Index Theorems for the Stability of Periodic Traveling Waves of KdV Type

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Outline

Introduction

Consider the KdV equation

$$u_t = u_{xxx} + f(u)_x$$

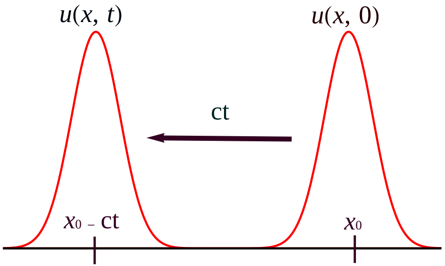
where $f(u)$ is "nice". Arise in applications with a variety of nonlinearities.

- $f(u) = u^2 \Rightarrow$ KdV equation. Canonical model for weakly dispersive nonlinear unidirectional wave propagation.
- $f(u) = \pm u^3 \Rightarrow$ focusing/defocusing mKdV equation. Arises naturally in plasma physics as a model for ion acoustic perturbations.
- $f(u) = \alpha u^{r+1/2}$ for $r \in \left(-\frac{1}{2}, \frac{1}{2}\right)$... has been derived in several plasma physics models.

Also interesting for mathematical study: $f(u) = u^5$ is L^2 critical, and KdV and mKdV are completely integrable PDE!

Traveling Waves of form $u(x,t) = \bar{u}(x+ct)$ are basic structures in nonlinear waves!

- Characteristics:
  1. (1) Constant velocity c
  2. (2) Same shape and profile!



$\therefore$ Traveling wave profile $\bar{u}$ is STATIONARY solution of PDE

$$u_t = u_{xxx} + f(u)_x - cu_x,$$

i.e. solves ODE

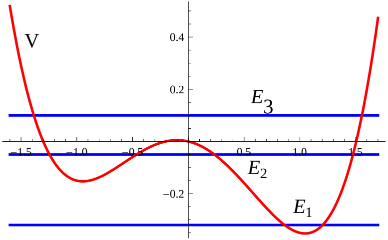
$$u''' + f(u)' - cu' = 0.$$

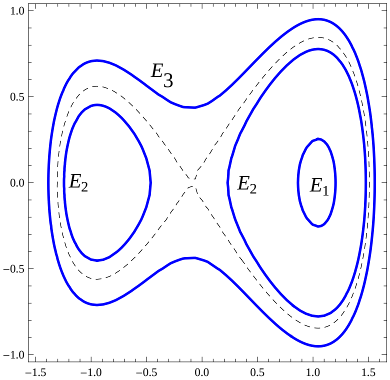
After one integration, this is a HAMILTONIAN ODE!!!

Wave profile u must satisfy ODE

$$\frac{u_x^2}{2} = E - \underbrace{F(u) - \frac{cu^2}{2} + au}_{V(u;a,c)}, \quad F' = f$$

where $a, E \in \mathbb{R}$ depend on boundary conditions imposed.





Summary of ∃ theory

Solitary Waves:

- If $\bar{u}(x) \to \textit{const}.$ when $x \to \pm\infty$, have "Solitary wave".
- In this case, a and E fixed by b.c.'s at $\pm\infty$, so have two parameter family of traveling waves:

$$\bar{u}(x + x_0 - ct; c).$$

Periodic Waves:

- If $\bar{u}(x + T) = \bar{u}(x)$ for some $T > 0$, have "periodic wave".
- In this case, a and E are "free", so have four parameter family of traveling waves:

$$\bar{u}(x + x_0 - ct; a, E, c), \quad \text{period } T = T(a, E, c).$$

- In special cases, $\bar{u}$ can be expressed in terms of elliptic functions. We make no use of this extra structure in our analysis...

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Solitary Wave Stability Theory:

Linearization of gKdV flow about solitary wave with $\bar{u}(x) \to 0$ as $x \to \pm\infty$:

$$-v_t = \partial_x \underbrace{\left(-\partial_x^2 - f'(\bar{u}) + c\right)}_{\mathcal{L}[\bar{u}]} v, \quad v \in L^2(\mathbb{R}).$$

Seek separated solution $v(x,t) = e^{-\lambda t} v(x)$ leads to spectral problem

$$\partial_x \mathcal{L}[\bar{u}] v = \lambda v.$$

Spectral stability iff $\sigma(\partial_x \mathcal{L}[\bar{u}]) = \sigma_{\text{ess}}(\partial_x \mathcal{L}[\bar{u}]) \cup \sigma_p(\partial_x \mathcal{L}[\bar{u}]) \subset i\mathbb{R}$.

