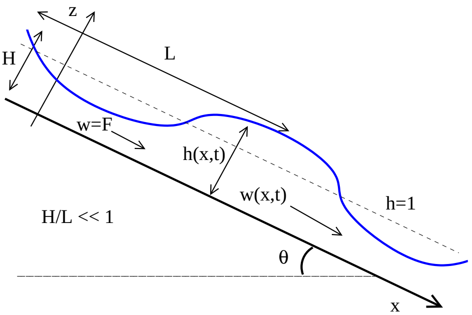
Stability of Periodic Traveling Waves in a KdV/Kuramoto-Sivashinsky Equation

Mathew A. Johnson
University of Kansas

SIAM Conference on Nonlinear Waves and Coherent Structures
6/15/2012

Joint work with
Kevin Zumbrun & Blake Barker (Indiana University),
Pascal Noble & L.-Miguel Rodrigues (Université Lyon 1)

Motivating Application: Roll Waves



In small amplitude limit

$$u := h - 1, \quad |u| \ll 1$$

can be modeled as periodic traveling wave solutions of generalized Kuramoto–Sivashinksi (gKS) equation

$$\partial_t u + u\partial_x u + \varepsilon\partial_x^3 u + \delta\left(\partial_x^2 u + \partial_x u^4\right) = 0, \quad \varepsilon^2 + \delta^2 = 1.$$

Here, $\theta = \frac{\pi}{2} \Rightarrow \varepsilon = 0$ (classic KS) while $\theta = 0 \Rightarrow \delta = 0$ (KdV).

Linear Dispersion

Seeking soln. of form $u(x,t) = u_0 + \nu e^{\lambda t + ikx}$, $u_0 \in \mathbb{R}$ yields

$$\lambda + iu_0 k - i\varepsilon k^3 + \delta\left(-k^2 + k^4\right) = \mathcal{O}(\nu).$$

- All constant solutions unstable to low-frequency perturbations.
- Dissipation stabilizes constant solutions at higher frequencies.
- Dispersion is neutral at this level.

BUT, long time dynamics of solutions seems dominated by traveling pulse trains of individually unstable solitary waves (whose separation distance is not too great).

Possible Interpretation: There should exist stable periodic waves. Interpretation further motivated by "observability" of such wave trains.

Traveling waves $u(x,t) = \bar{u}(x - ct)$ of (gKS)

$$u_t + uu_x + \delta u_{xx} + \varepsilon u_{xxx} + u_{xxxx} = 0$$

satisfy ODE

$$-c\bar{u} + \bar{u}''' + \varepsilon\bar{u}'' + \delta\bar{u}' + \frac{1}{2}\bar{u}^2 = q, \quad q \in \mathbb{R}.$$

Periodic orbits generated through Hopf bifurcation from one of the two constant states.

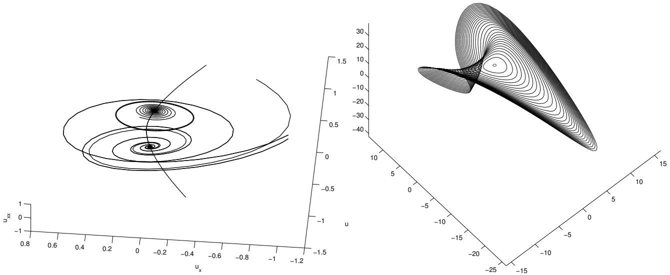
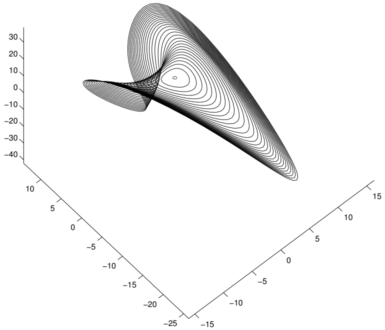
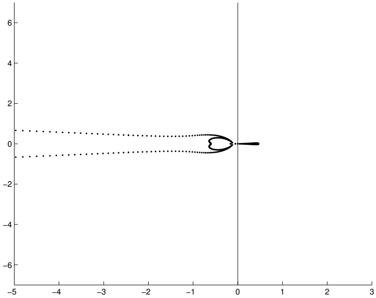


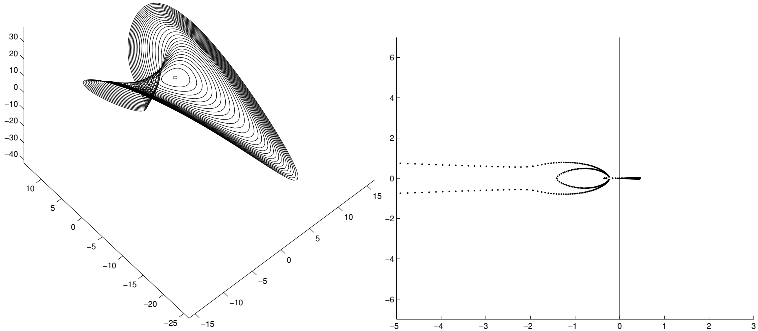
Figure: (Left) Here, period $X = 6.3$, $\varepsilon = 0.2$, $q = 0.04$. (Right) Same, but with $q \in [1, 30] \cap \mathbb{N}$ and δ free.

Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

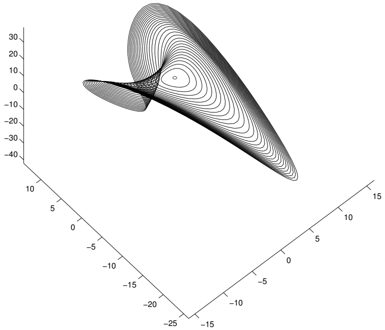


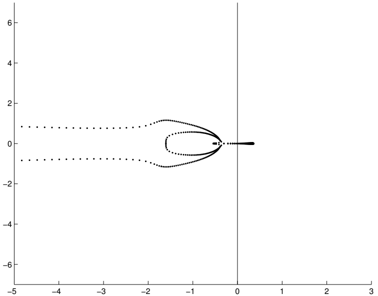


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

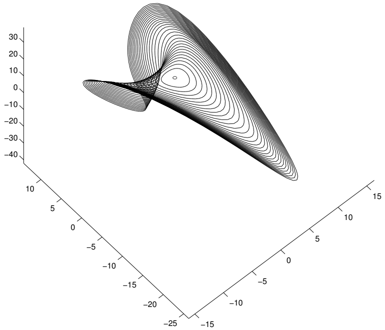


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

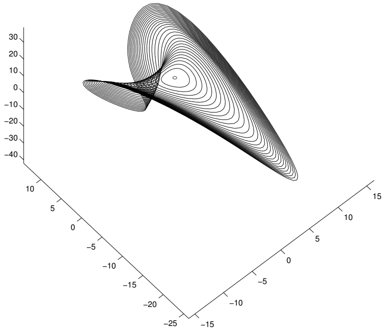


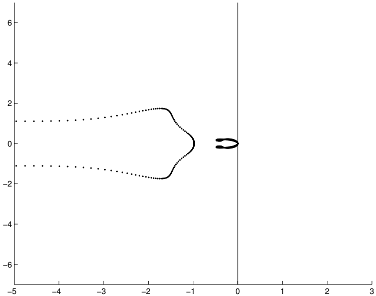


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

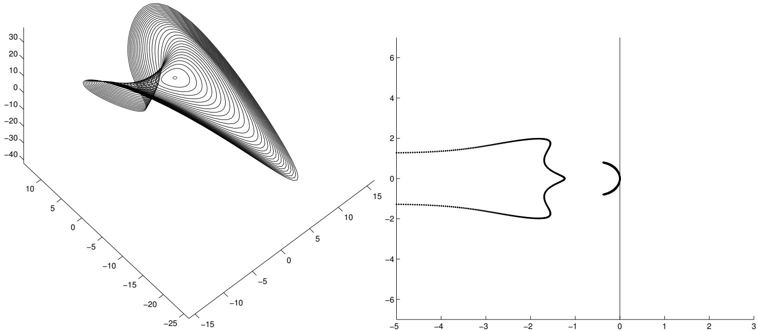


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

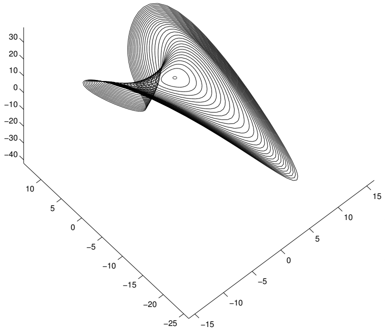


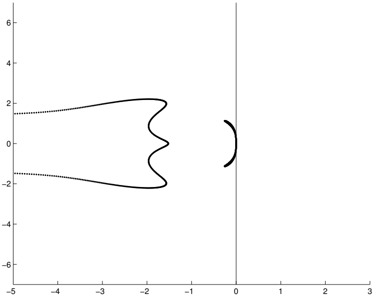


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

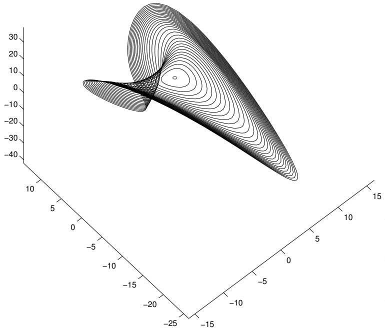


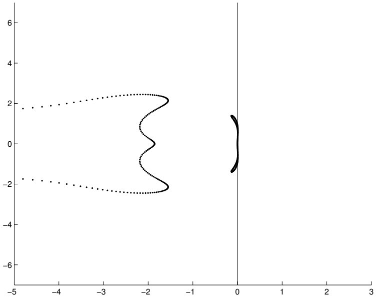
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



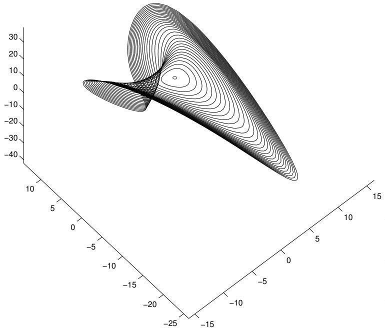


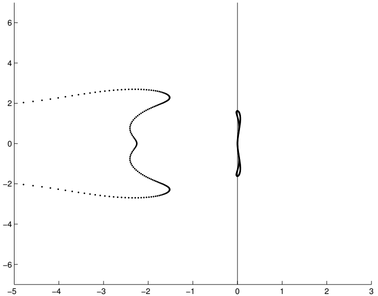
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



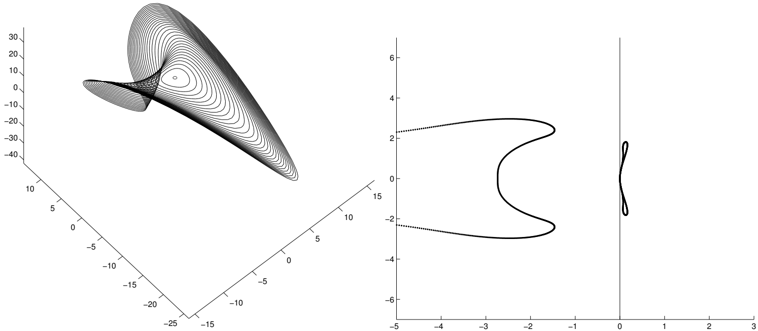


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

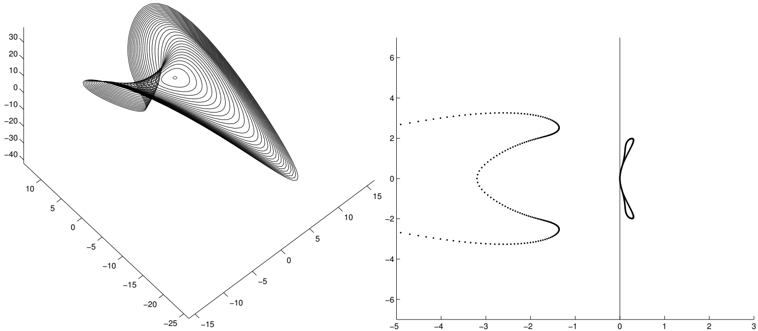




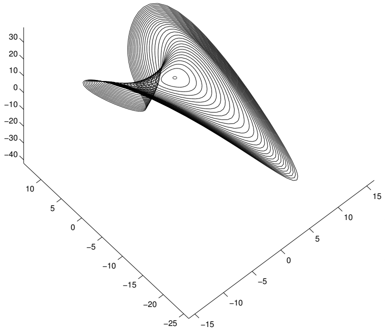
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

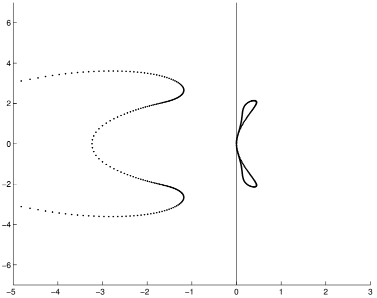


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

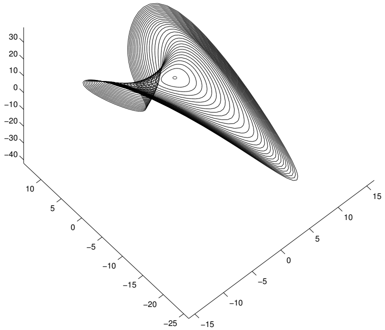


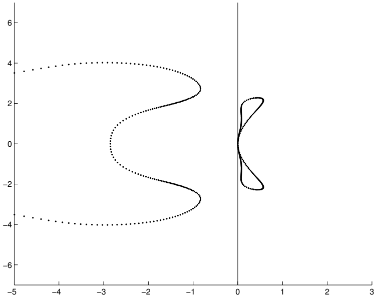
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



