SIAM J. MATH. ANAL.
Vol. 45, No. 5, pp. 3168–3193

STABILITY OF SMALL PERIODIC WAVES IN FRACTIONAL KdV-TYPE EQUATIONS*

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Abstract. We consider the effects of varying dispersion and nonlinearity on the stability of periodic traveling wave solutions of nonlinear PDEs of KdV type, including generalized KdV and Benjamin–Ono equations. In this investigation, we consider the spectral stability of such solutions that arise as small perturbations of an equilibrium state. A key feature of our analysis is the development of a nonlocal Floquet-like theory that is suitable to analyze the $L^2(\mathbb{R})$ spectrum of the associated linearized operators. Using spectral perturbation theory then, we derive a relationship between the power of the nonlinearity and the symbol of the fractional dispersive operator that determines the spectral stability and instability to arbitrary small localized perturbations.

Key words. KdV-type equations, fractional dispersion, periodic traveling waves, spectral stability

AMS subject classifications. 35Q53, 35B35, 35B10, 35P99

DOI. 10.1137/120894397

1. Introduction.

In this paper, we are concerned with the spectral stability and instability of periodic traveling wave solutions $u(x,t) = u(x-ct)$ of a class of scalar evolution equations of the form

$$u_t + \left(-\Lambda^\alpha u + u^{p+1}\right)_x = 0, \quad x, t \in \mathbb{R}, \tag{1.1}$$

where subscripts denote partial differentiation, $u = u(x,t)$ is a real-valued function, and the pseudodifferential operator $\Lambda = \sqrt{-\partial_x^2}$, referred to as Calderon's operator, is of order one and is defined by its Fourier multiplier as $\widehat{\Lambda u}(k) = |k|\hat{u}(k)$; throughout our analysis, the circumflex denotes the Fourier transform taken either on $\mathbb{R}$ or an appropriate one-dimensional torus, depending on the context. By inspection of the Fourier symbol, we see that Calderon's operator can alternatively be defined as $\Lambda = \mathcal{H}\partial_x$, where $\mathcal{H}$ denotes the Hilbert transform being applied on either the line or the torus, depending on the context, and in the x variable. Here, we consider $\alpha > \frac{1}{2}$ and either $p \in \mathbb{N}$ or $p = \frac{m}{n}$ with m and n being even and odd natural numbers, respectively; Remark A.1 in Appendix A discusses the purposes of these restrictions.

Equations of the form (1.1) arise naturally in the modeling of unidirectional propagation of weakly nonlinear dispersive waves of long wavelength, in which case u represents the wave profile or its velocity and the variables x and t are proportional to the distance in the direction of propagation and the elapsed time, respectively. In this context, the parameter $\alpha > 0$ characterizes the linear dispersion about the zero state; in particular, letting $c(\xi)$ denote the phase velocity of plane waves with frequency ξ, we find the dispersion relation $c(\xi) = |\xi|^\alpha$ for equations of the form (1.1). It is the goal of this paper to derive conditions on the dispersion parameter α and the power p of the nonlinearity for which the small amplitude periodic traveling wave solutions of (1.1) are stable.

*Received by the editors October 9, 2012; accepted for publication (in revised form) July 17, 2013; published electronically October 16, 2013. This work was supported by the National Science Foundation under NSF grant DMS-1211183 and by University of Kansas General Research Fund allocation 2302278.

http://www.siam.org/journals/sima/45-5/89439.html

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Arguably the most common and well-studied example of an equation of form (1.1) is the generalized KdV equation

$$u_t + u_{xxx} + \left(u^{p+1}\right)_x = 0 \tag{1.2}$$

corresponding to $\alpha = 2$. When $p = 1$, (1.2) was proposed by Korteweg and de Vries [30] in 1895 to model the unidirectional propagation of surface water waves of small amplitude and long wavelengths in a channel. When $p = 2$, (1.2) corresponds to the modified KdV equation, which arises as a model for large amplitude internal waves in a density stratified medium, as well as for Fermi–Pasta–Ulam lattices with bistable nonlinearity. In both these cases, the PDE is completely integrable and hence the Cauchy problem can, in principle, be completely solved via the inverse scattering transform.

Another important class of equations of form (1.1) arises when $\alpha = 1$, in which case we recover the generalized Benjamin–Ono equation

$$u_t + \left(-\mathcal{H}u_x + u^{p+1}\right)_x = 0. \tag{1.3}$$

When $p = 1$, Benjamin [5] and Ono [33] independently derived (1.3) as a model for the unidirectional propagation of internal waves in deep water. In this case, (1.3) is also completely integrable. Further examples with varying values of α can be derived in the context of shallow water theory by assuming different order relationships between the small quantities $\gamma := \frac{a}{h}$ and $\beta := \frac{h^2}{\lambda^2}$, where h denotes the depth of the water at rest and a and λ denote characteristic amplitudes and wavelengths of the waves searched for, respectively, corresponding to modeling in the small amplitude and small wavelength regime. Of particular interest, although outside the scope of our analysis, we point out that in the case[1] $\alpha = -\frac{1}{2}$, (1.1) was recently shown by Hur [23] to approximate up to quadratic order the surface water wave problem in two spatial dimensions in the infinite depth case, hence generalizing Whitham's equation in [36].

The stability of the solitary wave solutions of equations of the form (1.1) have a long and rich history, dating back to the novel work of Benjamin in [6], in which the stability of such waves in the case $\alpha = 2$ was established for $p < 4$. The stability analysis for $\alpha \geq 1$ was carried out by Bona, Souganidis, and Strauss in [7], where the authors extended the seminal work of Grillakis, Shatah, and Strauss [19, 20] to equations of form (1.1), where, most notably, the symplectic form in the Hamiltonian structure fails to be invertible. In [7], the authors establish the nonlinear orbital stability of the solitary wave solutions of (1.1) when $\alpha \geq 1$ for $p < 2\alpha$ and the instability of such solutions for $p > 2\alpha$.