Essential Spectrum: Linear dispersion relation about background $\bar{u} \equiv 0$ state is

$$ik\left(k^2 - f'(0) + c\right) = \lambda \Rightarrow \sigma_{\text{ess}}\left(\partial_x \mathcal{L}[\bar{u}]\right) = i\mathbb{R}.$$

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Point Spectrum: Eigenvalues of $\partial_x \mathcal{L}[\bar{u}]$, acting on $L^2(\mathbb{R})$, determined by roots of "Evans Function" (transmission coefficient) $D(\lambda)$.

- $D(\lambda)$ detects intersections of stable mfld. at $+\infty$ and unstable mfld at $-\infty$.
- Complex analytic in λ.
- Roots agree in location and (algebraic) multiplicity of e.v.'s of $\partial_x \mathcal{L}[\bar{u}]$.

Fact 1

1. $\text{sign}(D(\lambda)) = 1$ for $\lambda \gg 1$.
2. For some constant $A > 0$,

$$D(\lambda) = A \left(\partial_c \int_{\mathbb{R}} u(x;c)^2 dx|_{\bar{u}} \right) \lambda^2 + \mathcal{O}(|\lambda|^3).$$

Thus, have spectral instability if $\partial_c \int_{\mathbb{R}} u(x;c)^2 dx < 0$ at $\bar{u}$.

FACT 2: [Pego& Weinstein 1992] $0 \leq n_{-}\left(\partial_x \mathcal{L}[\bar{u}]\right) \leq n_{-}\left(\mathcal{L}[\bar{u}]\right)$.

FACT 3: For solitary waves, $n_{-}(\mathcal{L}[\bar{u}]) = 1$.
Proof: $\bar{u}'$ satisfies $\mathcal{L}[\bar{u}]\bar{u}' = 0$ and has only one root on $\mathbb{R}$. Sturm Liouville Theory $\Rightarrow$ 0 is second eigenvalue of $\mathcal{L}[\bar{u}]$.

$\therefore$ all unstable eigenvalues must be real!!

$\therefore$ Spectral stability iff $\partial_c \int_{\mathbb{R}} u(x;c)^2 dx > 0$ at $\bar{u}$.

Further, Fact 3 $\Rightarrow$ condition necessary/sufficient for nonlinear (orbital) stability!

FACT 2: [Pego& Weinstein 1992] $0 \leq n_-\left(\partial_x \mathcal{L}[\bar{u}]\right) \leq n_-\left(\mathcal{L}[\bar{u}]\right)$.

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Periodic Case?

Periodic Case is much more complicated:

(1) "More" of them: 4 parameter family, compared to 2 parameter family of solitary waves.

(2) More general classes of perturbations available:
   (a) Co-periodic $= L^2(\mathbb{R}/T\mathbb{Z})$.
   (b) Sub-harmonic $= L^2(\mathbb{R}/nT\mathbb{Z})$, $n \in \mathbb{N}$, $n > 1$.
   (c) Localized $= L^2(\mathbb{R})$... Most Physical!

(3) Structure of spec.: may be only eigenvalues, may be only essential spec... depends on class of perturbations.

(4) $n_-(\mathcal{L}[\bar{u}])$ can be *arbitrarially large* (or "uncountable") depending on class of perturbations.

Periodic Stability Theory

- Let $\bar{u}$ be T-periodic stationary solution of the nonlinear PDE

$$u_t = u_{xxx} + f(u)_x - cu_x.$$

Consider a perturbation of $\bar{u}$: $\psi(x,t) = \bar{u}(x) + \varepsilon v(x,t)$, $v \in X$.

$$\Rightarrow \partial_x \underbrace{\left(-\partial_x^2 - f'(\bar{u}) + c\right)}_{\mathcal{L}[\bar{u}]} v = -v_t$$

Decompose $v(x,t) = e^{-\lambda t} v(x)$ so v solves the spectral problem

$$\partial_x \mathcal{L}[u] v = \lambda v$$

considered on X.

Spectral stability to X-perturbations $\iff$ $\text{spec}_X\left(\partial_x \mathcal{L}[u]\right) \subset \mathbb{R}i$.

Goal: analyze spectral problem

$$\partial_x \mathcal{L}[\bar{u}]v = \lambda v, \quad v \in X. \quad (\star)$$

What is structure of $\text{spec}_X\left(\partial_x \mathcal{L}[u]\right)$?

