Krein Signatures for Eigenvalue Problems Associated to Integrable Systems

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(Joint work with Jared Bronski - UIUC Mathematics)

Abstract

There is a result of Klaus and Shaw which shows that the Zakharov-Shabat eigenvalue problem has discrete spectrum which lies on the imaginary axis if the potential has a single critical point and decays monotonically away from this point (the potential is monomodal). We put this calculation in the context of the Krein signature (a tool for studying the stability of symplectic matrices) and prove an analogous theorem for the eigenvalue problem which solves the Sine-Gordon equation (in laboratory coordinates). This is joint work with Jared Bronski.

Outline

Introduction

- We consider the Cauchy problem for the Sine-Gordon equation in laboratory coordinates:

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u_{xx} - u_{tt} &= \sin u \\
u(x,0) &= u(x) \\
u_t(x,0) &= v(x)
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- Toy Model: Consider an infinite wire with a continuum of coupled pendulums, with the pendulum at position x at time t hanging with angle $u(x,t)$ with respect to its rest position.

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- Define $Q_{top} := \frac{1}{2\pi} \int_{-\infty}^{\infty} u'(x)dx = k_+ - k_-$ to be the "topological charge" of the potential u.
- We consider only initial data with topological charge 0 (breathers) or ± 1 (kinks).

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- This equation is integrable via the Inverse Scattering Transform with corresponding spectral problem (Faddeev-Tahktajan / Kaup)

$$4\Phi_x = \left(z - \frac{1}{z}\right)\tau_1 \cos\left(\frac{u}{2}\right)\Phi + \left(z + \frac{1}{z}\right)\tau_2 \sin\left(\frac{u}{2}\right)\Phi - v\tau_3\Phi \quad (1)$$

considered on $L^2(dx;\mathbb{C})$, where $z \in \mathbb{C}$, $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$,

- $u = u(x,0)$
- $v = u_t(x,0)$

and

$$\tau_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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- "Main Result": Under certain conditions on u and v, one must have $z \in S^1$.

Known Results

- Characteristic coordinates $\chi = \frac{x+t}{\sqrt{2}}$, $\eta = \frac{x-t}{\sqrt{2}}$

$$\Rightarrow u_{\chi\eta} = \sin(u).$$

This is the isospectral flow for the Zhakarov-Shabat system

$$\begin{aligned} v_{1,\chi} &= -izv_1 + qv_2 \\ v_{2,\chi} &= izv_2 - q^* v_1 \end{aligned}$$

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- We have

Theorem (Klaus & Shaw (2001, 2002))

If $q : \mathbb{R} \to \mathbb{R}$ is in $L^1 \cap C^1$ and has only one critical point (a local max), then the discrete spectrum lies on the imaginary axis in the $z-$plane.

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- In laboratory coordinates, initial data with zero initial momentum corresponds to $v(x) = u_t(x,0) = 0$ and stationary solitons correspond to $z \in S^1$.
- Buckingham & Miller (2008): If one takes initial data which satisfies

$$\sin(u(x)/2) = \operatorname{sech} x, \quad \cos(u(x)/2) = \tanh(x), \quad v(x) = 0,$$

eigenvalues are confined to unit circle and are simple. Notice such a potential is monotone with $Q_{top} = -1$.

Main Theorem

- Recall (1) is given by

$$4\Phi_x = \left(z - \frac{1}{z}\right)\tau_1 \cos\left(\frac{u}{2}\right)\Phi + \left(z + \frac{1}{z}\right)\tau_2 \sin\left(\frac{u}{2}\right) - v\tau_3\Phi$$

Theorem (Bronski & M.J.)

Consider the Sine-Gordon equation in laboratory coordinates. Let $v = u_t(x,0) = 0$ and let $u = u(x,0)$ be such that one of the following conditions holds:

1. *u is monotone with $Q_{top} = \pm 1$, or*
2. *u has $Q_{top} = 0$ and is monomodal with a positive maximum $u_0 \in (0,\pi)$.*

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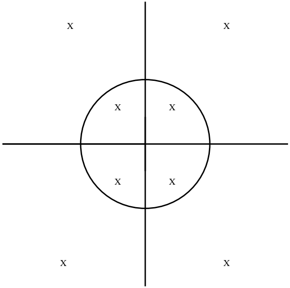
- Main analytical tool: Krein signatures, i.e. functionals on L^2 which encode information about eigenvalues and eigenspaces.

Symmetries

As we will see, Krein signatures are built in such a way to abuse the symmetries of the discrete spectrum. So, we begin with the following lemma:

Lemma

The symmetry group of the discrete spectrum of (1) *is* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, *corresponding to reflection across the real and imaginary axis, as well as the unit circle.*



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for $v \in L^2$.

- Basic Result: If $Mv = \lambda v$,

$$\kappa(v) \neq 0 \Rightarrow \lambda \in S^1$$

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- Thus,

$$\left(\lambda - \frac{1}{\overline{\lambda}}\right)\kappa(v) = 0$$

which proves the claim.

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- Construct Krein signature κ corresponding to reflection about S^1.
- Prove $\kappa(v) \neq 0$ for any eigenfunction v.