In contrast to its solitary wave counterpart, the stability theory for T-periodic traveling wave solutions of equations of the form (1.1) has received considerably less attention, even in the classical case $\alpha = 2$, and this theory is still far from complete. Within this context, stability results typically fall into one of two categories: spectral stability to perturbations in $C_b(\mathbb{R})$ or $L^2(\mathbb{R})$ (see [9, 11, 13, 15, 21]), and nonlinear orbital stability to perturbations in $L^2_{\text{per}}([0, nT])$ for some $n \in \mathbb{N}$ (see [1, 2, 3, 10, 15, 16, 14, 25]). The majority of the nonlinear stability results restrict to the co-periodic case, i.e., perturbations in $L^2_{\text{per}}([0, nT])$ when $n = 1$, a clearly very restrictive class of perturbations, in which case authors are often able to use adaptations of the stability theory in [19, 20] to establish orbital stability. The only examples the author is aware of that establishes orbital stability for $n > 1$ are due to Deconinck and Nivala [15, 16],

[1]Notice that the operator $\partial_x \Lambda^\alpha$ is nonsingular for all $\alpha \geq -1$.

where the authors consider the KdV and mKdV equations, relying heavily in both cases on the complete integrability of the governing equations.

Concerning the spectral stability of such solutions, Bottman and Deconinck [13] established the spectral stability in $L^2(\mathbb{R})$ of periodic traveling wave solutions of the classic KdV equation ($\alpha = 2$ and $p = 1$), while Deconinck and Nivala [15, 16] established such a result for the modified KdV equation ($\alpha = 2$ and $p = 2$). For more general power law nonlinearities, when $\alpha = 2$ Bronski and Johnson [9] analyzed the spectral stability of such periodic traveling waves of (1.2) and derived there a geometric index for the stability and instability of such waves to perturbations in $L^2_{\mathrm{per}}([0, nT])$ with $n \gg 1$, while Haragus and Kapitula [21] established the spectral stability of such waves of sufficiently small amplitude when $p < 2$ and spectral instability when $p > 2$; as mentioned previously, it is this class of small amplitude periodic waves that we will be concerned with here.

It is important to note that all the periodic stability analyses described above are valid only in the local case $\alpha = 2$. Such results for $\alpha \in (0, 2)$ seem to be very few. Most notably, in [3] Pava and Nabali investigate the nonlinear stability of periodic traveling wave solutions to equations of the form (1.1) when subject to perturbations with the same period as the underlying wave.[2] In particular, they establish the nonlinear stability of periodic traveling waves of the Benjamin–Ono equation, corresponding to $\alpha = 1$ and $p = 1$ in (1.1) to perturbations with the same period as the underlying wave. More recently, Hur and Johnson in [24] verified for $\alpha \in (\frac{1}{3}, 2]$ and $p = 1$ the nonlinear stability of T-periodic traveling wave solutions of (1.1) to T-periodic perturbations when the underlying wave arises as a constrained energy minimizer. As far as the author is aware, these analyses are the only rigorous results concerning the nonlinear stability of periodic waves in equations of the form (1.1) when the dispersive operator is nonlocal. Furthermore, the author is not aware of any rigorous results concerning the spectral stability of periodic waves in equations of the form (1.1) when $\alpha \neq 2$.

It is the intent of the current paper to investigate the spectral stability of periodic traveling wave solutions of (1.1) of sufficiently small amplitude when subject to arbitrarily small localized, i.e., integrable, perturbations. Our main result, stated in Theorem 3.4 below, states that such small amplitude periodic traveling wave trains are spectrally stable if $\alpha > 1$ and $1 \leq p < p^*(\alpha)$, where the function $p^*(\alpha)$ is defined explicitly in (3.7) below, and is spectrally unstable if either $\alpha \in (\frac{1}{2}, 1)$ or $\alpha > 1$ and $p > p^*(\alpha)$. A plot of $p^*(\alpha)$ is given in Figure 1.1, and from this it is evident that for p sufficiently large all such small periodic traveling wave solutions are necessarily spectrally unstable. Furthermore, for $p \in (1, P_{\max})$, where $P_{\max} := \max_{\alpha \geq 1} p^*(\alpha) \approx 2.19$, there exist numbers $\alpha_-(p) < \alpha_+(p)$ such that such small periodic traveling waves are spectrally stable provided $\alpha \in (\alpha_-(p), \alpha_+(p))$ and are spectrally unstable if $\alpha \notin [\alpha_-(p), \alpha_+(p)]$. The fact that there is an upper bound on the admissible dispersion parameters α corresponding to stability for a given $p \in (1, P_{\max})$ is a striking new feature in the periodic stability analysis of models of the form (1.1), and it stands in direct contrast with the solitary wave theory. It is likely that this is due to the fundamental difference between the nature of disperison on the line and "dispersion" on the circle: here, we do not attempt to give an explanation for this difference and leave this instead as an interesting open question.

Concerning specific models of the form (1.1), in the classical case $\alpha = 2$, Theorem 3.4 recovers the result of Haragus and Kapitula [21] that such waves are spectrally

[2] In fact, they consider more general nonlocal dispersive operators than what are considered here.

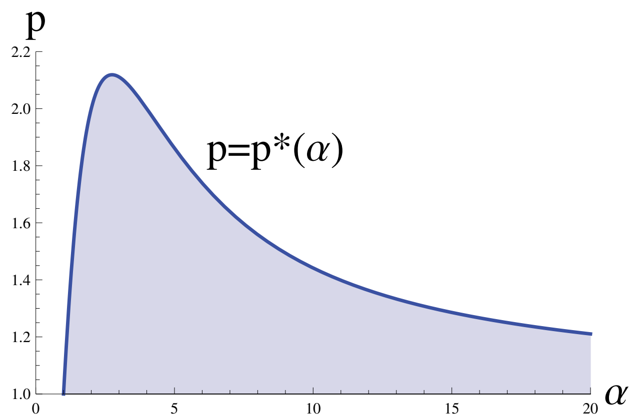

FIG. 1.1. *A plot of the critical nonlinearity $p^*(\alpha)$ as a function of the dispersion parameter α, plotted for $\alpha \geq 1$. Notice that $p^*(\alpha)$ decreases monotonically to 1 as $\alpha \to \infty$. The shaded area corresponds to the region of stability for the small amplitude periodic traveling waves considered here. For $\alpha \in (1/2, 1)$, we see $p^*(\alpha) < 1$ and hence all such small periodic traveling waves are spectrally unstable.*