(1) If $X = L^2(\mathbb{R}/nT\mathbb{Z})$, then

$$\text{spec}_X\left(\partial_x \mathcal{L}[u]\right) = \text{spec}_{X,p}\left(\partial_x \mathcal{L}[u]\right)$$

$\Rightarrow$ $(\star)$ is an eigenvalue problem!!

(2) If $X = L^2(\mathbb{R})$, then

$$\text{spec}_X\left(\partial_x \mathcal{L}[u]\right) = \text{spec}_{X,\text{ess}}\left(\partial_x \mathcal{L}[u]\right)$$

$\Rightarrow$ $(\star)$ has no eigenvalues... all instabilities come from essential sepc.!!!

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Localized Perturbations: $X = L^2(\mathbb{R})$

- To see this, wite spectral problem as first order system

$$Y'(x,\lambda) = \mathbf{H}(x,\lambda)Y(x,\lambda).$$

- Period Map (Monodromy): $\mathbf{M}(\lambda) = \Phi(T,\lambda)$, where $\Phi(x,\lambda)$ is the matrix solution such that $\Phi(0,\lambda) = \mathbf{I}$. Thus, $\mathbf{M}(\lambda)$ is an operator such that

$$\mathbf{M}(\lambda)v(x,\lambda) = v(x+T,\lambda)$$

for any $x \in \mathbb{R}$ and vector solution $v(x,\lambda)$. For simplicity, assume that $v(x,\lambda)$ satisfies

$$\mathbf{M}(\lambda)v(x,\lambda) = \mu v(x,\lambda)$$

Then for all $n \in \mathbb{Z}$ have

$$v(NT,\lambda) = \mathbf{M}(\lambda)^N v(0,\lambda) = \mu^N v(0,\lambda)$$

$\Rightarrow$ if $v(x,\lambda) \to 0$ as $x \to +\infty$, then $\lim_{x\to-\infty} |v(x,\lambda)| = +\infty$.

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Best you can hope for is for v to be uniformly bounded, i.e. $|\lambda| = 1$.

- Gives characterization of (continuous) spectrum:

$$\lambda \in \mathrm{spec}(\partial_x \mathcal{L}[u]) \iff \sigma(\mathbf{M}(\lambda)) \bigcap S^1 \neq \emptyset.$$

Following Gardner then, we define

$$D(\lambda, e^{i\kappa}) = \det\left(\mathbf{M}(\lambda) - e^{i\kappa}\mathbf{I}\right).$$

Then $\lambda \in \mathrm{spec}(\partial_x \mathcal{L}[u]) \iff D(\lambda, e^{i\kappa}) = 0$ for some $\kappa \in \mathbb{R}$.

- Moreover,

$$\mathrm{spec}_{L^2(\mathbb{R})}(\partial_x \mathcal{L}[u]) = \bigcup_{\kappa \in [-\pi, \pi)} \left\{\lambda \in \mathbb{C} : D(\lambda, e^{i\kappa}) = 0\right\}.$$

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Remark:

If $\kappa \in 2\pi\mathbb{Q}$, then bounded solution of

$$\mathbf{M}(\lambda)v = e^{i\kappa}v$$

is nT periodic for some $n \in \mathbb{N}$, i.e. $\lambda \in \sigma_{L^2(\mathbb{R}/nT\mathbb{Z})}(\partial_x \mathcal{L}[\bar{u}])$.

$$\Rightarrow \quad \text{spec}_{L^2(\mathbb{R})}(\partial_x \mathcal{L}[u]) = \overline{\bigcup_{n\in\mathbb{N}} \text{spec}_{L^2(\mathbb{R}/nT\mathbb{N})}(\partial_x \mathcal{L}[u])}.$$

$\therefore$ spec. stable in $L^2(\mathbb{R})$ iff spec. stable in $L^2(\mathbb{R}/nT\mathbb{Z})$ $\forall n \in \mathbb{N}$.

Analyaisis of Evans ftn.

$\exists$ theory gives *a lot* of information about spectrum at $\lambda = 0$.

