The Transverse Instability of Periodic Traveling Waves in the Generalized Kadomtsev-Petviashvili (KP) Equation

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Joint work with Kevin Zumbrun (IU)

gKP Equations

- The gKP equations are given by

$$(u_t - u_{xxx} - f(u)_x)_x + \sigma u_{yy} = 0, \quad \sigma = \pm 1.$$

Weakly two-dimensional version of the gKdV equation

$$u_t = u_{xxx} + f(u)_x.$$

Special Case: $f(u) = \frac{1}{2}u^2$ (KdV-nonlinearity)

  - KP-I if $\sigma = +1$: model for thin films with high surface tension.
  - KP-II if $\sigma = -1$: model for water waves with small surface tension.

- Other multi-d generalizations exist: gZK (Zakharov-Kuznetsov) eqns.

$$u_t = (u_{xx} + u_{yy})_x + f(u)_x$$

but KP has extra *degeneracy* which presents interesting mathematical difficulty.

- **Take Away**: Solutions of gKdV = unidirectional (y-independent) solution of *gKP*.

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- **Main Problem:** When are stable solutions of gKdV stable in the gKP equation?
- Have answers for solitary waves, i.e. when $\lim_{x\to\pm\infty} u(x) = 0$:
  - $\sigma > 0 \Rightarrow$ solitary waves stable in gKdV, but unstable in gKP. Instability is to low-frequency perturbations.
  - $\sigma < 0 \Rightarrow$ depends on nonlinearity. KdV: transversely stable, but $\exists$ nonlinearities with unstable solitary waves to low-frequency perturbations.
  - Results based on multi-scale analysis...
- Few answers for spatially periodic waves:
  - (Haragus-2010) Small amplitude limit in KdV: $\sigma < 0$ stable, $\sigma > 0$ unstable. (Finished after our work?)
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Seek traveling wave solutions of gKdV

$$u_t = u_{xxx} + f(u)_x.$$

Stationary solution in moving coordinate frame $x - ct$:

$$u_{xxx} + f(u)_x - cu_x = 0$$

Integrable: $\exists$ constants $a, E \in \mathbb{R}$ such that

$$\frac{u_x^2}{2} = E - \underbrace{\left(\int^x f(u(z))dz - \frac{c}{2}u^2 - au\right)}_{V(u;a,c)}$$

- Solitary waves: bdry conditions $\Rightarrow a = E = 0$.
- $\exists$ (mod translations) three parameter family of periodic traveling wave solutions of gKdV, parameterized by (a, E, c).

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Transverse Stability

- Given a $T = T(a, E, c)$ periodic solution $u(x; a, E, c)$ of gKdV, want to determine stability in gKP.
- Strategy: linearize gKP about wave $u(x, y) := u(x; a, E, c)$

$$\partial_x \left(\underbrace{\partial_x \mathcal{L}[u]}_{\text{lin. gKdV}} \right) v + \sigma v_{yy} = v_{xt}, \quad v(\cdot, y, t) \in L^2_{\text{per}}([0, T]),$$

and take transforms (Fourier in y, Laplace in t):

$$\partial_x \left(\partial_x \mathcal{L}[u] \right) v - \sigma k^2 v = \lambda v_x.$$

Corresponding to transverse perturbations of form

$$v(x, y, t) = e^{\lambda t + iky} v(x), \quad v(\cdot) \in L^2_{\text{per}}([0, T]).$$

- Spectral instability if $\exists$ T-periodic *eigenvalue* λ with $\Re(\lambda) > 0$. How do we locate these eigenvalues?

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Evans Ftn.

Write spec problem as

$$Y' = A(x;\lambda,k)Y$$

$\Psi(x;\lambda,k)$ =Solution matrix. Floquet Theory$\Rightarrow$ λ is in T-periodic spec. of gen. e.val. problem iff

$$D(\lambda;k) = \det\left(\Psi(T;\lambda,k)\Psi(0;\lambda,k)^{-1} - Id\right) = 0$$

Why? $\Psi(T)\Psi(0)^{-1}$ =Period Map...
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Strategy: Search for real unstable e.v.'s by comparing $D(+\infty; k)$ with $D(0, k)$ when $0 < |k| \ll 1$.

$$D(+\infty; k) D(0, k) < 0 \Rightarrow \exists \text{ unstable } \lambda > 0.$$

Note: Generally, one compares $D(+\infty, k)$ with slope at $\mu = 0$. BUT, $D(0, k) \neq 0$ for small k, so only need to compute $D(0, k)$.