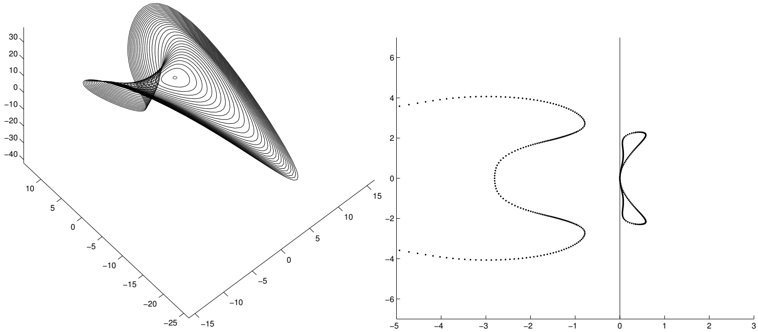


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

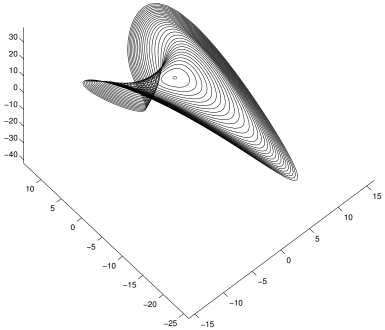


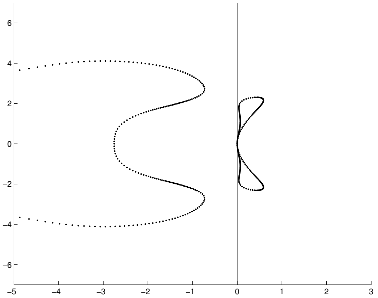


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

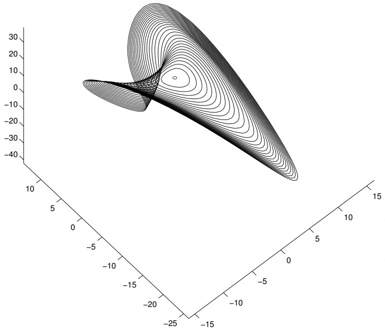


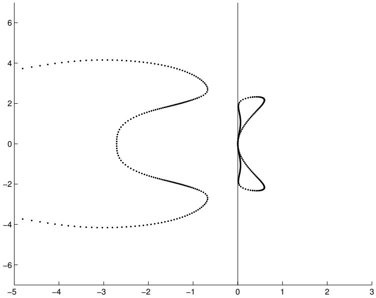
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



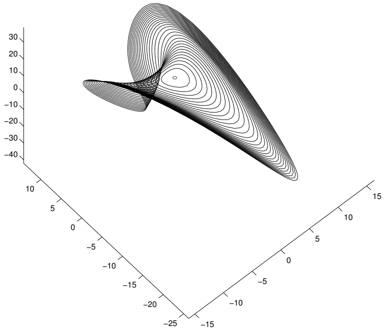


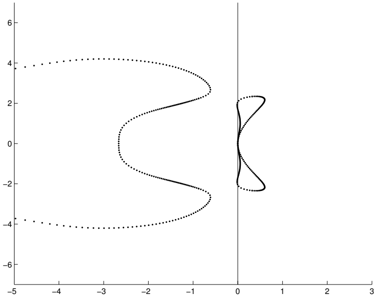
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



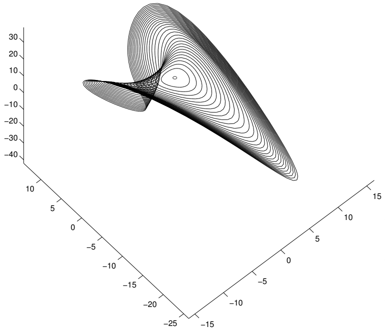


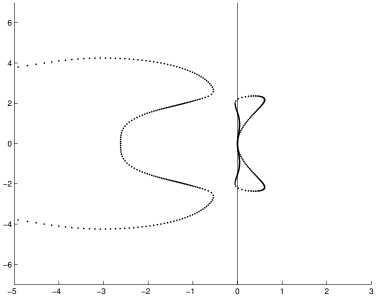
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



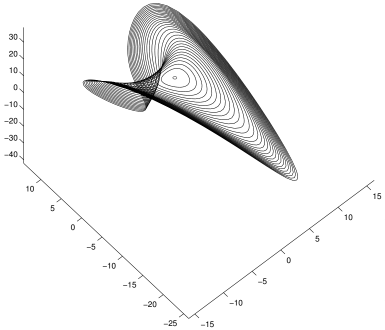


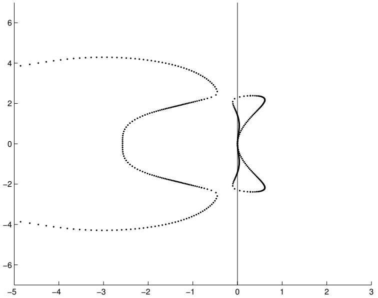
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



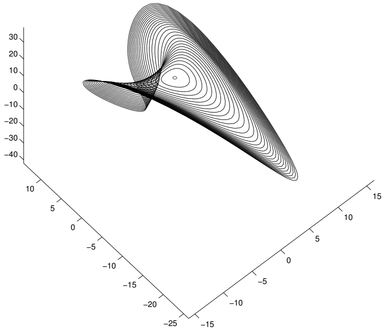


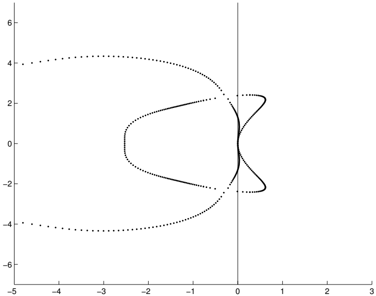
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



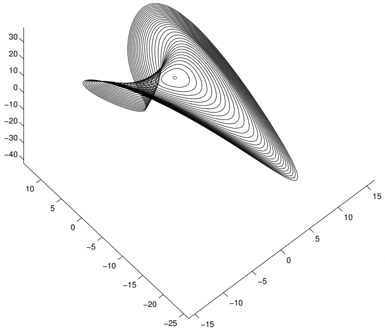


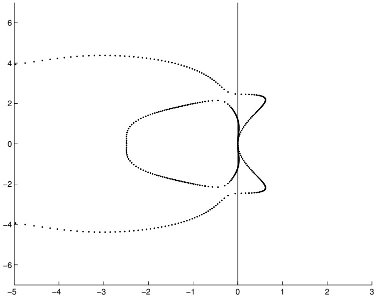
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



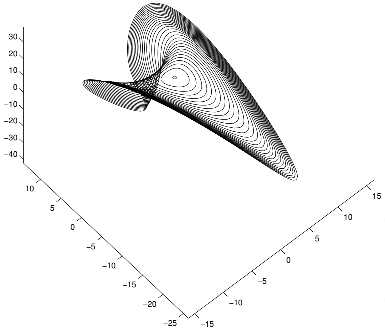


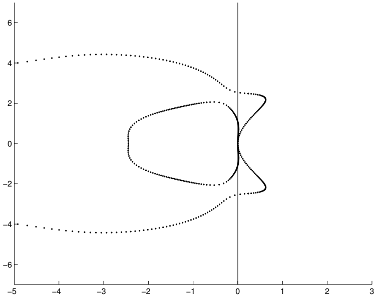
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



