On the Instability of Periodic Wave Trains in the Whitham Equation

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Joint work with
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Introduction:

We consider **SPECTRAL STABILITY** of periodic traveling wave solutions of scalar evolution equations of form

$$u_t + \mathcal{M} u_x + \left(u^2\right)_x = 0,$$

where

$$\widehat{\mathcal{M}u}(\xi) = m(\xi)\hat{u}(\xi).$$

- Note $\mathcal{M}$ is **NONLOCAL** *unless* the linear phase velocity $m(\xi)$ is a polynomial function of ξ.

Common Applications: Models of unidirectional propagation of small amplitude surface water waves / internal waves / plasmas / etc.

Theme: If stubbornly restrict to *local* theory, can not see important physical phenomena.

Common Examples:

KdV: $m(\xi) = 1 - \xi^2$.

- Models small amp., long-wavelength, surf. water waves.

Kawahara: $m(\xi) = 1 - \xi^2 + \xi^4$.

- Models small amp., long-wavelength, surf. water waves. for Weber numbers $\approx 1/3$.

Benjamin-Ono: $m(\xi) = |\xi|$.

- Models small amp. deep internal waves.

Intermediate Long Wave Equation: $m(\xi) = \xi \coth(\xi) - 1$.

- Models small amp. internal waves.

Fractional KdV: $m(\xi) = |\xi|^{-1/2}$.

- Models small amp., *infinite depth* **short** surface periodic water waves (Hur, 2012).

Known Results:

If only consider **local** equations....

- KdV wave trains are spectrally stable: Spector (1988), Bottman & Deconinck (2009).
- Small Kawahara wave trains are spectrally stable: Haragus, Lombardi, Scheel (2006).

But, both "model" finite depth surface water wave problem, which exhibits modulational instaiblity of short waves (Benjamin-Feir instability)!

To capture Benjamin-Feir instability, local equations may not be enough....

Naive Fix: Try BBM.... but these wavetrains are modulationally stable: Haragus (2008), M.J. (2010).

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Finite Depth Periodic Surface Water Waves:

Modeled as **periodic** traveling wave solutions of **water wave problem**, ie. a free-surface Euler equation under influence of gravity over flat bottom.

- Fact : Up to rescaling, seeking solutions of form $\varepsilon e^{i(\omega t - \xi x)}$, $\omega > 0$, of full water wave problem leads to linear phase velocity

$$c(\xi) = \frac{\omega(\xi)}{\xi} = \sqrt{\frac{\tanh(h\xi)}{\xi}}, \quad h = \text{undisturbed depth} == 1$$

for small amp. periodic surface water waves with frequency ξ.

Note: For long waves ($|\xi| \ll 1$) have

$$c(\xi) \approx \underbrace{\overbrace{1 - \frac{1}{6}\xi^2}^{\text{KdV}} + \frac{19}{360}\xi^4}_{\text{Kawahara}} + \mathcal{O}(\xi^6).$$

Truncation leads to **LOCAL** evolution equation.

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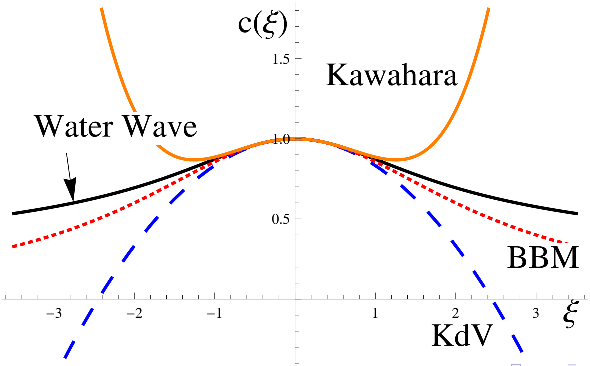
Another common long wave approx. is the BBM, with phase velocity

$$c(\xi) = \frac{1}{1 + \frac{1}{6}\xi^2}$$

Note: KdV, Kawahara are **poor** approximates for short waves.
BBM better, but $c(\xi)$ wrong for short waves...



