Rigorous Stability Theory for Nonlinear Modulated Waves (Justification of the Physicists Intuition)

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Joint work with Jared Bronski (UIUC) and Kevin Zumbrun (IU)

Outline

Introduction

Working definition of stability in ODE/PDE:

A solutions ability to persist when subject to slight perturbation

- Practically important: Unstable solutions do not (naturally) manifest in physical situations, except possibly as transient phenomena.
- Discriminates between physical solutions and mathematical oddities.
- Example: In a mathematical pendulum, the stationary solution $\theta = 0$ is stable, while $\theta = \pi$ is unstable .... how do we see this?

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- The equation for an (undamped) mathematical pendulum is

$$\partial_t^2 \theta + \sin(\theta) = 0.$$

Clearly $\theta_0 = 0$ and $\theta_0 = \pi$ solve this equation.Consider a nearby solution

$$\psi = \theta_0 + \varepsilon\theta_1 + \mathcal{O}(\varepsilon^2), \quad |\varepsilon| \ll 1$$

and note by Taylor expansion we have

$$\partial_t^2\left(\theta_0 + \varepsilon\theta_1\right) + \left(\sin(\theta_0) + \varepsilon\cos(\theta_0)\theta_1\right) = \mathcal{O}\left(\varepsilon^2\right)$$

The $\mathcal{O}(1)$ equation is clearly satisfied, and $\mathcal{O}(\varepsilon)$ equation reads

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$$\partial_t^2 \theta_1 + \theta_1 = 0$$

which has solutions $\theta_1(t) = A\cos(t) + B\sin(t)$, and hence nearby solutions oscillate around original stationary solution.

- If $\theta_0 = \pi$, linearized equation reads

$$\partial_t^2 \theta_1 - \theta_1 = 0$$

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- Therefore, $\theta_0 = \pi$ is (linearly) unstable while $\theta_0 = 0$ is (linearly) stable.

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- Stability is inherently a physical issue....

(1) By understanding of mechanism behind instability, one may be able to *stabilize* the solution!
Example: In the pendulum example above, the unstable solution $\theta_0 = \pi$ can be stabilized by addition of an appropriate periodic forcing term:

$$\partial_t^2 \theta + \sin(\theta) = \beta \cos(t) \sin(\theta).$$

(2) Helps us understand how solutions of our approximate model actually simulate real life.
Example: Light pulses through a fiber optic wire and the single-particle wavefunction in a Bose–Einstein condensate are modeled by the GrossPitaevskii equation (nonlinear Schrödinger equation)

$$i\psi_t + \psi_{xx} + \psi|\psi|^2 = 0,$$

but this equation does not take into effect impurities or higher order nonlinear effects in the wire. Will a solution of this simplified model persist under the influence of these defects?

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- Physicists often have a fantastic (and correct!) intuition about which solutions of a given model are stable, and which are unstable.
- As mathematicians though, we would like to develop a theory which makes this intuition rigorous!
- Example: It is well known in the physics/engineering community that solutions of a scalar reaction-diffusion equation

$$u_t + u_{xx} = f(u)$$

  which satisfy $\lim_{x\to\pm\infty} u_x(x,t) = 0$ are stable iff they are monotone. Thus, fronts are stable but pulses are not.
- How do we understand this as mathematicians?

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- Let $u = u(x)$ satisfy $\lim_{x\to\pm\infty} u_x(x,t) = 0$ and
$$u_t = u_{xx} + f(u)$$

- Consider "nearby" solution $w(x,t) = u(x) + \varepsilon v(x,t)$, $v \in L^2(\mathbb{R})$
$$\Rightarrow \qquad v_t = v_{xx} + f'(u)v.$$

- Decompose $v(x,t) = e^{\mu t} v(x)$, $\mu \in \mathbb{C}$:
$$\Rightarrow \qquad v_{xx} + f'(u)v = \mu v$$

- If u is monotone, can show $\operatorname{spec}\left(\partial_x^2 + f'(u)v\right) \subset (-\infty, 0]$ and hence perturbations remain bounded in time!
- If u is not monotone, then $\sigma_p\left(\partial_x^2 + f'(u)v\right) \bigcap (0,\infty) \neq \emptyset$.