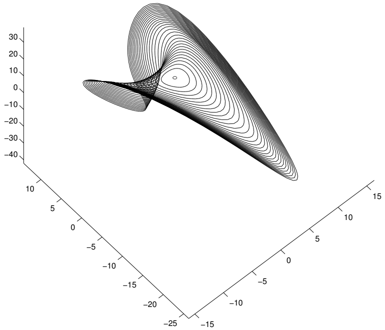


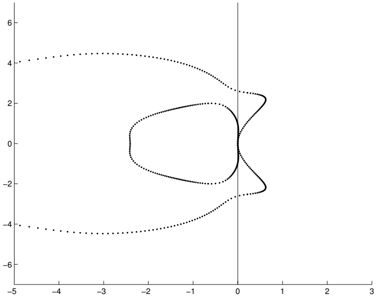
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



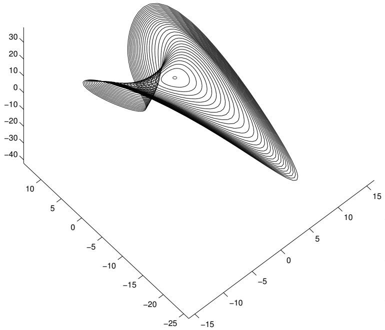


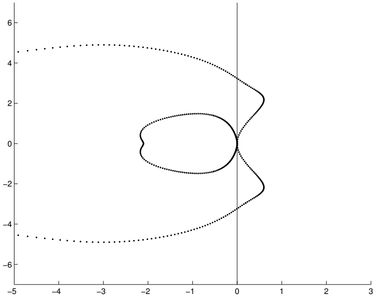
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



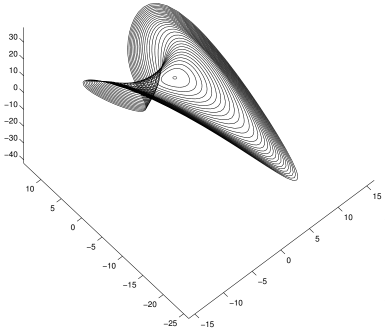


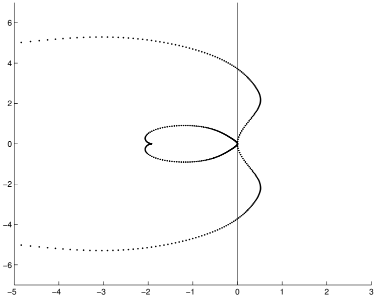
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



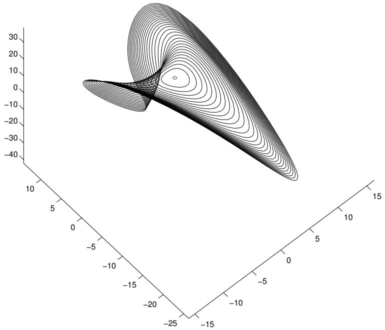


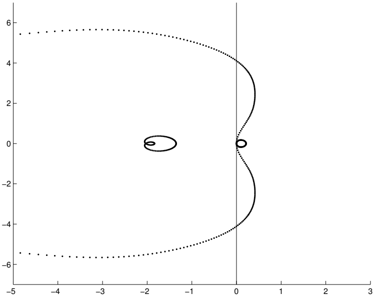
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



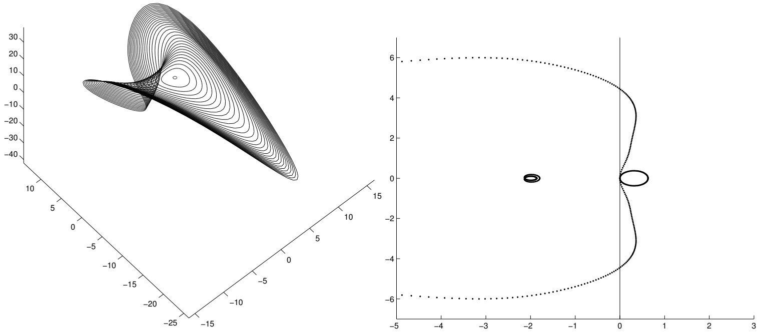


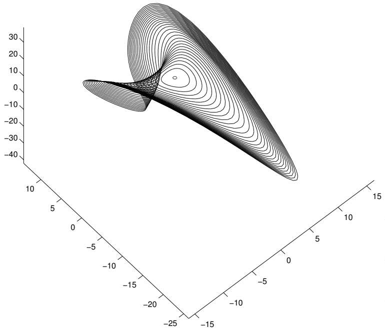
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

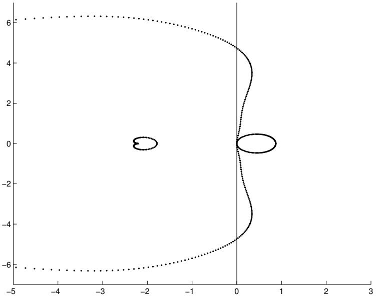




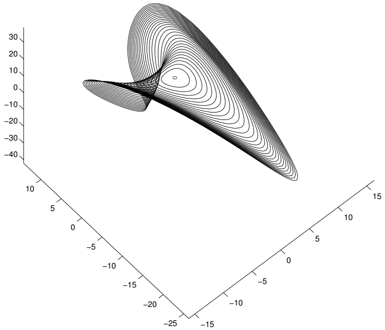
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

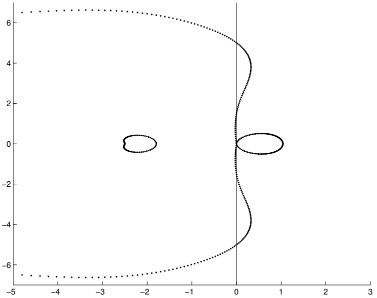



30
20
10
0
−10
−20
−30
−40
10
5
0
−5
−10
−15
−20
−25
−15
−10
−5
0
5
10
15

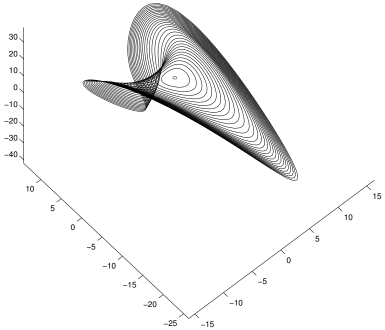

6
4
2
0
−2
−4
−6
−5
−4
−3
−2
−1
0
1
2
3

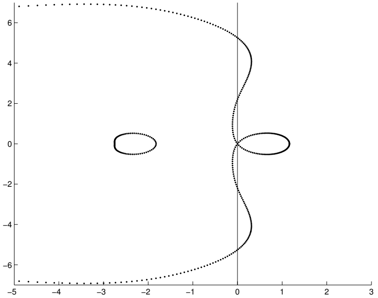
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