Whitham Equation:

To analyze stability of finite depth periodic surface water waves, propose to study the **Whitham Equation** (Whitham 1974)

$$u_t + \mathcal{M}u_x + \left(u^2\right)_x = 0,$$

where $\mathcal{M}$ has symbol

$$m(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}.$$

This "**fake**" water wave approx. introduced by Whitham to explain *wave breaking*.

Theorem (Vera Mikyoung Hur, M.J., preprint)

Whitham eqn. exhibits B.F. instability, i.e. small amplitude periodic traveling waves with frequency $k > 0$ are...

- *Spectrally stable if k is sufficiently small.*
- *Modulationally unstable if k is sufficiently large.*

Existence Theory:

- Seek traveling wave $u(x,t) = \phi(x - ct)$, $c > 0$ is wavespeed.
- Profile v satisfies

$$-c\phi + \mathcal{M}\phi + \phi^2 = (1-c)^2 b, \quad b \in \mathbb{R}.$$

<u>Known Results:</u>

- Ehrnström & Kalish (2009): When $b = 0$, $\exists$ small amplitude periodic traveling waves for each $c \in (0,1)$.
    - Proof uses Crandall-Rabinowitz bifurcation theorem.
- Ehrnström, Groves, & Wahlén (2012): $\exists$ solitary waves.
    - Proof uses constrained minimization principle & concentration compactness.

We need $\exists$ theory for small periodic traveling waves TOGETHER with dependence on b.

Existence Theory:

Seek periodic solutions $v(z) = P(kz)$, $P(\cdot + 2\pi) = P$.

$\Rightarrow$ P must satisfy

$$-cP + P^2 + \widetilde{\mathcal{M}}_k P = (1-c)^2 b, \quad \mathcal{F}\left(\widetilde{\mathcal{M}}_k v\right)(\xi) = m(k\xi)\hat{v}(\xi).$$

Using Lyapunov-Schmidt, find 3-parameter family of small amp. periodic traveling waves:

$$P_{a,b}(z) = Q_b + a\cos(z) + \frac{1}{2}\left(\frac{-1}{1-m(k)} + \frac{\cos(2z)}{m(k)-m(2k)}\right)a^2 + H.O.T$$

$$c_{a,b} = c_{0,b} + \left(\frac{-1}{1-m(k)} + \frac{1}{2(m(k)-m(2k))}\right)a^2 + \mathcal{O}\left(|a|(a^2+b^2)\right.$$

where

$$Q_b = b(1-m(k)) + \mathcal{O}(b^2), \quad c_{0,b} := m(k) + 2b(1-m(k)) + \mathcal{O}(b^2).$$

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Linearized Equations:

After rescaling, $P_{a,b}(z)$ is a 2π-periodic stationary solution of PDE

$$u_t - c_{a,b}u_z + \widetilde{\mathcal{M}}_k u_z + \left(u^2\right)_z = 0.$$

Linearizing about $P_{a,b}$ leads to spectral problem

$$\partial_z \left(\underbrace{\widetilde{\mathcal{M}}_k - c_{a,b} + 2P_{a,b}}_{\mathcal{L}_{a,b}} \right) v = \lambda v.$$

considered on $L^2(\mathbb{R})$ (localized perturbations).

- $P_{a,b}$ is spectrally stable if $\sigma_{L^2(\mathbb{R})}\left(\partial_z \mathcal{L}_{a,b}\right) \subset \mathbb{R}i$.
- First difficulty: spectrum is continuous & Floquet theory **does not** apply.