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GKdV

- The purpose of this talk is to consider the stability of spatially periodic waves of the generalized Korteweg-de Vries (gKdV) equation
$$u_t = u_{xxx} + f(u)_x$$
where f is "nice". Arise in applications with a variety of nonlinearities.
- $f(u) = u^2 \Rightarrow$ KdV equation. Canonical model for weakly dispersive nonlinear unidirectional wave propagation.
- $f(u) = \pm u^3 \Rightarrow$ focusing/defocusing mKdV equation. Arises naturally in plasma physics as a model for ion acoustic perturbations.
- $f(u) = \alpha u^{r+1/2}$ for $r \in \left(-\frac{1}{2}, \frac{1}{2}\right)$... has been derived in several plasma physics models.

Also interesting for mathematical study: $f(u) = u^5$ is L^2 critical, and KdV and mKdV are completely integrable!

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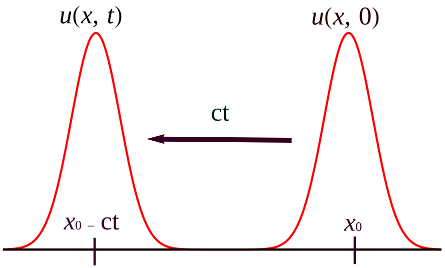
- Consider traveling wave profile $u(x,t) = u(x+ct)$.
- Characteristics:
  (1) Constant velocity c
  (2) Same shape and profile!



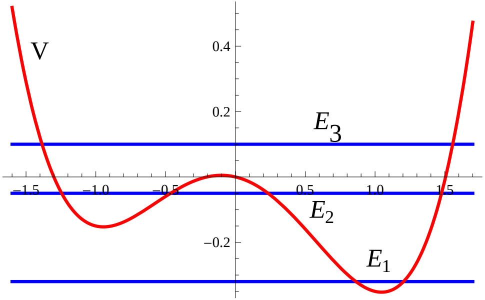
- Solution is STATIONARY solution of PDE

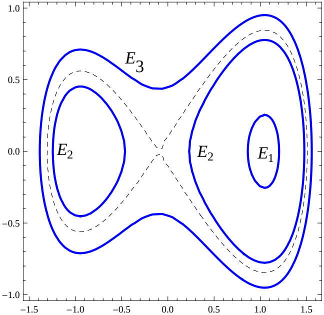
$$u_t = u_{xxx} + f(u)_x - cu_x.$$

- Wave profile u must satisfy ODE

$$u_{xxx} + f(u)_x - cu_x = 0$$

$$\Rightarrow \quad \frac{u_x^2}{2} = E - \underbrace{F(u) - \frac{cu^2}{2} + au}_{V(u;a,c)}, \quad F' = f, \quad a, E \in \mathbb{R}.$$





GKdV

- Periodic traveling waves form a four parameter family $\mathcal{P}$ of solutions

$$u(x) = u(x + x_0; a, E, c)$$

- Translation mode can be modded out: Consider quotient space $\mathcal{P}/\mathcal{R}$ where

$$u\mathcal{R}v \iff \exists \xi \in \mathbb{R} : u = v(\cdot + \xi).$$

Near any nonconstant solution then, the projection $\mathcal{P} \mapsto \mathcal{P}/\mathcal{R}$ is locally a fibration (where the fibers are circles) and hence $\mathcal{P}/\mathcal{R}$ is locally dimension three.

- **Henceforth**, we will identify $\mathcal{P}$ and $\mathcal{P}/\mathcal{R}$ and hence consider $\mathcal{P}$ as a manifold of dimension three.

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- **Objection:** True spatially periodic solutions can not exist in reality!!
- What one does see in experiment, however, are solutions which *locally in space-time* look spatially periodic, but on larger scales there is evident slow change in the physical characteristics of the wave (amplitude, frequency, phase, etc...).
- Thus, our spatially periodic solutions are *idealized* versions of these slowly modulated periodic waves!
- **Q**: How can one study the stability of these seemingly more physical nonlinear modulated waves?
- **A**: We treat them as perturbations of the idealized periodic solutions to slow modulations, ie. to long-wavelength perturbations.
- **Q**: OK..... how do we do that?!