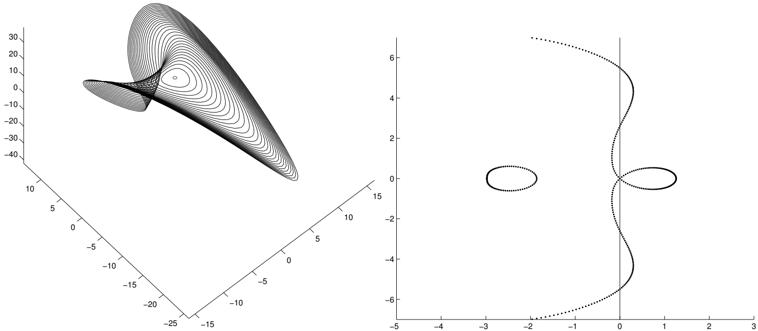


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

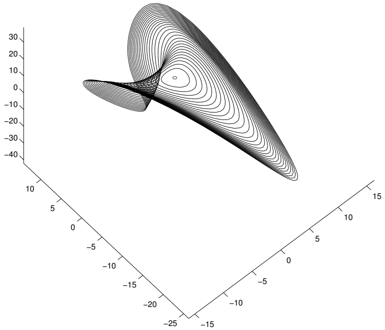


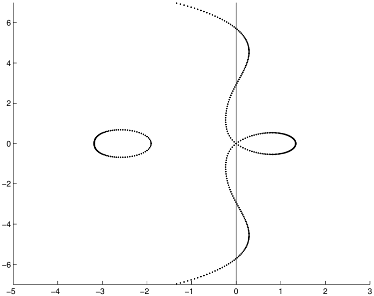


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

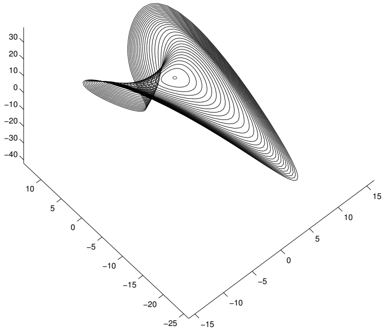


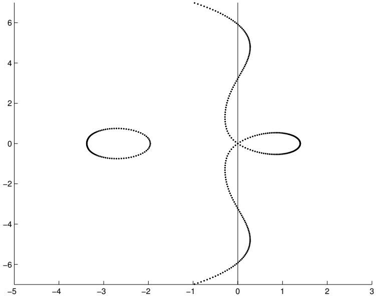
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



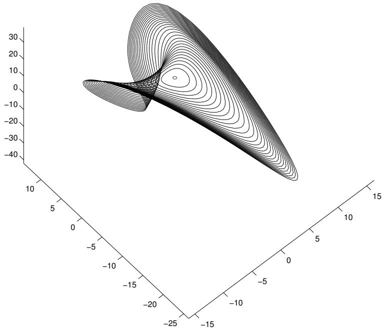


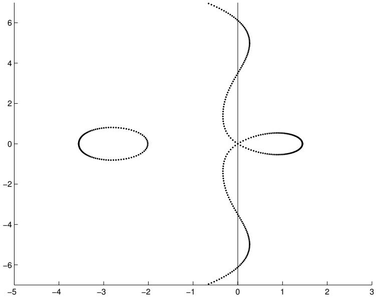
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



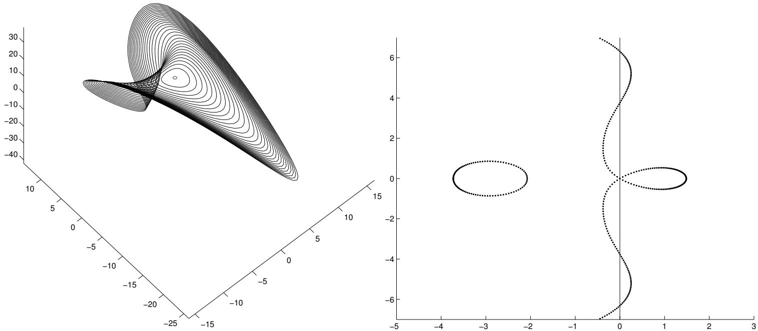


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

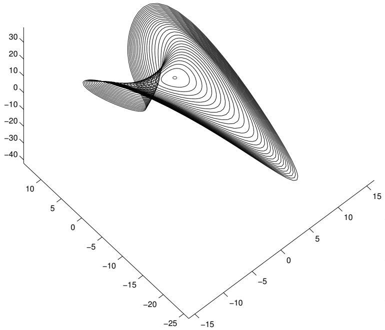


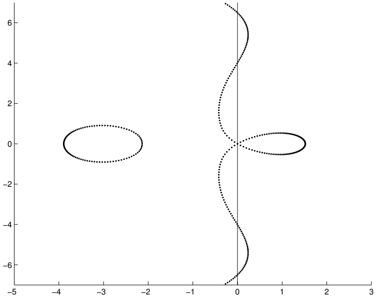


Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$

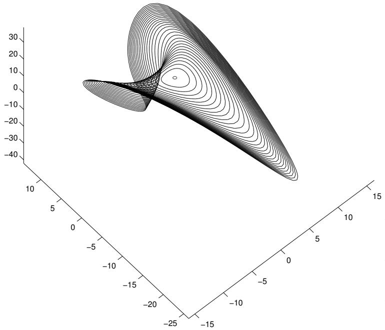


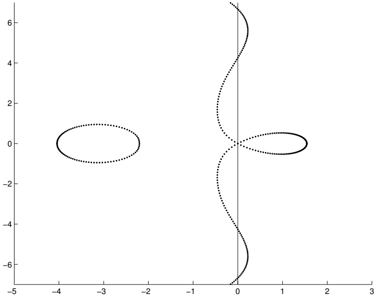
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



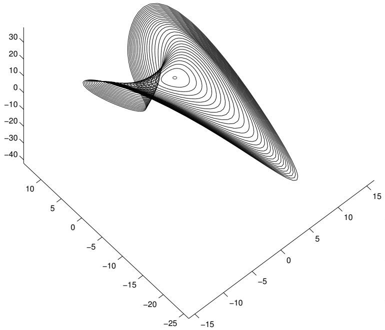


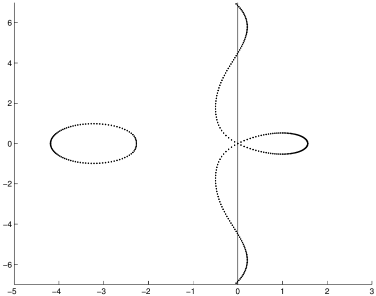
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$



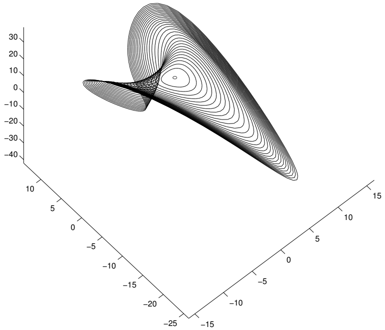


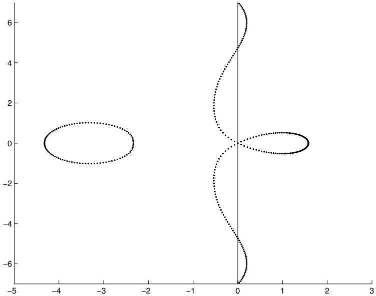
Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$





Numerical Determination of $\sigma(\mathcal{L})$ for $\varepsilon = 0.2$, $\delta = 0.04$





STOP!

Note: Spectral pictures generated with SpectrUW package based on Hill's method (Galerkin-based truncation).