- Have 4-dim manifold
$$\mathcal{M} = \{u(x + x_0 - ct; a, E, c)\}$$
of stationary solutions of gKdV
$$u_t - u_{xxx} - f(u)_x + cu_x = 0.$$
- Diff. Geometry $\Rightarrow$ equation
$$(\partial_t - \partial_x \mathcal{L}[\bar{u}])\, v = 0$$
defines tan. space of $\mathcal{M}$ at fixed solution $\bar{u}$.
- Tan. space at $\bar{u}$ generated by variations:
$$\mathcal{T}_{\bar{u}}(\mathcal{M}) = \mathrm{span}\{u_x, u_a, u_E, -tu_x + u_c\}$$
- Follows that (formally)
$$\partial_x \mathcal{L}[u]\{u_x, u_a, u_E\} = 0, \quad \partial_x \mathcal{L}[u] u_c = -u_x.$$

Periodic Stability Theory

$\therefore$ have full set of (formal) solutions to ODE

$$\partial_x \mathcal{L}[u]v = 0.$$

Moreover, $\bar{u}_x$ is T-periodic... follows that

$$D(0,1) = \det\left(\mathbf{M}(0) - \mathbf{I}\right) = 0.$$

Want to find curve $\kappa \to \lambda(\kappa)$ defined in neighborhood of $(\lambda, \kappa) = (0,0)$ such that

$$D(\lambda(\kappa), e^{i\kappa}) = \det\left(\mathbf{M}(\lambda(\kappa)) - e^{i\kappa}\right) = 0.$$

Would be easy if we could use implicit function theorem, i.e. if

$$\partial_\lambda D(\lambda, 1)\Big|_{\lambda=0} \neq 0.$$

Evaluate of $\partial_\lambda D(\lambda,1)\big|_{\lambda=0}$

- At $\lambda = 0$, $\{u_x, u_a, u_E\}$ provides three linearly independent solutions of the *formal* differential equation

$$\partial_x \mathcal{L}[u]v = 0.$$

Thus, can explicitly construct monodromy matrix at $\lambda = 0$.

- By analyticity of $\mathbf{M}(\lambda)$, have

$$\mathbf{M}(\lambda) = \mathbf{M}(0) + \lambda\mathbf{M}_\lambda(0) + \mathcal{O}(|\lambda|^2)$$

We use perturbation theory to find $\mathbf{M}_\lambda(\lambda)$....

- Variation of parameters formula yields first order variation in u_a and u_E columns. Moreover, u_c solves

$$\partial_x \mathcal{L}[u]u_c = -u_x$$

Follows that $-u_c$ gives first order λ-variation in translation (u_x) direction!!!! Thus, we have constructed $\mathbf{M}_\lambda(0)$.

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Thus, can explicitly construct monodromy matrix at $\lambda = 0$.

- By analyticity of $\mathbf{M}(\lambda)$, have

$$\mathbf{M}(\lambda) = \mathbf{M}(0) + \lambda\,\mathbf{M}_\lambda(0) + \mathcal{O}(|\lambda|^2)$$

We use perturbation theory to find $\mathbf{M}_\lambda(\lambda)$....

- Variation of parameters formula yields first order variation in u_a and u_E columns. Moreover, u_c solves

$$\partial_x \mathcal{L}[u]u_c = -u_x$$

Follows that $-u_c$ gives first order λ-variation in translation (u_x) direction!!!! Thus, we have constructed $\mathbf{M}_\lambda(0)$.

Asymptotic Expansion of $D(\lambda, 1)$

- Taking determinants then, we have

$$\frac{d}{d\lambda} D(\lambda, 1) = \det\left(\mathbf{M}(0) + \lambda\, \mathbf{M}_\lambda(0) - I + \mathcal{O}(|\lambda|^2) \right) \Big|_{\lambda=0} = 0$$

$\Rightarrow$ Implicit Function Theorem fails!!!!!

- We need to determine next order term $\mathbf{M}_{\lambda\lambda}(0)$. Can be done by using variation of parameters again!

Asymptotic Expansion for $D(\lambda, 1)$

- Ugly algebra yields

$$D(\lambda,1) = -\frac{1}{2}\underbrace{\frac{\partial(T,M,P)}{\partial(a,E,c)}}_{\{T,M,P\}_{a,E,c}}\lambda^3 + \mathcal{O}(|\lambda|^4).$$

  where T =period and M and P refer to the mass and momentum:

$$M = \int_0^T u(x)dx \qquad P = \int_0^T u(x)^2 dx.$$

  M and P are conserved quantities of the gKdV flow!

- Thus, $D(\lambda, 1) = \mathcal{O}(|\lambda|^3)$ and hence more care is needed to use the implicit function theorem.
- In particular, follows there are in general three branches of spectrum which bifurcate from the $\lambda = 0$ state for $|\kappa| \ll 1$.