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$$u_t = u_{xxx} + f(u)_x - cu_x.$$

Consider nearby solutions of form $\psi(x,t) = u(x) + \varepsilon v(x,t) + \mathcal{O}(\varepsilon^2)$, $v \in L^2(\mathbb{R})$.

$$\Rightarrow \partial_x \underbrace{\left(-\partial_x^2 - f'(u) + c\right)}_{\mathcal{L}[u]} v = -v_t$$

Decompose $v(x,t) = e^{-\mu t} v(x)$ so v solves the spectral problem

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$$\text{Spectral stability} \iff \text{spec}\left(\partial_x \mathcal{L}[u]\right) \subset \mathbb{R}i.$$

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$$Y'(x,\mu) = \mathbf{H}(x,\mu)Y(x,\mu).$$

- Period Map (Monodromy): $\mathbf{M}(\mu) = \Phi(T,\mu)$, where $\Phi(x,\mu)$ is the matrix solution such that $\Phi(0,\mu) = \mathbf{I}$. Thus, $\mathbf{M}(\mu)$ is an operator such that
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- Best you can hope for is for v to be uniformly bounded, which here corresponds to $|\lambda| = 1$.
- Gives characterization of (continuous) spectrum:

$$\mu \in \text{spec}(\partial_x \mathcal{L}[u]) \iff \sigma(\mathbf{M}(\mu)) \bigcap S^1 \neq \emptyset.$$

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- How does all this help determine modulational stability?
  "FACT": Stability to slow-modulations equivalent with spectral stability near $\mu = 0$. Need to study $\text{spec}\left(\partial_x \mathcal{L}[u]\right)$ near the origin.
- By translation invariance of original PDE, have

$$\partial_x \mathcal{L}[u] u_x = 0$$

  Since u_x is co-periodic with u, follows that

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Evaluate of $\partial_\mu D(\mu,1)\big|_{\mu=0}$

- At $\mu = 0$, $\{u_x, u_a, u_E\}$ provides three linearly independent solutions of the *formal* differential equation

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Thus, can explicitly construct monodromy matrix at $\mu = 0$.

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$\Rightarrow$ Implicit Function Theorem fails!!!!!

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- Ugly algebra yields

$$D(\mu,1) = -\frac{1}{2}\underbrace{\frac{\partial(T,M,P)}{\partial(a,E,c)}}_{\{T,M,P\}_{a,E,c}}\mu^3 + \mathcal{O}(|\mu|^4).$$

where T =period and M and P refer to the mass and momentum:

$$M = \int_0^T u(x)dx \qquad P = \int_0^T u(x)^2 dx$$

M and P are conserved quantities of the gKdV flow!

- Thus, $D(\mu,1) = \mathcal{O}(|\mu|^3)$ and hence more care is needed to use the implicit function theorem.
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Modulational Instability

- Continuing above computations, local analysis around $(\mu, \kappa) = (0, 0)$ yields

$$D(\mu, e^{i\kappa}) = i\kappa^3 + \frac{i\kappa\mu^2}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) - \frac{\mu^3}{2}\{T, M, P\}_{a,E,c} + \mathcal{O}(|\mu|^4 + \kappa^4)$$

  where the notation $\{f, g\}_{x,y}$ is used for two-by-two Jacobians.

- Defining $z = \frac{i\kappa}{\mu}$, we see z must be a root of

$$P(z) = -z^3 + \frac{z}{2}\left(\{T, P\}_{E,c} + 2\{M, P\}_{a,E}\right) - \frac{1}{2}\{T, M, P\}_{a,E,c}.$$

  and hence have modulational stability when P has three real roots!

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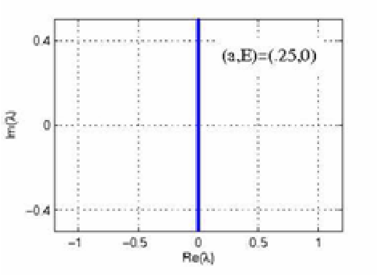
Modulational Instability: M.J. & Bronski (2008)

Define

$$\Delta_{MI} := \frac{1}{2}\left(\{T,P\}_{E,c} + 2\{M,P\}_{a,E}\right)^3 - \frac{27}{4}\{T,M,P\}^2_{a,E,c}.$$



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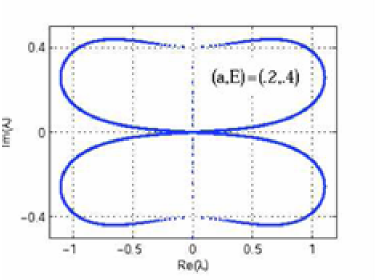

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Whitham Theory

- Physicists have had a formal approach of such modulational stability arguments for years (at least 1973) which has been dubbed Whitham theory!
- Introduce slow variables εx and εt and note the idealized is constant in the slow variables.
- Consider the original PDE in the slow variables and linearize about the idealized constant solution... after averaging, yields a constant coefficient system of PDE!
- Expectation: The stability of the constant (idealized) solution in the averaged-slow variable system should appropriately describe the stability of the original modulated wave.
- **Problem:** This argument has only been justified in a few (special) cases, but has never been seen to be wrong!