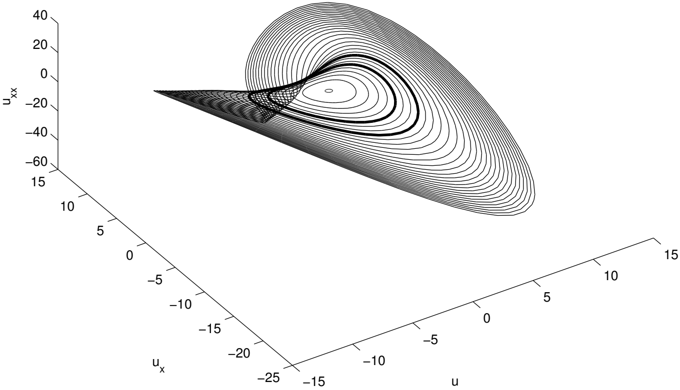


Figure: Depiction of lower (inner bold) and upper (outer bold) stability boundaries.

General Stability Boundaries

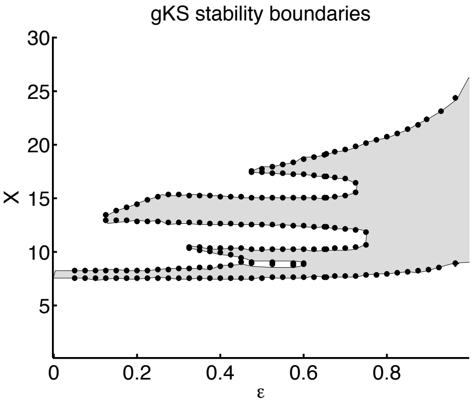
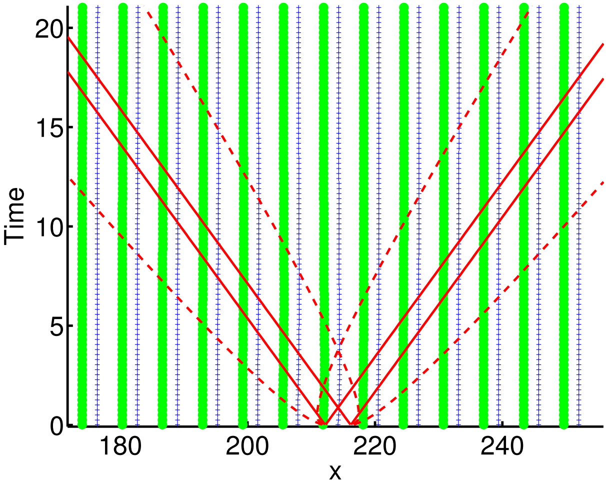
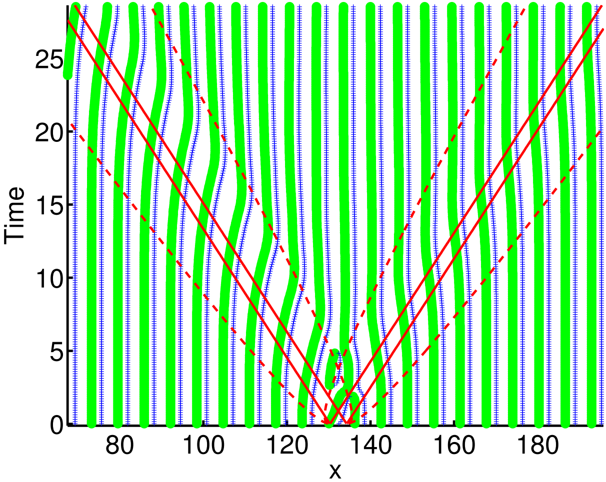


Figure: Plot of the stability boundaries (in the period X) versus the parameter $\varepsilon = \sqrt{1-\delta^2}$, with shaded regions correspond to spectrally stable periodic traveling waves. In the limits $\varepsilon \to 0$ and $\varepsilon \to 1$, we see the existence of only one band of spectrally stable periodic traveling waves.

Spectrally Stable Example:



Spectrally Stable Example (w/ "Stronger" Perturbation):



$u(x,t) \approx \bar{u}(x + \phi(x,t))$ where $\phi \approx \int$ "convecting Gaussians," i.e.

$$\phi(x,t) \approx \text{small amplitude, localized "bump"}$$

Theorem (Barker, J., Noble, Rodrigues, Zumbrun (preprint 2012))

Let $\bar{u}$ be a "spectrally stable" periodic traveling wave solution of gKS and let $\tilde{u}(x,t)$ be another solution of KS such that

$$E_0 := \|\tilde{u}(x,0) - \bar{u}\|_{L^1 \cap H^4}$$

is sufficiently small. Then $\exists\, \psi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{u}\|_{H^4(\mathbb{R})}, \;\; \|(\psi_t, \psi_x)\|_{H^4(\mathbb{R})} \lesssim (1+t)^{-1/4} E_0$$

Moreover, we have $L^1 \cap H^4 \to L^\infty$ nonlinear stability estimate

$$\|\tilde{u}(\cdot,t) - \bar{u}(\cdot)\|_{L^\infty(\mathbb{R})}, \quad \|\psi(\cdot,t)\|_{L^\infty(\mathbb{R})} \le CE_0.$$

Remark: "Spectral stability" carefully defined here.... see paper.

Verification of Spectral Stability Hypothesis:

(Joint w/ P. Noble, L.M. Rodrigues, & K. Zumbrun – preprint 2012)
Consider gKS

$$u_t + uu_x + u_{xxx} + \delta\left(u_{xx} + u_{xxxx}\right) = 0, \quad \delta > 0$$

In "thin film" regime $0 < \delta \ll 1$, gKS singular perturbation of KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

- Fact: KdV wave trains are marginally stable, i.e.

$$\mathrm{spec}_{L^2(\mathbb{R})}(L_{kdv}) \subset \mathbb{R}i.$$

- **Q:** Are "near-KdV wave trains" of gKS spectrally stable?
- **A:** Some (but certainly not all) are!!!
  - Result relies on numerical computation of elliptic integrals.... only precise up to machine error... not a thm :-(

Expansion of "near KdV" waves (Formally)

Seek traveling wave solutions of form $u(x,t) = \bar{u}(x-ct)$ of

$$u_t + uu_x + u_{xxx} + \delta\left(u_{xx} + u_{xxxx}\right) = 0.$$

Profile $\bar{u}$ satisfies ODE

$$\left(\bar{u} - c\right)\bar{u}' + \bar{u}''' + \delta\left(\bar{u}'' + \bar{u}''''\right) = 0.$$

When $\delta = 0$, $\bar{u}$ satisfies KdV, so is of form

$$U_0(x;\phi,\kappa,k,u_0) = u_0 + 12k^2\kappa^2\,\mathrm{cn}^2\left(\kappa(\omega+\phi),k\right)$$
$$c_0 = u_0 + 8\kappa^2 k^2 - 4\kappa^2$$

Now, expand

$$c = c_0 + \delta c_1 + \mathcal{O}(\delta^2), \quad \bar{u} = U_0(x) + \delta U_1(x) + \mathcal{O}(\delta^2)$$

Expansion of "near KdV" waves (Formally)

$\mathcal{O}(1)$ eqn. holds by choice of U_0.