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- Continuing above computations, local analysis around $(\lambda, \kappa) = (0, 0)$ yields

$$D(\lambda, e^{i\kappa}) = i\kappa^3 + \frac{i\kappa\lambda^2}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) - \frac{\lambda^3}{2}\{T, M, P\}_{a,E,c} + \mathcal{O}(|\lambda|^4 + \kappa^4)$$

where the notation $\{f, g\}_{x,y}$ is used for two-by-two Jacobians.

- Defining $z = \frac{i\kappa}{\lambda}$, we see z must be a root of

$$P(z) = -z^3 + \frac{z}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) - \frac{1}{2}\{T, M, P\}_{a,E,c}.$$

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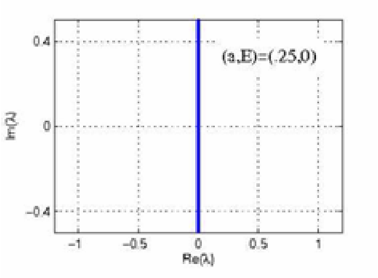
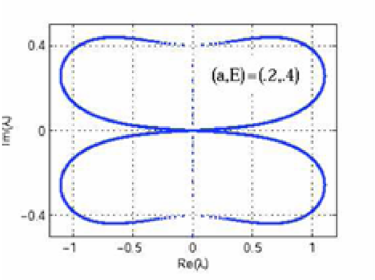
Define

$$\Delta_{MI} := \frac{1}{2}\left(\{T,P\}_{E,c} + 2\{M,P\}_{a,E}\right)^3 - \frac{27}{4}\{T,M,P\}^2_{a,E,c}.$$



$\Delta_{MI} > 0$



$\Delta_{MI} < 0$

Yields "normal form" for spectrum near origin for gKdV equations!

Index Δ_{MI} detects instabilities to "long-wavelength" perturbations, i.e. to $\tilde{T}$-periodic perturbations with

$$0 < |T - \tilde{T}| \ll 1.$$

Such instabilities sometimes called "modulational" or "side-band".

Can also use above computations to detect "co-periodic" ($\kappa = 0$) instabilities, i.e. stabilities in $L^2(\mathbb{R}/T\mathbb{Z})$.

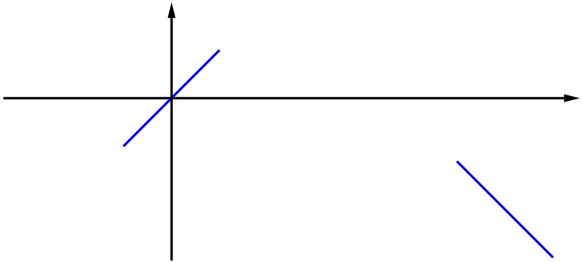
Recall, from above, that $D(\lambda, 1) = -\frac{1}{2}\{T, M, P\}_{a,E,c}\lambda^3 + \mathcal{O}(|\lambda|^4)$. Also, can prove that

$$\lim_{\lambda \to \infty} \operatorname{sign}\left(D(\lambda, 1)\right) < 0$$

M.J. & Bronski (ARMA – 2010)

Yields orientation index

$$\lim_{\lambda\to 0^+} \operatorname{sign}\left(D(\lambda,1)\right) \lim_{\lambda\to\infty} \operatorname{sign}\left(D(\lambda,1)\right) = \operatorname{sign}\left(\{T,M,P\}_{a,E,c}\right).$$



If $\{T,M,P\}_{a,E,c} < 0$, then $D(\lambda,1) = 0$ for some $\lambda > 0$.

M.J. & Bronski (ARMA – 2010)

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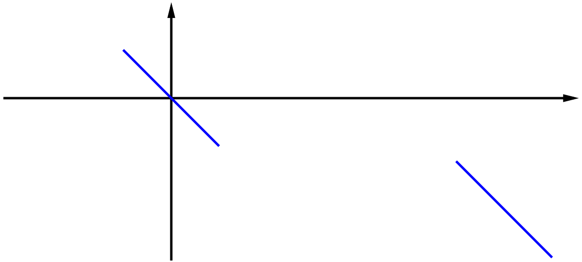

If $\{T, M, P\}_{a,E,c} > 0$, then $D(\lambda, 1) < 0$ for all $\lambda > 0$?

Yes, if for $\mathcal{L}[u]$ considered on $L^2(\mathbb{R}/T\mathbb{Z})$, $n_-(\mathcal{L}[u]) = 1$. (Same argument as in Solitary wave case!)