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Whitham Theory for GKdV

- In slow variables, gKdV reads

$$u_t = \varepsilon^2 u_{xxx} + f(u)_x$$

Consider WKB approximation

$$u_\varepsilon(x,t) = u^0\left(x, t, \frac{\phi(x,t)}{\varepsilon}\right) + \varepsilon u^1\left(x, t, \frac{\phi(x,t)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2)$$

where $y \to u^0(x,t,y)$ is an unknown 1-periodic function. Substitute this into rescaled gKdV and collect powers of ε.

- $\mathcal{O}(\varepsilon^{-1})$: $\phi_x^3 \partial_y^3 u^0 + \phi_x \partial_y f(u^0) - \phi_x \partial_y u^0 = 0$. Defining $\omega = \phi_x$ and $s = -\frac{\partial_t}{\partial_x}$, may choose

$$u^0(x,t,y) = \bar{u}\left(\omega y; a(u^0), E(u^0), -s(u^0)\right), \quad \bar{u} \in \mathcal{P}$$

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Averaging over single period in y yields conservation law

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- Whitham's Trick: Restrict to mean zero waves, i.e. consider $u^0 = \partial_x v^0$ for some $v_0 \in \mathcal{P}$ (provides Lagrangian formulation).
- This ends up providing additional conservation law that closes the Whitham system.
- This trick works fine for KdV waves (all waves can be made mean zero by Galilean invariance), but there are lots of physical solutions of gKdV which are not mean zero..... now what?!

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$$(M\omega)_t - G_x = 0, \qquad \omega_t + (s\omega)_x = 0$$

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- This gives us motivation for how to close the Whitham system: From the gKdV we find that

$$\left(\frac{u^2}{2}\right)_t = \left(uf(u) + uu_{xx} - F(u) - \frac{u_y^2}{2}\right)_x$$

and using the WKB expansion as before we find upon averaging the $\mathcal{O}(\varepsilon^0)$ equation the third conservation law

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- With the addition of this extra conservation law, we now have three equations for the three dimensional unknown $u^0 \in \mathcal{P}$.
- Assuming (a, E, c) are good local corrdinates on $\mathcal{P}$, we can write the Whitham system as

$$\partial_t \left(M\omega, P\omega, \omega \right) - \partial_x \left(a - sM\omega, -sP\omega - 2E, -s\omega \right) = 0$$

  where now these are considered as functions of $(a, E, s) \in \mathbb{R}$.
- To determine the stability of a particular constant solution corresponding to $(a, E, s) = (a_0, E_0, -c_0)$, following the physicists intuition we linearize the above system at this point.

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- The resulting linear system is has constant coefficients, and hence we can determine its stability by Fourier Transform techniques: need the characteristic polynomial

$$P(\mu,\kappa) = \det\left(\mu\frac{\partial\left(M\omega, P\omega, \omega\right)}{\partial(a,E,s)} - \frac{i\kappa}{T}\frac{\partial\left(a - sM\omega, -sP\omega - 2E, -s\omega\right)}{\partial_(a,E,s)}\right)$$

  have three real roots in the variable $\frac{i\kappa}{\mu T}$ at $(a,E,s) = (a_0, E_0, -c_0)$.

- Equivalent to hyperbolicity (i.e. local well-posedness) of Whitham system!

- **Q:** Does this polynomial agree with the leading order behavior of the Evans function?

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Whitham Theory Vs. Evans Function Techniques: M.J. & Zumbrun (2009)

- Direct (ugly) calculation shows that

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- Direct (ugly) calculation shows that

$$D(\mu, e^{i\kappa}) = \Gamma_0 P(-\mu, \kappa) + \mathcal{O}(|\mu|^4 + \kappa^4)$$

- Thus, just as the physicists said, Whitham theory correctly describes modulational stability of a given periodic traveling wave of the gKdV!!!!!

Computation?