$\mathcal{O}(\delta)$ eqn. gives

$$\mathcal{M}U_1 = c_1 U_0' - U_0'' - U_0''''$$

where $\mathcal{M}$ is Fredholm w/ index zero and

$$\ker\left(\mathcal{M}^\dagger\right) = \operatorname{span}\{1, U_0\}.$$

Have $2K(k)/\kappa$ per. soln. provided that

$$\left\langle (U_0')^2 \right\rangle = \left\langle (U_0'')^2 \right\rangle .$$

Gives selection criterion

$$\kappa = \mathcal{G}(k):$$

only KdV cnoidal waves with κ and k functionally dependent as above extend to "near KdV" periodic waves.

Q: How to verify spectral stability assumption for above "near KdV" waves?

A: Given $\delta > 0$, fix (WLOG) 1-periodic traveling wave $\bar{u} = \bar{u}_\delta$, analyze $L^2(\mathbb{R})$ spectrum of

$$\mathcal{L}[\bar{u}] := \partial_x(c - \bar{u}) - \partial_x^3 - \delta\left(\partial_x^2 + \partial_x^4\right)$$

Have spectral stability iff

$$\sigma\left(\mathcal{L}[\bar{u}]\right) \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}.$$

First problem: spectrum is purely continuous.

Characterization: $\lambda \in \sigma(\mathcal{L}[\bar{u}])$ iff $\exists \xi \in [-\pi, \pi)$ such that

$$\begin{cases} e^{-i\xi x}\mathcal{L}[\bar{u}]e^{i\xi x}v = \lambda v \\ v(x+1) = v(x) \end{cases}$$

has a non-trivial solution, i.e. if λ is a 1-periodic eigenvalue of "Bloch operator"

$$\mathcal{L}_\xi[\bar{u}] := e^{-i\xi x}\mathcal{L}[\bar{u}]e^{i\xi x} : H^4_{\rm per}([0,1]) \Subset L^2_{\rm per}([0,1]) \to L^2_{\rm per}([0,1]).$$

Bar & Nepomnyashchy (Phys. D, 1995, hereafter BN) did following:

- For fixed ξ and "near KdV" wave train u_δ assume spectral curves of Spectral problem
$$\mathcal{L}_\xi[\bar{u}_\delta]v = \lambda v$$
can be expanded as
$$\lambda(\delta,\xi) = \lambda_0(\xi) + \delta\lambda_1(\xi) + \mathcal{O}(\delta^2)$$
where $\lambda_0(\xi) \in \text{spec}(\mathcal{L}_{KdV,\xi})$.
- Using...
  (1) expansions of KdV (cnoidal) wave trains in δ, and
  (2) fact that e-values and e-ftns. of $\mathcal{L}_{kdv,\xi}$ are EXPLICITLY known $\forall\xi$,

  find EXPLICIT formula for $\lambda_1(\xi)$ in terms of elliptic integrals and show (numerically)
$$\max_\xi \Re(\lambda_1(\xi)) < 0$$
for periods $X \in (8.49, 26.17)$.
- BN conclude spectral stability for "near KdV" wave trains of these periods!

Our own numerics suggest similar stability boundaries:

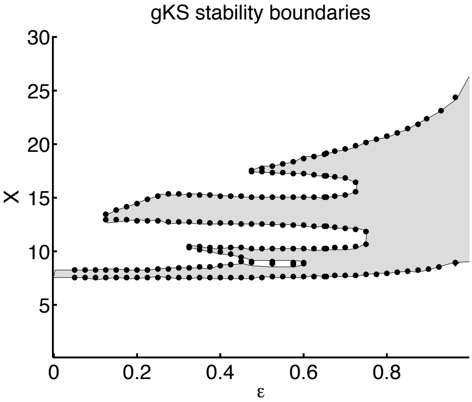


Figure: Plot of the stability boundaries (in the period X) versus the parameter $\varepsilon = \sqrt{1-\delta^2}$, with Here, $\delta = \sqrt{1-\varepsilon^2}$ is fixed by the choice of ε and the shaded regions correspond to spec. stable wave trains.

$$\text{For } 0 < 1-\varepsilon \ll 1,\ X_L(\varepsilon) \approx 8.5 \text{ and } X_U(\varepsilon) \approx 26.$$

Problems...

Computations of BN and our numerics BOTH omit small neighborhood of the origin!!!!

- In BN, expansion

$$\lambda(\delta,\xi) = \lambda_0(\xi) + \delta\lambda_1(\xi) + \mathcal{O}(\delta^2)$$

  only valid for $0 < \delta \ll |\xi|$ near $\lambda = 0$.
- Further, BN only "formally" do existence theory: requires geometric singular perturbation theory to make rigorous...
- We give high-precision computation down to very small, but positive, $\delta \Rightarrow$ omits $O(\delta \times \mathrm{TOL})$ nbhd. of origin.

To show stability, need to resolve neighborhood of origin: proceed in three steps...

(1) Study region $0 < \delta \ll |\xi|$ (BN region).
(2) Study region $0 < |\xi| \ll \delta$ (weak, large scale perturbations).
(3) Study in between, i.e. $C^{-1}|\xi| \le \delta \le C|\xi|$.

Evans Function Computations:

$\exists$ $\mathbb{C}$-analytic ftn. E (Evans ftn.) such that

$$\lambda \in \text{spec}\left(\mathcal{L}_\xi[\bar{u}_\delta]\right) \iff E(\lambda,\xi,\delta) = 0.$$

<u>Fact:</u> For $(|\lambda|,|\xi|,\delta) \ll 1$, up to non-zero factor

$$E(\lambda,\xi,\delta) = E_{kdv}(\lambda,\xi) + \delta E_1(\lambda,\xi) + \mathcal{O}\left(\delta^2(\xi^2+\lambda^2)\right).$$

Up to non-zero factor,

$$E(\lambda,\xi,\delta) = \underbrace{\prod_{j=1}^{3}(\lambda - i\alpha_j(\xi)\xi)}_{E_{kdv}(\lambda,\xi)} + \gamma\delta\prod_{k=1}^{2}(\lambda - i\beta_k^0\xi) + H.O.T.$$

where $\gamma \in \mathbb{R}$, $i\alpha_j(\xi)\xi \in \mathbb{R}i$ are *distinct* roots of $E_{kdv}(\cdot,\xi)$ near $(\lambda,\xi) = (0,0)$, and β_j^0 are real or $\mathbb{C}$-conjugate.