stable if $1 \leq p < 2$ and are spectrally unstable if $p > 2$. Furthermore, since $p^*(1) = 1$, it follows that our analysis is not sufficient to conclude spectral stability or instability in the classical Benjamin–Ono equation, corresponding to $\alpha = p = 1$ in (1.1): see Remark 3.3 for more details. Finally, fixing the nonlinearity p instead, we see that when $p = 1$ all such small periodic traveling waves are spectrally stable for all $\alpha > 1$, while when $p = 2$ such waves are spectrally stable if $\alpha \in (2, 4)$ and are spectrally unstable if $\alpha \notin [2, 4]$. It is important to note that our analysis demonstrates that *all* such small T-periodic traveling wave solutions of (1.1) are spectrally stable to small T-periodic perturbations, so that the spectral instability detected in Theorem 3.4 is necessarily of sideband type, occurring in $L^2_{\mathrm{per}}([0, nT])$ for $n > 1$, and is hence impossible to detect by co-periodic stability analyses.

While our analysis is similar to that given in [21] in the local case $\alpha = 2$, relying on perturbation arguments of the spectrum from the easily analyzed constant state, it is complicated by the facts that (1) the existence theory no longer follows by elementary phase plane analysis, and (2) the absence of a suitable Floquet theory for nonlocal differential equations with periodic coefficients, which is necessary to analyze the essential spectrum of the linearized operators obtained from linearizing (1.1) about a given periodic traveling wave. In the forthcoming analysis, we resolve (1) by using a Lyapunov–Schmidt reduction argument similar to that given in [22] in the context of the fifth order Kawahara equation. For (2), we utilize the inverse Bloch–Fourier representation of functions in $L^2(\mathbb{R})$, which is well known in the analysis of Schrödinger operators with periodic potentials [34] and has been extensively used in the stability analysis of periodic wave trains in dissipative systems (see [4, 26, 27, 28, 18, 32] and references therein), to show that, even in this nonlocal setting, the essential spectrum of the linearized operators acting on $L^2(\mathbb{R})$ can be continuously parameterized by the eigenvalues of a one-parameter family of Bloch operators acting on a periodic domain. This spectral characterization extends that introduced by Gardner [18] in the local setting and is valid for considerably more general nonlocal operators than what is considered here.

The outline of this paper is as follows. In section 2, we prove the existence and determine asymptotic expansions of periodic traveling wave solutions of (1.1) of sufficiently small amplitude. The existence argument is based on an appropriate Lyapunov–Schmidt reduction argument, the details of which are included in Appendix A. Section 3 contains our main stability results, beginning with a careful characterization of the essential spectrum of the linearized operators acting on $L^2(\mathbb{R})$ in terms of the eigenvalues of a one-parameter family of Bloch operators acting on periodic functions. We then use spectral perturbation arguments to analyze the spectrum of the small amplitude periodic wave by considering the associated Bloch operators as small perturbations of those with constant coefficients obtained from linearizing about the nearby constant state. As such, the restriction to periodic waves of sufficiently small amplitude is essential in our argument. In particular, our analysis gives no information about the stability or instability of periodic waves with large amplitude when subject to small localized perturbations. Finally, we conclude with an appendix in which we give the proofs of the existence result and the Bloch-wave decomposition.

2. Existence of periodic traveling waves. In this section we analyze the set of periodic traveling wave solutions of (1.1) of the form

$$u(x,t) = u(x-ct), \quad c \in \mathbb{R},$$

where the function $u(\cdot)$ is a real-valued periodic function of its argument. Due to the scaling properties of (1.1) we can without loss of generality assume that $c = 1$, in which case such solutions of (1.1) arise as stationary solutions of the PDE

$$u_t + \left(-\Lambda^\alpha u - u + u^{p+1}\right)_x = 0, \tag{2.1}$$

or, equivalently, after performing a single integration, solutions of the nonlocal profile equation

$$-\Lambda^\alpha u - u + u^{p+1} = b, \tag{2.2}$$

where $b \in \mathbb{R}$ is a constant of integration taken to be in general nonzero.

We begin by considering the equilibrium solutions of (2.2) for $|b| \ll 1$. Notice when $b = 0$, there are in general two nonnegative equilibrium solutions $u = 0$ and $u = 1$. In the classical case when $\alpha = 2$, it follows by straightforward phase plane analysis that in the (u, u') phase plane, $u = 0$ is a saddle point associated with a homoclinic orbit (solitary wave), while the equilibrium $u = 1$ is a nonlinear center. Thus, when $\alpha = 2$ and $b = 0$ it is clear that there exists a one-parameter family of periodic orbits $\left\{(u_\gamma, u_\gamma')\right\}_{\gamma\in[0,1)}$ of period T_γ such that

$$\lim_{\gamma\to 0^+} T_\gamma = \frac{2\pi}{\sqrt{p}}, \quad \lim_{\gamma\to 1^-} T_\gamma = +\infty.$$

Moreover, these qualitative features persist for $|b| \ll 1$.

When $\alpha \neq 2$, however, this classical picture breaks down as we can no longer make sense of the phase space in the same way. Nevertheless, for $|b| \ll 1$ there exists a unique equilibrium solution $u = Q_b$ of (2.2) continuing from $Q_0 = 1$; indeed, a straightforward calculation provides the expansion

$$Q_b = 1 + \frac{1}{p}b - \frac{p+1}{2p^2}b^2 + \mathcal{O}(|b|^3),$$

valid for $|b| \ll 1$. Seeking nearby periodic solutions of (2.2), we set $u(x) = P(kx)$, where P is 2π-periodic and $k > 0$ denotes the wave number, and require that P and k satisfy the rescaled nonlocal profile equation

$$-k^\alpha \Lambda^\alpha P - P + P^{p+1} = b$$

posed on a 2π-periodic domain. Here, we consider for each $s > 0$ the operator Λ^s as being defined on the torus. In particular, we consider Λ^s as a closed operator on $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with dense domain $H^s(\mathbb{R}/2\pi\mathbb{Z})$ being defined via Fourier series as

$$\Lambda^s f(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^s e^{ikx} \hat{f}(k), \quad s \geq 0. \tag{2.3}$$