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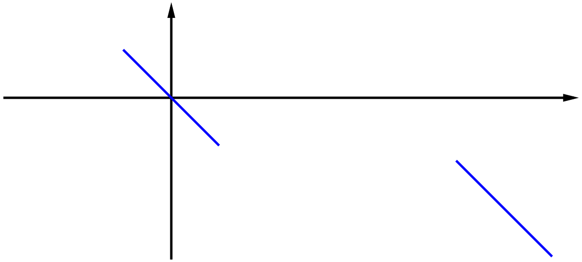

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Q: How does this compare with Solitary Wave theory when $f(u) = u^{p+1}$, $c > 0$.

Using standard asymptotic methods, can show that in "homoclinic limit"

$$\{T, M, P\}_{a,E,c} \sim -T_E M_a P_c$$

and where $T_E > 0$ (clearly) and $M_a < 0$ (computation).

Thus, in homoclinic limit,

$$\mathrm{sign}(\{T, M, P\}_{a,E,c}) = \mathrm{sign}(P_c) = \mathrm{sign}\left(\partial_c \int_0^T u^2 dx\right).$$

Moreover, by scaling, have

$$\mathrm{sign}(P_c) = \mathrm{sign}\left(\frac{2}{pc} - \frac{1}{2c}\right) = \mathrm{sign}(4 - p).$$

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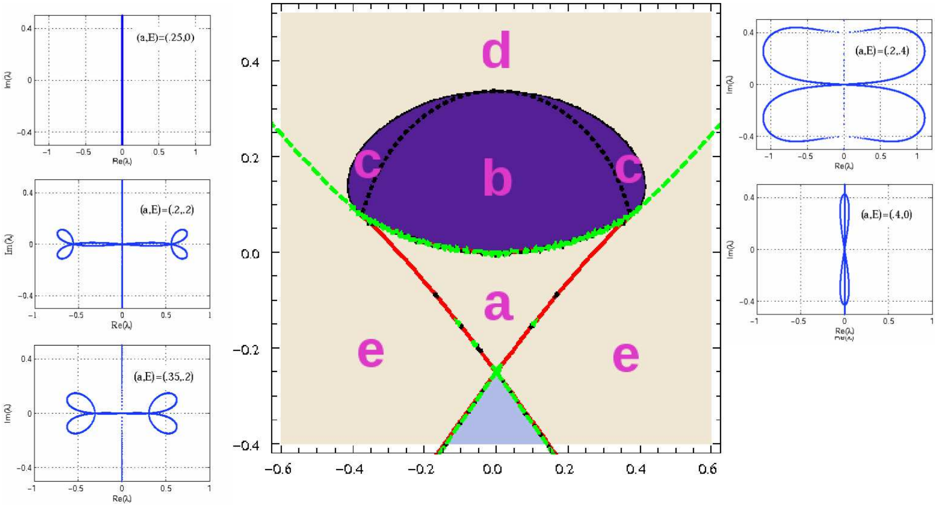
Computation?

- OK, so you have an expression which "determines" when a particular wave is modulationally stable..... can you compute it?!
- YES!!!
  (1) For power-law nonlinearities ($f(u) = u^{p+1}$) with $p \in \mathbb{N}$, can determine explicit formula for MI index in terms of moments of the underlying wave.
  (2) For non-power-law, must rely on numerics..... but at least you now have a determined quantity to do numerics on!

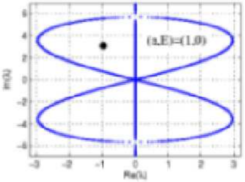
Modulational Theory for KdV

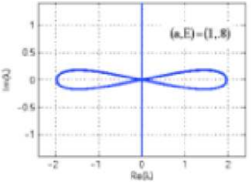
- In case of KdV
$$u_t = u_{xxx} + \left(\frac{u^2}{2}\right)_x,$$
can express conserved quantities and period as integrals of closed cycles over a Riemann surface, and hence we can compute MI index using elliptic function calculations (Picard-Fuchs system).
- Get
$$\Delta_{MI} = C_0 \cdot \frac{N^2}{\mathrm{disc}(E - V(\cdot; a, E, c))}$$
where $C_0 > 0$.
- Notice $\mathrm{disc}(E - V(\cdot; a, E, c)) > 0$ iff the corresponding solution is periodic, so all periodic waves of KdV are modulatioanlly stable!!!!