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High Freq. Analysis

Fix k, and rescale $\tilde{x} = |\lambda|^{1/3}x$ to obtain

$$\left(-\partial_x^4 - |\lambda|^{-2/3}\partial_x^2(f'(u)+c) - \sigma k^2|\lambda|^{-4/3}\right) v = v_x$$

Write as 1st order system

$$Y' = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{H_0} Y + B(\lambda)Y$$

where $B(\lambda) = \mathcal{O}(|\lambda|^{-2/3})$. Expect for $\lambda \gg 1$

$$D(\lambda;k) \approx \det\left(e^{H_0|\lambda|^{1/3}T} - Id\right) = 0.$$

Can we determine the limiting sign?

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GOAL: determine effect of $B(\lambda)$ on neutral subspace of H_0 for $\lambda \gg 1$.

FACT: $\exists$ (T-periodic) linear transformation $Q = Q_0 + \mathcal{O}(\varepsilon)$ such that

$$Q^{-1}H_0 Q = \underbrace{\mathrm{diag}(-1, \omega, \omega^*, 0)}_{Q_0^{-1}H_0 Q_0} + \begin{pmatrix} \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon^{3/2}) \end{pmatrix},$$

with $\omega = \frac{1}{2}(1 + i\sqrt{3})$, and

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Block-triangular tracking lemma: $\exists$ a T-periodic change of coordinates $W = ZY$ of form

$$Z = \begin{pmatrix} I_3 & 0 \\ \Phi & 1 \end{pmatrix}$$

where $\Phi = \mathcal{O}(\varepsilon^{3/2})$, taking system to an **exact** upper block triangular form with diagonal blocks

$$-1 + \mathcal{O}(\varepsilon), \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix} + \mathcal{O}(\varepsilon)$$

$$\text{and} \quad \frac{1}{2}A_x\varepsilon + \varepsilon^2\left(\frac{1}{2}AA_x - \sigma k^2\right) + \mathcal{O}(\varepsilon^{5/2}).$$

Block-triangular form *plus* periodicity of coordinate changes $\Rightarrow$

$$\text{Evans ftn.} = \prod \text{Evans ftn. for blocks.}$$

Stable block:

$$e^{-|\lambda|^{1/3}T} - 1 < 0.$$

Similarly, unstable block gives > 0. Neutral block gives approximately

$$\exp\left(\int_0^{|\mu|^{1/3}T}\left(\frac{1}{2}A_x\varepsilon + \varepsilon^2\left(\frac{1}{2}AA_x - \sigma k^2\right)\right)(s)ds\right) - 1$$
$$= \exp\left(-\sigma k^2|\lambda|^{-1}T\right) - 1$$
$$\approx -\sigma k^2|\lambda|^{-1}.$$

$\therefore \forall k \neq 0$, $\lim_{\lambda\to+\infty} \operatorname{sgn} D(\lambda; k) = \operatorname{sgn}(\sigma)$.
Remark: Same proof works in Solitary wave case.... never been done this way(?).

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Block-triangular form *plus* periodicity of coordinate changes $\Rightarrow$

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Stable block:

$$e^{-|\lambda|^{1/3}T} - 1 < 0.$$

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Goal: Compare sign of σ to sgn $D(0,k)$ for $0 < |k| \ll 1$, where

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Ψ =Soln. Matrix in *arbitrary* basis.

Choose useful basis at $(\lambda,k)=(0,0)$: recall periodic traveling waves parameterized by (a,E,c).

Noether's thm $\Rightarrow \{u_x, u_a, u_E\}$ (formally) solve $\partial_x^2\mathcal{L}[u]v = 0$.

Fourth soln. found by variation of parameters:

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Periodic traveling wave soln. of gKdV is transversely (spectrally) unstable in gKP if

$$\sigma \det\left(\frac{\partial(T,M)}{\partial(a,E)}\right) > 0.$$

Remark: From numerics, above det. is generically non-zero....

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- KdV:

$$\det\left(\frac{\partial(T,M)}{\partial(a,E)}\right) = \frac{-T^2 V'(M/T)}{12\,\mathrm{disc}(E - V(\cdot; a, c))} > 0$$

  by Jensen's inequality and fact that $E - V$ =cubic w/ 3 real roots (required for $\exists$ periodic orbits).

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$$u_t = (u_{xx} + u_{yy})_x + f(u)_x.$$

Result: for $|(\mu, k)| \ll 1$,

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Yields spectral instability criterion for small k. Criterion is verified for cnoidal waves of KdV, dnoidal waves of mKdV, etc...

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