Spectral Stability: Setup

Using Bloch decompositions, can derive “nonlocal Floquet theory”:

$$\sigma_{L^2(\mathbb{R})}\left(\partial_z \mathcal{L}_{a,b}\right) = \bigcup_{\xi\in[-1/2,1/2)} \underbrace{\sigma_{L^2(\mathbb{R}/2\pi\mathbb{Z})}\left(J_\xi \mathcal{L}_{a,b,\xi}\right)}_{\text{Isolated e.v.'s}},$$

where $J_\xi := e^{-i\xi z}\partial_z e^{i\xi z} = \partial_z + i\xi$ and

$$\mathcal{L}_{a,b,\xi} := e^{-i\xi z}\mathcal{L}_{a,b}e^{i\xi z}.$$

Here, ξ “$\approx$” relative frequency of perturbation to $P_{a,b}$.

- $\xi = 0 \Rightarrow$ co-periodic perturbations.
- $|\xi| \ll 1 \Rightarrow$ long-wavelength perturbations (regime of MI).

Next difficulty: How to determine spectrum of $J_\xi \mathcal{L}_{a,b,\xi}$ for $|(a,b)| \ll 1$?

Spectral Stability: Equilibrium solution

Stability of $P_{0,0} = 0$ solution governed by $L^2(\mathbb{R})$ spectrum of operator

$$\underbrace{\partial_z \left(\widetilde{\mathcal{M}}_k - m(k)\right)}_{\mathcal{L}_{0,0}}.$$

Fourier Analysis $\Rightarrow$

$$\sigma\left(\mathcal{L}_{0,0,\xi}\right) = \{\omega_{n,\xi} : n \in \mathbb{Z}\} \subset \mathbb{R}$$

where $\omega_{n,\xi} = [m(k(n+\xi)) - m(k)]$. In particular, for $\xi \in [-1/2, 1/2)$ have

$$\sigma\left(\mathcal{L}_{0,0,\xi}\right) = \underbrace{\{\omega_{0,\xi}, \omega_{\pm 1,\xi}\}}_{\sigma_1(\mathcal{L}_{0,0,\xi})} \bigcup \underbrace{\{\omega_{n,\xi} : |n| \geq 2\}}_{\sigma_2(\mathcal{L}_{0,0,\xi}) \geq C_k},$$

where $\forall\ v$ in spectral subspace for $\sigma_2(\mathcal{L}_{0,0,\xi})$, have

$$\langle v, \mathcal{L}_{0,0,\xi} v\rangle \geq C_k \|v\|^2.$$

Fact: Spectral properties persist for $|(a,b)| \ll 1$.

Spectral Stability: Equilibrium solution

Stability of $P_{0,0} = 0$ solution governed by $L^2(\mathbb{R})$ spectrum of operator

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$$\sigma\left(\partial_z \mathcal{L}_{0,0,\xi}\right) = \{i(n+\xi)\omega_{n,\xi} : n \in \mathbb{Z}\} \subset \mathbb{R}i$$

In particular, for $\xi \in [-1/2, 1/2)$ have

$$\sigma\left(J_\xi \mathcal{L}_{0,0,\xi}\right) = \underbrace{\{i\xi\omega_{0,\xi}, i(\pm 1+\xi)\omega_{\pm 1,\xi}\}}_{\sigma_1(J_\xi \mathcal{L}_{0,0,\xi})} \bigcup \underbrace{\{i(n+\xi)\omega_{n,\xi} : |n| \geq 2\}}_{\sigma_2(J_\xi \mathcal{L}_{0,0,\xi})}.$$

- All e.v.'s in $\sigma_2\left(J_\xi \mathcal{L}_{0,0,\xi}\right)$ have positive Krein signature $\Rightarrow$ they remain purely imaginary for $|(a,b)| \ll 1$.
- At $\xi = 0$, $\omega_{0,0} = \omega_{\pm 1,0} = 0$... more delicate analysis needed here.