It is clear from this definition that Λ^s is invertible for each $s > 0$ when restricted to the mean-zero subspace of $H^s(\mathbb{R}/2\pi\mathbb{Z})$, and we define its inverse Λ^{-s} on this subspace via Fourier series as

$$\Lambda^{-s} f(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{ikx}\hat{f}(k)}{|k|^s}, \quad s > 0;$$

see [35] for a recent interesting discussion of Λ^s acting on the torus. With these definitions, for $\alpha > \frac{1}{2}$ one can use a Lyapunov–Schmidt reduction to verify the existence of a two-parameter family of small amplitude even periodic traveling wave solutions $u_{a,b}$ of (2.2) existing in a neighborhood of the equilibrium solution $u = Q_b$; see Appendix A for details.

In the parameterization of the periodic solutions $u_{a,b}(x) = P_{a,b}(k_{a,b}x)$ of (2.2) given by Theorem A.1, the functions $P_{a,b}$ are 2π-periodic even solutions of the rescaled nonlocal profile equation

$$-k_{a,b}^\alpha \Lambda^\alpha v - v + v^{p+1} = b \tag{2.4}$$

such that $P_{0,b} = Q_b$ and $k_{0,b}^\alpha = (p+1)Q_b^p - 1$; here, we have used the fact that $k_{a,b} > 0$ for $|(a,b)| \ll 1$. Furthermore, the parameter a is precisely the first Fourier coefficient $P_{a,b}$, and we have the relations $P_{a,b}(z+\pi) = P_{-a,b}(z)$ and $k_{a,b} = k_{-a,b}$ valid for all $|(a,b)| \ll 1$. Taking into account the translation invariance of (2.2), it follows that for a fixed $\alpha > \frac{1}{2}$ we can find a three-parameter family of small amplitude periodic traveling wave solutions of (2.1).

A key feature of the functions $P_{a,b}$ and $k_{a,b}$ described in Theorem A.1 is that these depend analytically on a and b for $|(a,b)| \ll 1$. The next lemma exploits this fact to provide us with explicit expansions of the functions $P_{a,b}$ and $k_{a,b}$ valid for sufficiently small a and b. These expansions are crucial in the forthcoming stability analysis.

LEMMA 2.1. *For sufficiently small $a, b \in \mathbb{R}$ and $\alpha > \frac{1}{2}$, the functions $P_{a,b}$ and $k_{a,b}$ in Theorem* A.1 *can be expanded as*

$$P_{a,b}(x) = Q_b + \cos(z)a + \frac{p+1}{4}\left(\frac{1}{2^\alpha - 1}\cos(2z) - 1\right)a^2 + \mathcal{O}(|a|(a^2+b^2)),$$
$$k_{a,b}^\alpha = k_{0,b}^\alpha - \frac{p(p+1)(2^\alpha(p+3) - 2(p+2))}{8(2^\alpha - 1)}a^2 + \mathcal{O}(|a|^3 + |b|^3),$$

where $k_{0,b}^\alpha = (p+1)Q_b^p - 1$.

Proof. Taking $b = 0$ for now and recalling Theorem A.1, we begin with the following small amplitude ansatz for $P_{a,0}$ and $k_{a,0}$:

$$\begin{aligned} P_{a,0}(z) &= 1 + a\cos(z) + a^2 v_2(z) + a^3 v_3(z) + \mathcal{O}(a^4), \\ k_{a,0}^{\alpha} &= p + a^2 k_1 + \mathcal{O}(a^4), \end{aligned}$$

where each v_j is even and 2π-periodic in the z variable. Substituting these expansions into the profile equation (2.4) yields a hierarchy of compatibility conditions. The $\mathcal{O}(1)$ equation is trivially satisfied, while the $\mathcal{O}(|a|)$ equation reads

$$(-\Lambda^{\alpha} + 1)\cos(z) = 0,$$

which again holds. The $\mathcal{O}(a^2)$ equation now reads

$$p\,(-\Lambda^{\alpha} + 1)\,v_2 = -\frac{p(p+1)}{4}\,(1 + \cos(2z)),$$

which, using (2.3), is seen to have even solutions of the form

$$v_2(z) = \frac{p(p+1)}{4}\left(\frac{1}{2^{\alpha} - 1}\cos(2z) - 1\right) + A\cos(z)$$

for any $A \in \mathbb{R}$. Notice, however, that we must take $A = 0$ by the definition of the parameter a. Continuing to the $\mathcal{O}(a^3)$ equation, we find

$$\begin{aligned} p\,(-\Lambda^{\alpha} + 1)\,v_3 = k_1\cos(z) &- \frac{p(p+1)^2}{8(2^{\alpha} - 1)}\,(\cos(z) + \cos(3z)) + \frac{p(p+1)^2}{4}\cos(z) \\ &- \frac{p(p-1)(p+1)}{24}\,(3\cos(z) + \cos(3z))\,, \end{aligned}$$

which is readily seen to have a resonant solution containing a term proportional to $z\sin(z)$ unless

$$k_1 = \frac{p(p+1)^2}{8(2^{\alpha} - 1)} - \frac{p(p+1)^2}{4} + \frac{p(p-1)(p+1)}{8},$$

i.e., unless k_1 is chosen so that

$$k_1 = -\frac{p(p+1)\,(2^{\alpha}(p+3) - 2(p+2))}{8(2^{\alpha} - 1)}.$$

This completes the expansions to the desired order when $b = 0$. The expansions when $|b| \ll 1$ are obtained similarly. ∎

3. Spectral stability to localized perturbations. We are now ready to begin discussing the stability analysis of the small amplitude periodic solutions $P_{a,b}(k_{a,b}\cdot)$ constructed in the previous section under the flow induced by the PDE (2.1). As mentioned in the introduction, we are interested here in the spectral stability of such waves when subject to small localized, i.e., integrable, perturbations on $\mathbb{R}$. In the classical case when $\alpha \in 2\mathbb{N}$, the associated spectral problem obtained from linearizing about a given wave $P_{a,b}$ is that for an ordinary differential equation with periodic coefficients. As such, it is an easy calculation to see that the spectrum of this linearized operator, considered as an operator on $L^2(\mathbb{R})$, is purely essential and agrees with the continuous spectrum. In this classical case, a common method for analyzing the essential spectrum of the linearization is to use Floquet theory to provide a continuous parameterization of the essential spectrum by the eigenvalues of an associated one-parameter family of linear operators considered with periodic boundary conditions; see [9, 11, 21], for example.