- OK, so you have an expression which “determines” when a particular wave is modulationally stable..... can you compute it?!
- YES!!!
  (1) For power-law nonlinearities ($f(u) = u^{p+1}$) with $p \in \mathbb{N}$, can determine explicit formula for MI index in terms of moments of the underlying wave.
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Modulational Theory for KdV

- In case of KdV
$$u_t = u_{xxx} + \left(\frac{u^2}{2}\right)_x,$$
can express conserved quantities and period as integrals of closed cycles over a Riemann surface, and hence we can compute MI index using elliptic function calculations (Picard-Fuchs system).
- Get
$$\Delta_{MI} = C_0 \cdot \frac{N^2}{\mathrm{disc}(P(a,E,c))}$$
where $C_0 > 0$ and
$$P(a,E,c) = E + au + \frac{c}{2}u^2 - \frac{u^3}{6}.$$
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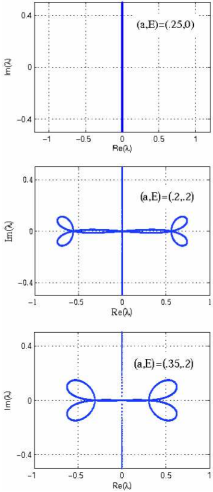
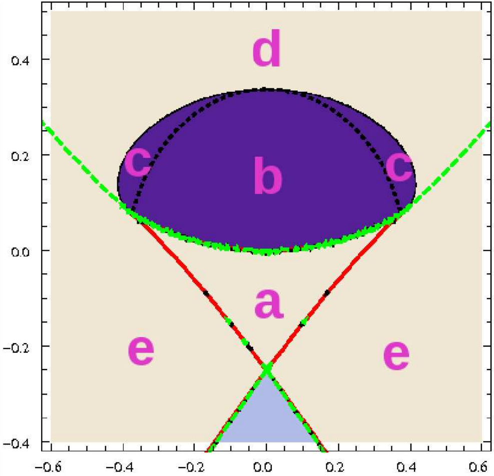
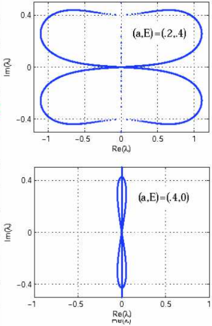
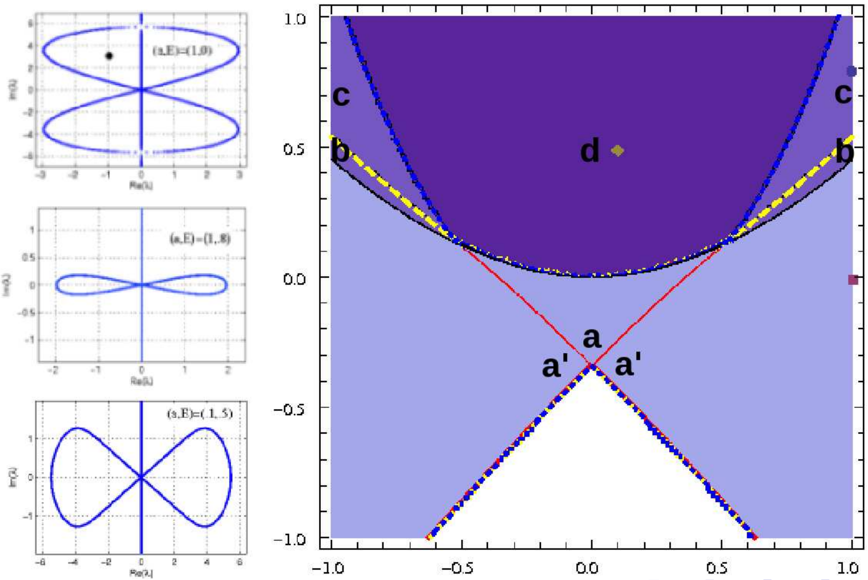
Modulational Theory for mKdV $f(u) = u^3$ (with positive wavespeed)







L^2-Critical KdV $f(u) = u^5$(with positive wavespeed)



Conclusions:

- Have extended modulation arguments of Whitham for KdV to non-zero mean waves of gKdV.
- Outline of Rigorous Theory: Integrability of traveling wave ODE $\Rightarrow$ generalized null-space of linearized operator can be "explicitly computed". Once this is in hand, perturbation theory and elbow grease does the rest!
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