*

$$\|u_K\|_{L^\infty(0,T;L^2(\Omega))} + \|u_K\|_{L^2(0,T;H_0^1(\Omega))} + \|\left(u_K\right)_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}\right).$$

Note here the space $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$, i.e. it is the space of all bounded linear functionals on $H_0^1(\Omega)$. In the precise statement of the theorem, the function $(u_K)_t \in H_0^1(\Omega)$ is regarded as an element of $H^{-1}(\Omega)$ through pairing under the $L^2$ inner product, that is, via the mapping

$$H_0^1(\Omega) \ni v \mapsto \langle (u_K)_t, v \rangle_{L^2(\Omega)} \in \mathbb{R}.$$

The point of the above estimates is that the approximate solutions are controlled *uniformly in $K$* by the inhonogeneity $f$ and the initial condition $u_0$. Consequently, it follows that the sequence $\{u_K\}$ is bounded in $L^2(0,T; H_0^1(\Omega))$ while its sequence of time-derivatives $\{(u_K)_t\}$ is bounded in $L^2(0,T; H^1{-}1(\Omega))$. The following abstract topological result allows us then extract convergent subsequences.

**Theorem 3** (Banach-Alaoglu)**.** *If $\{x_n\}$ is a bounded sequence in a reflexive Banach space $X$, then there exists a subsequence $\{x_{n_k}\}$ and an $x \in X$ such that $x_{n_k}$ converges to $x$ weakly, i.e. for every $F \in X^*$ we have*

$$\lim_{k\to\infty} F(x_{n_k}) = F(x).$$

Using Banach-Alaoglu, we may extract a weakly convergent subsequence $\{u_{K_j}\}$ that converges to some function $u$, and then one shows the weak limit $u$ is our weak solution. Notice this approach is fairly natural since weak convergence means convergence under continuous linear functionals. Since the entire weak formulation is defined in an integral sense, i.e. in the context of continuous linear functionals, these theories mesh together well. In summary, we get the following result.

**Theorem 4** (Parabolic Existence and Uniqueness)**.** *The linear parabolic problem* (7) *has a unique weak solution.*

For a precise statement of the theorem, see Evans. Note the uniqueness component of the theorem follows by an energy estimate as described in Observation 3 in Section 2. Indeed, note if $u_1$ and $u_2$ are two weak solutions of (7) then the difference $v = u_1 - u_2$ satisfies (7) with $f = 0$ and $u_0 = 0$. From the weak formulation, it follows that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} + B[v, v; t] = 0.$$

Now, using that the operator $\partial_t + L$ is uniformly parabolic we have as in the proof of Garding's inequality (i.e. the elliptic energy estimates)

$$\begin{aligned} B[v, v; t] &\geq \theta \int_\Omega |Dv|^2 dx - n\|b^i\|_{L^\infty(\Omega_T)} \int_\Omega |Dv||v| dx - \|c\|_{L^\infty(\Omega_T)} \int_\Omega v^2 dx \\ &\geq \frac{\theta}{2} \int_\Omega |Dv|^2 dx - \gamma \int_\Omega v^2 dx, \end{aligned}$$

where the final inequality follows by using Cauchy-with-$\epsilon$. It now follows that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} = -B[v(t), v(t); t] \leq \gamma \|v(t)\|^2_{L^2(\Omega)}$$

which, by Gronwall's inequality gives

$$\|v(t)\|_{L^2(\Omega)} \leq e^{-\gamma t}\|v(0)\|_{L^2(\Omega)}.$$

Since $v(0) = 0$ by construction, this establishes uniqueness.

## 5 Final Thoughts

The above discussion has primarily concerned the existence of weak solutions to linear uniformly parabolic PDE on bounded domains. Clearly there is lots more one could say, including various qualitative properties and regularity of solutions. For example, just as for the classical heat equation one can prove sufficiently smooth solutions of (7) obey maximum principles (both weak and strong) and exhibit infinite speed of propagation.

As one might expect, the regularity theory (proving our weak solutions are actually classical solutions) is quite detailed and delicate. Some form of parabolic smooth may be seen directly from Theorem 2, which implies if $f = 0$ then then the initial data $g \in L^2(\Omega)$ control both $u$ and $Du$ in $L^2(\Omega)$ for $t > 0$. That is, the relatively "rough" initial data $g \in L^2(\Omega)$ instantaneously in time becomes at least $H^1$. One can also establish higher order regularity results. The big idea is that if the coefficients of $L$ and the data $f$ and $g$ are smooth, one can derive parabolic energy estimates (similar to Theorem 2 for the space- and time-derivatives of the Galerkin approximations, and then take $K \to \infty$ to show the limiting solution inherits this regularity. The necessary energy estimate is given as follows.