When $\alpha > \frac{1}{2}$ is not an even natural number, however, the linearized operator associated with $P_{a,b}$ is nonlocal and hence the classical Floquet theory does not apply. Nevertheless, we find in the next section that we can still reduce the spectral stability problem on $L^2(\mathbb{R})$ to a one-parameter family of eigenvalue problems with periodic boundary conditions. This characterization of the essential spectrum is given by utilizing the Floquet–Bloch transform defined on $L^2(\mathbb{R})$. Once this characterization is established, we use an appropriate spectral perturbation theory to analyze the eigenvalues of the associated one-parameter family of Bloch operators when $|(a,b)| \ll 1$.

3.1. Characterization of the essential spectrum. To analyze the spectral stability of the small amplitude periodic waves obtained in the previous section, we fix $\alpha > \frac{1}{2}$ and set $z = k_{a,b}x$ and $s = k_{a,b}t$ in (2.1) to get

$$v_s + (-|k_{a,b}|^\alpha \Lambda^\alpha v - v + v^{p+1})_z,$$

where now Λ acts on $\mathbb{R}$ in the z-variable. Linearizing about $P_{a,b}$ and considering solutions of the linearized equation of the form $v(z,t) = e^{\lambda t} v(z)$, with $\lambda \in \mathbb{C}$ and $v(\cdot) \in L^2(\mathbb{R})$, leads to the spectral problem

$$\mathcal{M}_{a,b} v := \partial_z \mathcal{L}_{a,b} v = \lambda v$$

considered on $H^{\alpha+1}(\mathbb{R})$, where here λ denotes the spectral parameter and

$$\mathcal{L}_{a,b} := |k_{a,b}|^\alpha \Lambda^\alpha + 1 - (p+1) P_{a,b}^p$$

is considered as a closed, densely defined operator acting on $L^2(\mathbb{R})$. In this case, spectral stability is defined by the condition that the operator $\mathcal{M}_{a,b}$ has no spectrum in the open right half plane. Notice, however, that the Hamiltonian form of the spectral problem[3] implies the spectrum is symmetric with respect to the real and imaginary axes, and hence spectral stability is equivalent with all the spectrum of the operator $\mathcal{M}_{a,b}$ being confined to the imaginary axis; we will discuss this more in the next section.

In order to analyze the spectrum of the linear operator $\mathcal{M}_{a,b}$ acting on $L^2(\mathbb{R})$, we recall now some facts about the Floquet–Bloch decomposition of $L^2(\mathbb{R})$; as mentioned above, the standard Floquet theory does not apply for $\alpha \notin 2\mathbb{N}$ since the spectral problem for $\mathcal{M}_{a,b}$ is not in the form of an ordinary differential equation. To this end, notice that given any $v \in L^2(\mathbb{R})$ we can express v in terms of its inverse Bloch representation as

$$v(x) = \int_{-1/2}^{1/2} e^{i\xi x} \check{v}(\xi, x) d\xi,$$

where $\check{v}(\xi, x) := \sum_{k\in\mathbb{Z}} e^{ikx} \hat{v}(\xi + k)$ are 2π-periodic functions of x and where $\hat{v}(\omega) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega z} v(z) dz$ denotes the standard Fourier transform of v. Indeed, the above

[3]More precisely, the fact that the spectral problem takes the form $J\mathcal{L}v = \lambda v$, where v belongs to some Hilbert space X and where J is skew symmetric and $\mathcal{L}$ is self-adjoint when acting on X.

formulas may be easily checked on the Schwartz class by grouping frequencies which differ by one in the standard Fourier transform representation of v:

$$v(z) = \sum_{j\in\mathbb{Z}} \int_{-1/2}^{1/2} e^{i(k+j)z}\hat{v}(k+j)dk = \int_{-1/2}^{1/2} e^{ikz}\check{v}(\xi,z)dz.$$

The Bloch transform $\mathcal{B} : L^2(\mathbb{R}) \to L^2([-1/2,1/2); L^2(\mathbb{R}/2\pi\mathbb{Z}))$ given by $\mathcal{B}(v)(\xi,x) := \check{v}(\xi,x)$ is then well defined, bijective, and continuous; in fact, using the classical Parseval theorem we find for all $v \in L^2(\mathbb{R})$ that

$$\|v\|^2_{L^2(\mathbb{R})} = 2\pi \int_{-1/2}^{1/2}\int_0^{2\pi} |\mathcal{B}(v)(\xi,z)|^2 \, dz \; d\xi \tag{3.1}$$

so that the rescaled Bloch transform $\sqrt{2\pi}\mathcal{B}$ is an isometry on $L^2(\mathbb{R})$.[4]

Furthermore, we find given any $v \in L^2(\mathbb{R})$ that

$$\mathcal{B}\left(\mathcal{M}_{a,b}v\right)(\xi,x) = \mathcal{M}_{a,b,\xi}\left(\check{v}(\xi,\cdot)\right)(x),$$

where $\mathcal{M}_{a,b,\xi} : H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z}) \subset L^2(\mathbb{R}/2\pi\mathbb{Z}) \to L^2(\mathbb{R}/2\pi\mathbb{Z})$ is defined by

$$\mathcal{M}_{a,b,\xi} := e^{-i\xi x}\mathcal{M}_{a,b}e^{i\xi x}, \quad \xi \in [-1/2, 1/2).$$

The operators $\mathcal{M}_{a,b,\xi}$ are called the Bloch operators associated to $\mathcal{M}_{a,b}$, and from above they correspond to operator-valued symbols of $\mathcal{M}_{a,b}$ under $\mathcal{B}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$. The next result relates the spectrum of $\mathcal{M}_{a,b}$ acting on $L^2(\mathbb{R})$ to that of the corresponding one-parameter family of Bloch operators $\{\mathcal{M}_{a,b,\xi}\}_{\xi\in[-1/2,1/2)}$, providing us with a Floquet-like theory that is suitable for our needs.