In BN Range ($0 < \delta \ll |\xi|$):

Equation $E(\lambda, \xi, \delta) = 0$ reads

$$\prod_{j=1}^{3} (\lambda - i\alpha_j(\xi)\xi) + \gamma\delta \prod_{k=1}^{2} (\lambda - i\beta_k^0 \xi) + H.O.T. = 0.$$

Setting $\bar{\delta} = \delta/\xi$, $\bar{\lambda} = \lambda/\xi$ gives

$$\prod_{j=1}^{3} \left(\bar{\lambda} - i\alpha_j(\xi)\right) + \gamma\bar{\delta} \prod_{k=1}^{2} (\bar{\lambda} - i\beta_k^0 \xi) + H.O.T. = 0$$

When $\bar{\delta} = 0$, get $\prod_{j=1}^{3} \left(\bar{\lambda} - i\alpha_j(\xi)\right) = 0$. Since $\alpha_j(\xi)$ are distinct, implicit function theorem gives expansions

$$\bar{\lambda}_j(\xi, \bar{\delta}) = i\alpha_j(\xi) - \gamma\bar{\delta} \frac{(\alpha_j(\xi) - \beta_1^0)(\alpha_j(\xi) - \beta_2^0)}{\prod_{k \neq j} (\alpha_j(\xi) - \alpha_k(\xi))} + \mathcal{O}(\bar{\delta}\xi).$$

In BN Range ($0 < \delta \ll |\xi|$):

Using even symmetry of $\Re(\lambda_j(\xi,\delta))$ and $\alpha_j(\xi)$, follows

$$\Re(\lambda_j(\xi,\delta)) = -\gamma \frac{(\alpha_j(0) - \beta_1^0)(\alpha_j(0) - \beta_2^0)}{\prod_{k \neq j}(\alpha_j(0) - \alpha_k(0))} + \mathcal{O}(\delta\xi^2).$$

Fact: $\Re(\lambda_j(\xi,\delta)) < 0$ for each iff "Subcharacteristic Conditions" hold

(S1) $\beta_1^0, \beta_2^0 \in \mathbb{R}$ are distinct.

(S2) $\alpha_1(0) < \beta_1^0 < \alpha_2(0) < \beta_2^0 < \alpha_3(0)$.

(S3) $\gamma > 0$.

Fact: Numerics (elliptic function calculations) of BN $\Rightarrow$ for periods $X \in (8.49, 26.17)$, have

$$\max_{\xi} Re(\lambda_j(\xi,\delta)) < 0$$

Thus, (S1)-(S3) hold for these periods!!!!!

In Range $0 < |\xi| \ll \delta$:

As before, equation $E(\lambda, \xi, \delta) = 0$ reads

$$\prod_{j=1}^{3} (\lambda - i\alpha_j(\xi)\xi) + \gamma\delta \prod_{k=1}^{2} (\lambda - i\beta_k^0 \xi) + H.O.T. = 0.$$

Setting $\bar{\lambda} = \lambda/\delta$, $\bar{\xi} = \xi/\delta$ gives

$$\prod_{j=1}^{3} (\bar{\lambda} - i\alpha_j(\delta\bar{\xi})\bar{\xi}) + \gamma \prod_{k=1}^{2} (\bar{\lambda} - i\beta_k^0 \bar{\xi}) + H.O.T. = 0.$$

When $\bar{\xi} = 0$, get $\bar{\lambda}^2 \left(\bar{\lambda} + \gamma\right) = 0$.

In Range $0 < |\xi| \ll \delta$:

Implicit Function Theorem gives three roots:

$$\Re\left(\lambda_k(\xi,\delta)\right) = (-1)^{k+1}\frac{\xi^2}{\gamma\delta}\frac{\prod_{j=1}^{3}(\beta_k^0 - \alpha_j(0))}{\beta_1^0 - \beta_2^0} + \mathcal{O}(\xi^2), \;\; k = 1,2$$

$$\lambda_3(\xi,\delta) = -\gamma\delta + o(\delta).$$

if (S1)-(S3) hold, where recall

(S1) $\beta_1^0, \beta_2^0 \in \mathbb{R}$ are distinct.

(S2) $\alpha_1(0) < \beta_1^0 < \alpha_2(0) < \beta_2^0 < \alpha_3(0)$.

(S3) $\gamma > 0$.

Fact: $\Re(\lambda_j(\xi,\delta)) < 0$ for each j if (S1)-(S3) hold!!!

$\therefore \Re\left(\lambda_k(\xi,\delta)\right) \leq 0$ for $k = 1,2,3$ by numerics of BN!!!

In Between... Have No Crossing!

- Fact: If (S1)-(S3) hold, then $\forall C > 1\ \exists \eta > 0$ such that

$$C^{-1}|\xi| \leq \delta \leq C|\xi|$$

  implies

$$\mathrm{spec}_p(\mathcal{L}_\xi) \cap B(0,\eta) \cap \mathbb{R} \neq \emptyset$$

  iff $\lambda = \xi = 0$.
- Proof: Simple symmetry argument.

Thus, numerics of BN $\Rightarrow$ have spectral stability near origin!!!

$\therefore$ some near KdV-wavetrains are spectrally stable!!!!

In fact, they satisfy hypothesis of our nonlinear stability theorem

$\therefore$ some near KdV-wavetrains are nonlinearly stable!!!!

Introduce slow coordinates $(X, T) = (\varepsilon x, \varepsilon t)$, $\varepsilon \ll 1$, set $\delta = \bar{\delta}\varepsilon$ and notice in slow coordinates KdV-KS reads

$$\partial_T u + u\partial_X u + \varepsilon^2 \partial_X^3 u + \bar{\delta}\left(\varepsilon^2 \partial_X^2 u + \varepsilon^4 \partial_X^4 u\right) = 0.$$

Performing WKB expansion in $u(X, T)$, Noble & Rodrigues showed Whitham system for KdV-KS in singular $\delta \to 0^+$ limit is

$$\begin{aligned}
&\partial_T \kappa + \partial_X \left(\kappa c_0(u_0, \kappa, k)\right) = 0 \\
&\partial_T \langle U_0 \rangle + \partial_X \left\langle \frac{U_0^2}{2} \right\rangle = 0 \\
&\partial_T \left\langle \frac{U_0^2}{2} \right\rangle + \partial_X \left\langle \frac{U_0^3}{3} - \frac{3(U_0')^2}{2} \right\rangle = \bar{\delta}\left(\langle (U_0')^2 \rangle - \left\langle (U_0'')^2 \right\rangle\right).
\end{aligned}$$

$\bar{\delta} \to 0 \Rightarrow$ characteristics $\alpha_j(u_0, \kappa, k) \in \mathbb{R}$, $j = 0, 1, 2$ distinct (strict hyperbolicity of KdV Whitham!).

Relation to Whitham Equations

$\bar{\delta} \to \infty$ obtain relaxed system

$$\partial_T \kappa + \partial_X \left(\kappa c_0(u_0, \kappa, k) \right) = 0$$
$$\partial_T \langle U_0 \rangle + \partial_X \left\langle \frac{U_0^2}{2} \right\rangle = 0$$

where $\kappa = \mathcal{G}(k)$. This is exactly Whitham for KdV-KS (fixed δ) in limit $\delta \to 0$.

Hyperbolicity of above a necessary condition for spectral stability "near-KdV" waves for δ, i.e. above should have eigenvalues

$$\beta_0(u_0, k) < \beta_1(u_0, k).$$

Relation to Whitham Equations

Relaxation theory $\Rightarrow$ necessary condition for stability of "near KdV" wave is

$$\alpha_0 \leq \beta_0 \leq \alpha_1 \leq \beta_1 \leq \alpha_2$$

where above α_j, β_j are evaluated at limiting KdV cnoidal wave U_0.

This motivates subcharacteristic condition (S1). Can also motivate (S2)-(S3), see paper.

Thank you!

Papers & references available at
http://www.math.ku.edu/ matjohn