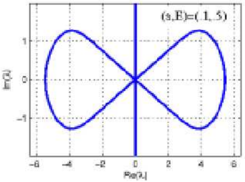
Modulational Theory for mKdV $f(u) = u^3$ w/ $c > 0$

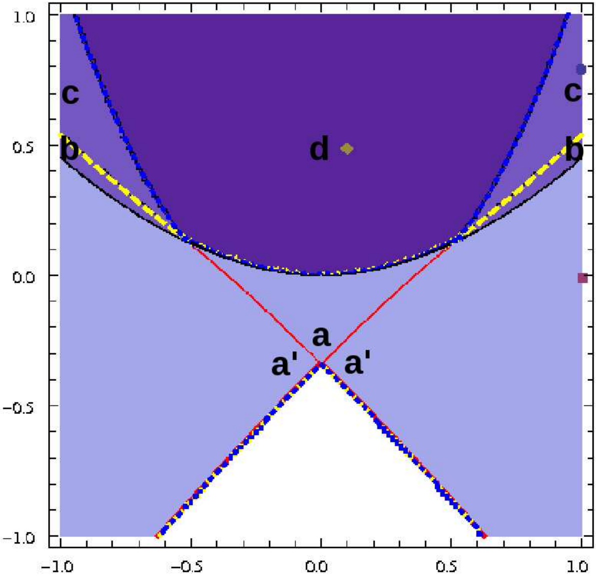


L^2-Critical KdV $f(u) = u^5$(with positive wavespeed)









Nonlinear (Orbital) Stability

Q: Does spectral stability $\Rightarrow$ Orbital Stability?

- True for solitary waves!
- Depends on class of perturbations for periodic case!!

In periodic case, if consider perturbations in...

- $L^2(\mathbb{R}/T\mathbb{Z})$, then

$$0 \leq n_-(\partial_x \mathcal{L}[\bar{u}]) \leq n_-(\mathcal{L}[u]) = 1 \text{ or } 2,$$

  so, sometimes, spec. stable $\Rightarrow$ orbital stable (same argument from solitary wave case).

- $L^2(\mathbb{R}/nT\mathbb{Z})$, then

$$n \leq n_-(\mathcal{L}[\bar{u}]) \leq n+1$$

  so solitary wave argument goes out window!
  - For KdV/mKdV, can work around this for any $n \in \mathbb{N}$!
  - KdV/mKdV proof follows "multi-soliton" stability approach.
  - Nothing known outside integrable context...
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Result of Deconinck & Kapitula: $\forall n \in \mathbb{N}$,

$$k_u^+(n) + k_i^-(n) = n_-\left(\mathcal{L}[\bar{u}]|_{H^1(n)}\right) - n(D)$$

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Theorem: [Bronski, M.J., Kapitula (to appear)] We have

$$k_u^+(n) + k_i^-(n) = 2n - p\left(\partial^2 K(\bar{u})\right)$$

where $K = K(a, E, c)$ is the classical action (in sense of action-angle variables) of the traveling wave ODE

$$\frac{u_x^2}{2} = E - V(u; a, E, c),$$

and $p(A)$ denotes the number of positive eigenvalues of a given matrix A.

Here $K(a, E, c) = \oint_\Gamma p\, dq = \oint_\Gamma \sqrt{E - V(u; a, E, c)}du$ is a *generating function* for the conserved quantities of the gKDV flow:

$$K_a = M, \quad K_E = T, \quad K_c = P.$$

So, $\partial^2 K(\bar{u})$ is expressed in terms of derivatives of T, M, P with respect to (a, E, c).

In particular, for $f(u) = u^p$ can express $p(\partial^2 K(\bar{u}))$ in terms of moments of the wave $\bar{u}$ itself!

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Idea of Proof...

The major steps:

(1) $n(\mathcal{L}[\bar{u}]|_{H^1(n)}) = n(\mathcal{L}[\bar{u}])+$ "fudge factor".

(2) $n(\mathcal{L}[\bar{u}]) = 2n - 1 + n_-(T_E)$... so unstable e.v.'s may be complex!!!

(3) Determine $n_-(D)$... relates to Jordan co-periodic Jordan block at $\lambda = 0$.

Proofs are new to literature, but based on VERY classical ideas.