**Theorem 5** (Improved Parabolic Regularity)**.** *Suppose $g \in H_0^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$ and that $u$ is the weak solution of* (7)*. Then we have*

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)), \quad \text{and } u_t \in L^2(0,T;L^2(\Omega))$$

*and, moreover, there exists a constant $C = C(\Omega, T, L)$ such that*

$$\begin{aligned} \text{ess sup}_{0\le t\le T} \|u(t)\|_{H^1(\Omega)} + \|u\|_{L^2(0,T;H^2(\Omega))} + \|u_t\|_{L^2(0,T;L^2(0,T))} \\ \le C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)}\right). \end{aligned}$$

**Remark 3.** *Note that in order to get this improved regularity result we must require that $g \in H_0^1(\Omega)$, i.e. that the initial data $g$ satisfy the given Dirichlet boundary conditions.*

Coupling this procedure with the Sobolev Embedding Theorem, one can show that weak solutions of (7) are indeed smooth provided the coefficients of $L$, $f$ and $g$ are smooth **AND IF** $f$ and $u_0$ satisfy certain compatability conditions. To see the source of the compatability conditions, note that if $u$ solves (7) then the function $\tilde{u} := u_t$ satisfies

$$\begin{cases} \tilde{u}_t + L\tilde{u} = f_t, \quad x \in \Omega_T, \\ \tilde{u}(x,t) = 0, \quad x \in \partial\Omega, \ t \in [0,T]. \\ \tilde{u}(x,0) = f(0,x) - L(0)u_0(x), \quad x \in \Omega \end{cases}$$

So, using Theorem 5 we would need to require that the compatability condition $f(x,0) - L(0)u_0 \in H_0^1(\Omega)$ in order to guarantee $u_t(t) \in H^2(\Omega)$. For more information, see Evans.

Finally, the same "Galerkin approximation" scheme can be used to the existence and uniqueness of second-order hyperbolic PDE, i.e. generalizations of the wave equation. For more information, again see Evans.

# 6 Exercises

Please complete the following exercises.

1. Let $U \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary and consider the following generalized linear diffusion equation:

$$\begin{cases} u_t = -\Delta^2 u, \quad \text{in } U \\ u = \dfrac{\partial u}{\partial n} = 0, \quad \text{on } \partial U \\ u = u_0, \quad \text{on } U \times \{t = 0\} \end{cases} \tag{11}$$

where $u_0 \in L^2(U)$ is given. Using methods analogous to those of Problem # 3 of HW2, it is possible to show that the differential operator[7] $L = \Delta^2$ has a countably infinite number of positive eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of finite multiplicity that, when listed with respect to multiplicity, can be listed as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty.$$

Furthermore, the associated eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ in form a orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^2(U)$ with respect to the inner product $\langle v_1, v_2 \rangle_* := \int_U \Delta v_1 \Delta v_2 dx$. With these preparations in mind, our goal here is to show that

$$u(t) = \sum_{j=1}^{\infty} \langle u_0, \phi_j \rangle_{L^2(U)} e^{-\lambda_j t} \phi_j, \tag{12}$$

is the unique weak solution of the above IVBVP.

(a) We say $u : [0, \infty) \to L^2(U)$ is a weak solution of the above IVBVP if

(A) $u \in C([0, \infty); L^2(U))$ with $u(0) = u_0$.

(B) $u \in C((0, \infty); H_0^2(U))$.

(C) for all $t \in (0, \infty)$ and for all $v \in H_0^2(\Omega)$, we have

$$\frac{d}{dt} \langle u(t), v \rangle_{L^2(U)} = - \langle u(t), v \rangle_* \tag{13}$$

Prove that the function defined in equation (12) is a weak solution of (11).

(b) Prove that weak solutions of the given IVBVP are unique.

(c) (Suggested) Verify the claims about the structure of the eigenvalues and eigenfunctions of the operator $L$ discussed above.

2. (Galerkin's Method for Elliptic BVP[8].) Suppose $U \subset \mathbb{R}^n$ is open and bounded and consider the Poisson equation

$$\begin{cases} -\Delta u = f \text{ in } U \\ \quad u = 0 \text{ on } U, \end{cases}$$

where $f \in L^2(U)$. Furthermore, let $\{\phi_j\}_{j=1}^{\infty} \subset C^{\infty}(\bar{U})$ be the eigenfunctions of $-\Delta$ taken with Dirichlet boundary conditions, chosen to be an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^1(U)$.

[7]Here, we consider $L$ as a closed densely defined operator on $L^2(U)$ with form domain $H_0^2(U)$. Recall by the Trace Theorem functions in $H_0^2(U)$ the given boundary conditions.

[8]Based on Problem 7.4 from Evans

(a) Prove that for each $m \in \mathbb{N}$ there exists constants $d_m^k$ such that the function $u_m := \sum_{k=1}^m d_m^k \phi_k$ satisfies

$$\int_U Du_m \cdot D\phi_j \; dx = \int_U f\phi_j \; dx, \quad \forall j = 1, 2, \ldots, m.$$

(b) Now, show there exists a subsequence of $\{u_m\}$ which converges weakly[9] in $H_0^1(U)$ to a weak solution $u$ of the above Poisson problem.
*Hint: Use the Banach-Alaoglu Theorem to show that $\{u_m\}$ converges weakly in $H_0^1(U)$ to some $u \in H_0^1(U)$. Then, argue that this weak limit $u$ is a weak solution of the given Dirichlet problem.*

[9]Here, we say a sequence $\{f_j\}$ converges weakly to $f$ in $H_0^1(U)$ if $\lim_{j\to\infty} F(f_j) \to F(f)$ for every bounded linear functional $F$ on $H_0^1(U)$.