PROPOSITION 3.1. *Consider the operator $\mathcal{M}_{a,b}$ acting on $L^2(\mathbb{R})$ with domain $H^{\alpha+1}(\mathbb{R})$ and the associated Bloch operators $\{\mathcal{M}_{a,b,\xi}\}_{\xi\in[-1/2,1/2)}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with domain $H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$. Then for any $\lambda \in \mathbb{C}$, the following statements are equivalent:*

(i) *λ belongs to the spectrum of the closed operator $\mathcal{M}_{a,b}$ acting on $L^2(\mathbb{R})$.*

(ii) *There exists a $\xi \in [-1/2, 1/2)$ such that λ belongs to the spectrum of the closed operator $\mathcal{M}_{a,b,\xi}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$.*

(iii) *There exists a nonzero function $V \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ of the form $V(z) = e^{i\xi z}v(z)$ for some $\xi \in [-1/2, 1/2)$ and $v \in H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ such that $(\mathcal{M}_{a,b} - \lambda\mathbf{I})\, V = 0$.*

The proof of Proposition 3.1 is contained in Appendix B. Using this result, it follows that

$$\sigma_{L^2(\mathbb{R})}\left(\mathcal{M}_{a,b}\right) = \bigcup_{\xi\in[-1/2,1/2)} \sigma_{L^2_{\mathrm{per}}([0,2\pi])}\left(\mathcal{M}_{a,b,\xi}\right)$$

so that, in particular, the Bloch transform provides a continuous parametrization of the essential spectrum of the operator $\mathcal{M}_{a,b}$ by the discrete spectrum of a one-parameter family of Bloch operators. As a result, rather than analyzing the essential spectrum of the operator $\mathcal{M}_{a,b}$ directly, we can instead choose to study the point spectrum of the operators $\mathcal{M}_{a,b,\xi}$ for each $\xi \in [-1/2, 1/2)$. This investigation is the subject of the next section.

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[4] This is in fact a special case of a more general class of generalized Hausdorff–Young-type inequalities, following from interpolating (3.1) with the triangle inequality, that are known to be satisfied by the Bloch transform; see [29] for details.

3.2. Analysis of the unperturbed operators. We begin by considering the stability of the constant state $P_{0,0} = Q_0 = 1$. Later, we will treat the linearized operators about the nearby periodic solutions as small perturbations of those linearized about Q_0. Indeed, it is straightforward to establish the estimate

$$\|\mathcal{M}_{a,b,\xi} - \mathcal{M}_{0,0,\xi}\| = \mathcal{O}(|a| + |b|)$$

as $(a, b) \to (0, 0)$ uniformly in the Bloch parameter $\xi \in [-1/2, 1/2)$. A standard perturbation argument then guarantees the spectrum of $\mathcal{M}_{a,b,\xi}$ and $\mathcal{M}_{0,0,\xi}$ stays close for sufficiently small (a, b). More precisely, we have the following result.

LEMMA 3.2. *Let $p \geq 1$ and $\alpha > \frac{1}{2}$. For any $\delta > 0$ there exists an $\varepsilon > 0$ such that for any $\xi \in [-1/2, 1/2)$ and any $(a, b) \in \mathbb{R}^2$ with $\|(a, b)\| \leq \varepsilon$, the spectrum of $\mathcal{M}_{a,b,\xi}$ satisfies*

$$\sigma\left(\mathcal{M}_{a,b,\xi}\right) \subset \left\{\lambda \in \mathbb{C} : \operatorname{dist}\left(\lambda, \sigma\left(\mathcal{M}_{0,0,\xi}\right)\right) < \delta\right\}.$$

We now analyze the spectrum of $\mathcal{M}_{0,0,\xi}$ posed on $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Since $\mathcal{M}_{0,0,\xi}$ has constant coefficients, we find by a straightforward Fourier analysis argument that

$$\sigma(\mathcal{M}_{0,0,\xi}) = \{\lambda = i\omega_{n,\xi} : n \in \mathbb{Z}\} \subset i\mathbb{R}$$

for each fixed $\xi \in [-1/2, 1/2)$, where the $\omega_{n,\xi}$ are determined by the linear dispersion relation

$$\omega(k) := kp\left(|k|^{\alpha} - 1\right) \tag{3.2}$$

through $\omega_{n,\xi} := \omega(n+\xi)$. Notice that every $\lambda \in \sigma(\mathcal{M}_{0,0,\xi})$ is a semisimple eigenvalue with algebraic and geometric multiplicity given by the number of distinct $n \in \mathbb{Z}$ such that $\lambda = i\omega_{n,\xi}$ with associated eigenfunction $e_n := e^{inz}$.

To study the behavior of these eigenvalues for small $(a, b) \in \mathbb{R}^2$, notice that the spectrum of the operator $\mathcal{M}_{a,b}$ is symmetric with respect to both the real and the imaginary axis. Indeed, since the coefficients of $\mathcal{M}_{a,b}$ are real-valued it follows that its spectrum is symmetric with respect to the real axis. In terms of the Bloch operators this implies that $\sigma(\mathcal{M}_{a,b,\xi}) = \overline{\sigma(\mathcal{M}_{a,b,-\xi})}$. Furthermore, noting that $\mathcal{M}_{a,b}$ anticommutes with the isometry $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$Sv(z) = v(-z),$$