Step 1: Relate $n_-(\mathcal{L}[\bar{u}]|_{H^1(n)})$ and $n_-(\mathcal{L}[\bar{u}])$

Lemma: Spse. T is invertible, bounded below with compact resolvant. Let S be a subspace with $\dim(S) = d$. Then

$$n_-(T|_S) + n_-(T^{-1}|_{S^\perp}) = n_-(T).$$

- $(H^1(n))^\perp = \mathrm{span}(1)$, so

$$n_-(\mathcal{L}^{-1}(H^1(n))^\perp) = n_-\left(\langle 1, \mathcal{L}^{-1}(1)\rangle\right)$$

- Traveling waves satisfy

$$u_{xx} + f(u) - cu = a \quad \Rightarrow \quad \mathcal{L}u_a = 1, \quad \mathcal{L}u_E = 0.$$

- Thus (since $T_E u_a - T_a u_E$ is T-periodic),

$$\langle 1, \mathcal{L}^{-1}(1)\rangle = \int_0^T \left(u_a - \frac{T_a}{T_E}u_E\right) dx = \frac{\{T, M\}_{a,E}}{T_E}.$$

Lemma: Spse. T is invertible, bounded below with compact resolvant. Let S be a subspace with $\dim(S) = d$. Then

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Step 1: Relate $n_-(\mathcal{L}[\bar{u}]|_{H^1(n)})$ and $n_-(\mathcal{L}[\bar{u}])$

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Step 2: Compute $n_-(\mathcal{L})$

Notice $\mathcal{L} = -\partial_x^2 - f'(u) + c$ is a linear Schrodinger operator w/ periodic coefficients.

Facts: Considered as an operator on $L^2(\mathbb{R}/nT\mathbb{Z})$,

- Spec. of $\mathcal{L}$ determined by Floquet discriminate $k(\lambda)$:

$$\lambda \in \mathrm{spec}(\mathcal{L}) \text{ iff } k(\lambda) = 2.$$

- Translation invariance of gKdV $\Rightarrow \mathcal{L}u' = 0 \Rightarrow k(0) = 2$.
- u' has 2n roots on $[0, nT) \Rightarrow$ Sturm Liouville Theory says zero is either n^{th} or $(n+1)$st eigenvalue of $\mathcal{L}$.
- To see which, look at $k'(0)$...

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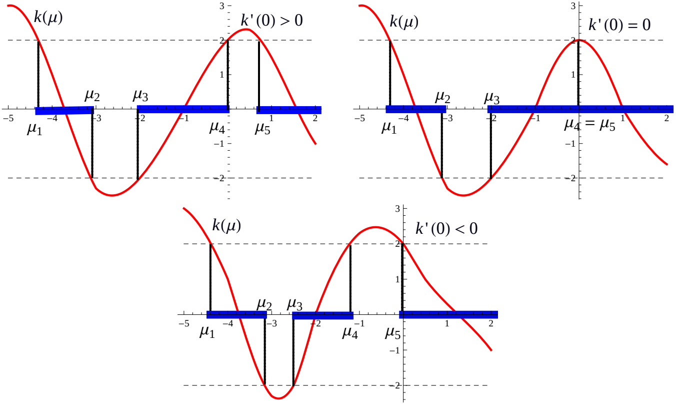
Step 2: Compute $n_-(\mathcal{L})$ (Cont.)



Fact: $\operatorname{sign}(k'(0)) = \operatorname{sign}(T_E)$.

$$\therefore \quad n_-(\mathcal{L}) = 2n - 1 + n_-(T_E).$$

Follows that

$$\begin{aligned} n_-(\mathcal{L}|_{H^1(n)}) &= n_-(\mathcal{L}) - n_-(\mathcal{L}^{-1}|_{H^1(n)^\perp}) \\ &= 2n - 1 + n_-(T_E) - n_-(T_E\{T, M\}_{a,E}). \end{aligned}$$

Fact: Recall, D comes from Jordan block...

$$n_-(D) = n_-\left(\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}\right).$$

So, by "Jacobi-Sturm rule",

$$\begin{aligned} n_-(\mathcal{L}|_{H^1(n)}) - n_-(D) &= 2n - p\begin{pmatrix} T_E & T_a & T_c \\ M_E & M_a & M_c \\ P_E & P_a & P_c \end{pmatrix} \\ &= 2n - p(\partial\langle T, M, P\rangle(a, E, c)) \\ &= 2n - p(\partial^2 K(a, E, c)). \end{aligned}$$

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$$

Conclusions:

- Have presented new set of techniques to analyze stability of periodic waves to more general classes of perturbations.
- Techniques can be used in variety of other problems:
  1. Ideally, want traveling wave ODE to be completely integrable (gBBM, NLS, etc.).
  2. Transverse instability analysis.
  3. Rigorous justifications of Whitham modulation theory (Formal physical theory for modulational instabilities).

Thank you!