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- Claim: For 2nd order ODE eigenvalue problems, semi-simple implies simple.
- Proof: The existence of a semi-simple eigenspace of multiplicity higher than one implies $\exists$ two linearly independent solutions decaying as $x \to \infty$. This contradicts the asymptotic behavior of the Jost solutions.$\square$

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- The Zakharov-Shabat eigenvalue problem (for real potentials) is given by $Mv = \lambda v$ on $L^2(dx; \mathbb{R})$, where

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$$\kappa = \langle \phi, U\phi \rangle_{L^2} = \int_{\mathbb{R}} (\phi_1^* \phi_2 + \phi_2^* \phi_1)\, dx.$$

This is exactly the quantity Klaus and Shaw consider in their papers.

Krein Signatures

Lemma

The eigenvalue problem (1) can be written as $Mv = \lambda v$, where the operator M above satisfies $M^\dagger UM = U$ where U is of the form

$$\begin{pmatrix} 0 & 0 & \cos\left(\frac{u(x)+\pi}{2}\right) & -\sin\left(\frac{u(x)+\pi}{2}\right) \\ 0 & 0 & \sin\left(\frac{u(x)+\pi}{2}\right) & \cos\left(\frac{u(x)+\pi}{2}\right) \\ -\cos\left(\frac{u(x)+\pi}{2}\right) & -\sin\left(\frac{u(x)+\pi}{2}\right) & 0 & 0 \\ \sin\left(\frac{u(x)+\pi}{2}\right) & -\cos\left(\frac{u(x)+\pi}{2}\right) & 0 & 0 \end{pmatrix}.$$

$z = re^{i\theta} \Rightarrow$

$$\begin{aligned} \kappa &= i\left\langle \Phi, U\Phi \right\rangle_{L^2} \\ &= \sin\theta \int_{\mathbb{R}} \sin\left(\frac{u}{2}\right) |\Phi|^2 dx - i\cos\theta \int_{\mathbb{R}} \cos\left(\frac{u}{2}\right) \left\langle \Phi, \tau_3\Phi \right\rangle dx \end{aligned}$$

Topological Charge ± 1: Confinement Result

- First Main Result:

Theorem

Let $u(x)$ be a monotone potential satisfying $u(x) \to 0$ as $x \to -\infty$ and $u(x) \to 2\pi$ as $x \to \infty$, i.e. u has topological charge $Q_{top} = 1$. Then the discrete spectrum of (1) lies on the unit circle and is simple.

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- Using original eigenvalue problem, eventually get

$$\kappa = 2\left(r + \frac{1}{r}\right)\int_{\mathbb{R}} \frac{\sin(\theta)}{\sin(u/2)}|\phi_2|^2 dx + 2\int_{\mathbb{R}} \frac{u_x}{\sin^2(u/2)}|\phi_2|^2 dx > 0.$$

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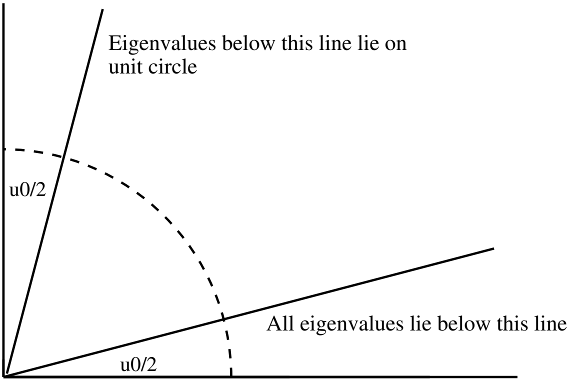
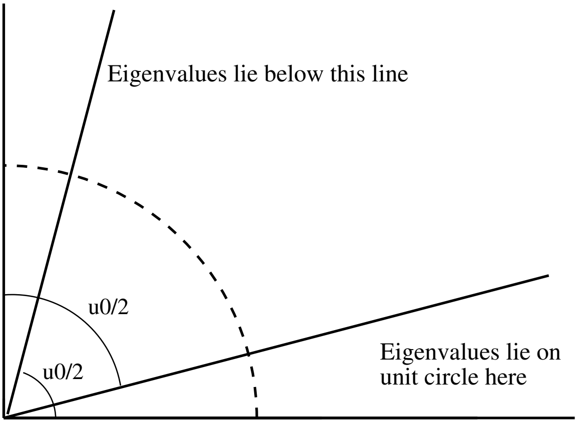


$\Rightarrow$ All eigenvalues lie on unit circle.

Topological Charge 0: $\frac{\pi}{2} < u_0 < \pi$



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- We will count the number of eigenvalues in two different ways:
  1. As you vary u_0, count how many times an eigenvalue either emerges from or gets absorbed into the continuous spectrum (homotopy argument). Yields upper bound.
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- Upper and lower bound agree!!!!

Theorem

If $Q_{top} = 0$ and $u_0 \in [0, \pi)$, let N be the largest non-negative integer such that

$$\int_{-\infty}^{\infty} \sin\left(\frac{u}{2}\right) dx > (2N-1)\pi$$

Then there exists exactly N eigenvalues of (1), all of which are on the unit circle and are simple.

Possible Extensions

- NUMERICS: Tentative numerical experiments indicate the result for $Q_{top} = \pm 1$ is tight: monotone kinks of higher topological charge and non-monotone kinks of topological charge ± 1 frequently have eigenvalues off S^1.

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- Thank You!