we find that the spectrum of $\mathcal{M}_{a,b}$ is symmetric with respect to the origin. It follows that the Bloch operators satisfy $\mathcal{M}_{a,b,\xi}S = -S\mathcal{M}_{a,b,\xi}$ so that $\sigma(\mathcal{M}_{a,b,\xi}) = -\sigma(\mathcal{M}_{a,b,-\xi})$. Finally, recalling that $k_{a,b}$ is even in a and that $P_{a,b}(z+\pi) = P_{-a,b}(z)$ for all $|(a, b)| \ll 1$ we see that $\sigma(\mathcal{M}_{a,b}) = \sigma(\mathcal{M}_{-a,b})$ and that $\sigma(\mathcal{M}_{a,b,\xi}) = \sigma(\mathcal{M}_{-a,b,\xi})$. As a result, we find that the spectrum of a given Bloch operator $\mathcal{M}_{a,b,\xi}$ is symmetric with respect to the imaginary axis. It follows then that when eigenvalues of $\mathcal{M}_{a,b,\xi}$ bifurcate from the imaginary axis they must bifurcate in pairs resulting from collisions of eigenvalues on the imaginary axis. A well-known result from the study of Hamiltonian systems tells us that when two purely imaginary eigenvalues collide, the collision will not result in a pair of eigenvalues bifurcating from the imaginary axis provided both eigenvalues have the same Krein signature; see [37].

We now consider the location of the eigenvalues more carefully, in particular watching for sets of eigenvalues that collide for a fixed ξ. Notice by the symmetry property $\sigma\left(\mathcal{M}_{a,b,\xi}\right) = \overline{\sigma\left(\mathcal{M}_{a,b,-\xi}\right)}$ we may restrict our consideration to Bloch frequencies $\xi \in [0, 1/2]$. Now, when $\xi = 0$ we find that

$$\omega_{-1,0} = \omega_{0,0} = \omega_{1,0} = 0$$

and

$$\cdots < \omega_{-3,0} < \omega_{-2,0} < 0 < \omega_{2,0} < \omega_{3,0} < \cdots .$$

Furthermore, for $\xi \in [0,1/2]$ we have

$$\omega_{n,\xi} \subset \left(-\infty, -\frac{3p}{2}\left(\left(\frac{3}{2}\right)^{\alpha} - 1\right)\right] \cup [2p(2^{\alpha}-1), \infty), \quad |n| \geq 2,$$

and that

$$\omega_{n,\xi} \subset \left[\omega^*(\alpha), \frac{3p}{2}\left(\left(\frac{3}{2}\right)^{\alpha} - 1\right)\right], \quad |n| \leq 1,$$

where

$$\omega^*(\alpha) = \begin{cases} -\alpha p(1+\alpha)^{-(1+1/\alpha)} & \text{if} \quad \alpha \in (1/2,1), \\ \frac{p}{2}\left(\left(\frac{1}{2}\right)^{\alpha} - 1\right) & \text{if} \quad \alpha \geq 1. \end{cases}$$

This naturally provides us with a spectral decomposition

$$\sigma\left(\mathcal{M}_{0,0,\xi}\right) = \sigma_1\left(\mathcal{M}_{0,0,\xi}\right) \cup \sigma_2\left(\mathcal{M}_{0,0,\xi}\right)$$

for $\mathcal{M}_{0,0,\xi}$ with

$$\begin{cases} \sigma_1\left(\mathcal{M}_{0,0,\xi}\right) = \left\{\lambda \in \mathbb{C} : \lambda = i\omega_{n,\xi} \text{ for some } |n| \geq 2\right\}, \\ \sigma_2\left(\mathcal{M}_{0,0,\xi}\right) = \{i\omega_{-1,\xi}, i\omega_{0,\xi}, i\omega_{1,\xi}\}, \end{cases}$$

with the property that for any v in the infinite-dimensional spectral subspace associated with $\sigma_1\left(\mathcal{M}_{0,0,\xi}\right)$ satisfies

$$\langle \mathcal{L}_{0,0,\xi} v, v\rangle \geq \left(\left(\frac{3}{2}\right)^{\alpha} - 1\right) p\|v\|^2$$

uniformly for $\xi \in [0,1/2]$, where

$$\mathcal{L}_{0,0,\xi} := e^{-i\xi x}\mathcal{L}_{0,0}e^{i\xi x}$$

is considered on $L^2(\mathbb{R}/2\pi\mathbb{Z})$; in particular, it follows that all eigenvalues in $\sigma_1(\mathcal{M}_{0,0,\xi})$ have positive Krein signature for $\xi \in [0,1/2]$. By a standard perturbation argument, we find that the above properties persist for sufficiently small a and b. More precisely, one has that for a and b sufficiently small we have a spectral decomposition

$$\sigma\left(\mathcal{M}_{a,b,\xi}\right) = \sigma_1\left(\mathcal{M}_{a,b,\xi}\right) \cup \sigma_2\left(\mathcal{M}_{a,b,\xi}\right)$$

such that

$$\sigma_1\left(\mathcal{M}_{a,b,\xi}\right) \cap \sigma_2\left(\mathcal{M}_{a,b,\xi}\right) = \emptyset$$

for $|(a,b)| \ll 1$ and $\xi \in [0,1/2]$, where the spectral subspace associated with $\sigma_2\left(\mathcal{M}_{a,b,\xi}\right)$ is three-dimensional, and the infinitely many eigenvalues associated with $\sigma_1\left(\mathcal{M}_{a,b,\xi}\right)$ all have positive Krein signature. Notice this latter property implies that all the eigenvalues in $\sigma_1\left(\mathcal{M}_{a,b,\xi}\right)$ are purely imaginary for $|(a,b)|$ sufficiently small.

It now remains to determine the location of the eigenvalues in $\sigma_2(\mathcal{M}_{a,b,\xi})$. These are smooth continuations for small $a, b \in \mathbb{R}$ of the eigenvalues $i\omega_{-1,\xi}$, $i\omega_{0,\xi}$, and $i\omega_{1,\xi}$ of $\mathcal{M}_{a,b,\xi}$. First, notice that two such eigenvalues collide if and only if $\xi = 0$, when $\omega_{\pm 1,0} = \omega_{0,0} = 0$. Furthermore, for any $\xi_0 \in (0, 1/2)$ there exists a constant $c_0 > 0$ such that

$$|\omega_{j,\xi} - \omega_{k,\xi}| \geq c_0 p, \quad j,k \in \{-1,0,1\},\ j \neq k,\ \xi \in [\xi_0, 1/2].$$

Consequently, these eigenvalues will remain simple and distinct under small perturbations (for sufficiently small a and b) for all $\xi \in [0, 1/2]$. In particular, it follows that for any $\xi_0 \in (0, 1/2]$ we have

$$\sigma_2(\mathcal{M}_{a,b,\xi}) \subset i\mathbb{R}, \quad \xi \in [\xi_0, 1/2]$$

for sufficiently small a and b. Thus, we have reduced the problem to locating the eigenvalues $i\omega_{\pm 1,\xi}$ and $i\omega_{0,\xi}$ for a, b, and ξ small.