Determination of $\sigma_1(J_\xi \mathcal{L}_{a,b,\xi})$:

- $|\xi| > \xi_0 > 0$:
  - The three eigenvalues are simple.
  - The three eigenvalues are symmetric about $\mathbb{R}i$.

$$\Rightarrow \sigma_1\left(J_\xi \mathcal{L}_{a,b,\xi}\right) \subset \mathbb{R}i.$$

- $\xi \approx 0$:
  - Determine basis $\{\eta_j(z; a, b, \xi)\}_{j=0,1,2}$ for spectral subspace for $\sigma_1(J_\xi \mathcal{L}_{a,b,\xi})$.
  - Critical eigenvalues for $|(a, b, \xi)| \ll 1$ found by solving

$$P(\lambda, a, b, \xi; k) = \det\left(\left[\left\langle \frac{\eta_j}{\|\eta_j\|^2}, \left(J_\xi \mathcal{L}_{a,b,\xi} - \lambda \mathbf{I}\right)\eta_l \right\rangle\right]_{j,l=0,1,2}\right) = 0$$

for λ.

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Determination of $\sigma_1(J_\xi \mathcal{L}_{a,b,\xi})$:

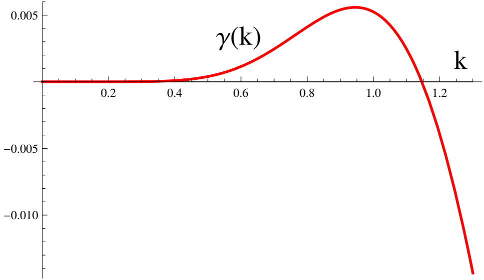
Fact: $P(\lambda, a, b, \xi) =$ cubic poly. in $X = -i\lambda/\xi$ with discriminant

$$\Delta_{a,b,\xi,k} = \Delta_{0,0,\xi,k} + \gamma(k)a^2 + \mathcal{O}(a^2(a^2 + \xi^2 + |b|)).$$

For $|\xi| \ll 1$ have

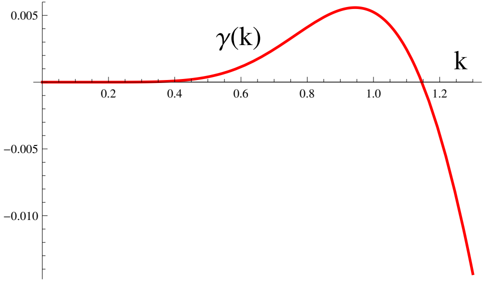
$$\Delta_{0,0,\xi,k} \approx 0.0625k^{12}\xi^2 + H.O.T.$$

$\Rightarrow$ Stability determined by sign of $\gamma(k)$.



Note: $\gamma(k) = \frac{1}{4}k^8 + \mathcal{O}(k^9) > 0$ for $|k| \ll 1$.

MI Index:



- $\gamma(k) > 0$ for $k < k^* \approx 1.146$

$$\Rightarrow \quad \text{sufficiently long waves are stable!!}$$

- $\gamma(k) < 0$ for $k > k^*$

$$\Rightarrow \quad \text{sufficiently short waves are unstable!!}$$

This (qualitatively) is the finite depth Benjamin-Feir instability!!!

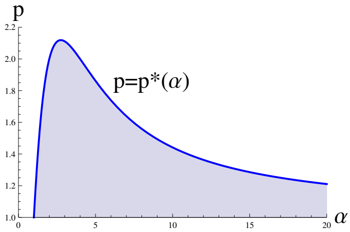
Generalizations:

Calculation is (nearly) independent of $m(\xi)$....

Ex: Small periodic wave trains in fractional KdV equation

$$u_t + \sqrt{-\partial_x^{2\alpha}}u_x + \left(u^{p+1}\right)_x = 0, \quad \alpha \geq 1, \quad p \geq 1$$

are spectrally stable for $1 < p < p^*(\alpha)$ and modulationally unstable if $p > p^*(\alpha)$.



Result illustrates difference between dispersion on line and "dispersion" on circle.

What's Next?

NO IDEA!!!..... but stability for nonlocal equations poorly understood.

- We like ODE things... but this can be too specialized.
- What tools fundamentally rely on ODE structure of spectral stability problem and which ones do not?
- What are appropriate generalizations of these tools that are "too specialized"?

Also, *instability* theory often neglected....

- What happens dynamically to unstable solutions? Poorly understood problem even in local theory!

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