3.3. Location of the critical Bloch spectrum. To this end, our strategy is to project, for $|(a,b,\xi)| \ll 1$, the infinite-dimensional spectral problem

$$\mathcal{M}_{a,b,\xi} v = \lambda v, \quad v \in L^2(\mathbb{R}/2\pi\mathbb{Z})$$

onto the three-dimensional critical eigenspace corresponding to the three eigenvalues bifurcating from the $(\lambda, \xi) = (0,0)$ state. More precisely, for $|(a,b,\xi)| \ll 1$ we compute a suitable basis $\{\eta_j(z; a, b, \xi)\}_{j=0,1,2}$ for the three-dimensional spectral subspace associated to $\sigma_2(\mathcal{M}_{a,b,\xi})$ and compute the 3×3 matrices[5]

$$\mathcal{B}_{a,b,\xi} := \left[\left(\left\langle \frac{\eta_j(z;a,b,\xi)}{\langle \eta_j(z;a,b,\xi), \eta_j(z;a,b,\xi)\rangle}, \mathcal{M}_{a,b,\xi}\eta_k(z;a,b,\xi)\right\rangle\right)\right]_{j,k=0,1,2}$$

and

$$I_{a,b,\xi} := \left[\left(\frac{\langle \eta_j(z;a,b,\xi), \eta_k(z;a,b,\xi)\rangle}{\langle \eta_j(z;a,b,\xi), \eta_j(z;\alpha,b,\xi)\rangle}\right)\right]_{j,k=0,1,2},$$

noting then that the eigenvalues $\mathcal{M}_{a,b,\xi}$ in a neighborhood of $\lambda = 0$ are given as the roots of the cubic polynomial

$$\det(\mathcal{B}_{a,b,\xi} - \lambda I_{a,b,\xi}) = 0. \tag{3.3}$$

This method of determining asymptotic expansions of the critical eigenvalues bifurcating from the $(\lambda, \xi) = (0,0)$ is well established and is applicable outside the small amplitude regime considered here; see [8, 11], for example. This is accomplished in three steps. First, since the operator $\mathcal{M}_{a,b,\xi}$ has constant coefficients when $a = 0$, the matricies $\mathcal{B}_{0,b,\xi}$ and $I_{0,b,\xi}$ can be explicitly computed using Fourier analysis. Second, setting $\xi = 0$ we study the (co-periodic) eigenvalues of $\mathcal{M}_{a,b,0}$. In particular, by differentiating the underlying wave profile we construct a basis for the generalized kernel of the operator $\mathcal{M}_{a,b,0}$ from which we can compute the matrices $\mathcal{B}_{a,b,0}$ and $I_{a,b,0}$ directly. Finally, we analyze the interactions between $|a| \ll 1$ (nearly constant solutions) and $|\xi| \ll$ (nearly co-periodic perturbations). This final step requires us to expand the Bloch operator $\mathcal{M}_{a,b,\xi}$ in the Bloch-frequency ξ; this is the content of Lemma 3.3

[5]Here and throughout, we are using $\langle f, g\rangle = \int_0^{2\pi} \overline{f(z)} g(z) dz$.

below. With this information, we find that (3.3) is a homogeneous cubic polynomial in the variables λ and ξ. Analyzing the discriminant of this cubic polynomial then yields an explicit relationship that must hold between the parameters p and α for the underlying wave to be spectrally stable in a small neighborhood of the origin.

First, at $a = 0$ the operator $\mathcal{M}_{0,b,\xi}$ has constant coefficients, and a basis for the critical spectral subspace in this case is spanned by the functions 1 and $e^{\pm iz}$ associated with the eigenvalues $i\widetilde{\omega}_{0,\xi,b}$ and $i\widetilde{\omega}_{\pm 1,\xi,b}$, respectively, where the $\widetilde{\omega}_{j,\xi,b}$ are defined in terms of the $\omega_{j,\xi}$ via

$$\widetilde{\omega}_{j,\xi,b} := n|k_{0,b}|^{\alpha}\left(|j|^{\alpha} - 1\right) = \frac{|k_{0,b}|^{\alpha}}{p}\,\omega_{j,\xi}.$$

Working instead with the real basis

$$\eta_0(z;0,0,\xi) = \cos(z), \quad \eta_1(z;0,0,\xi) = \sin(z), \quad \eta_2(z;0,0,\xi) = 1$$

a direct calculation yields

$$\mathcal{B}_{0,b,\xi} = \begin{pmatrix} \frac{i}{2}\left(\widetilde{\omega}_{1,\xi,b} + \widetilde{\omega}_{-1,\xi,b}\right) & \frac{1}{2}\left(\widetilde{\omega}_{1,\xi,b} - \widetilde{\omega}_{-1,\xi,b}\right) & 0 \\ -\frac{1}{2}\left(\widetilde{\omega}_{1,\xi,b} - \widetilde{\omega}_{-1,\xi,b}\right) & \frac{i}{2}\left(\widetilde{\omega}_{1,\xi,b} + \widetilde{\omega}_{-1,\xi,b}\right) & 0 \\ 0 & 0 & i\widetilde{\omega}_{0,\xi,b} \end{pmatrix},$$

valid for any $\xi \in [0, 1/2]$. Similarly, by the same reasoning we find

$$I_{0,b,\xi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for all b and ξ sufficiently small.

Next, at $\xi = 0$ we claim that $\lambda = 0$ is an eigenvalue of $\mathcal{M}_{a,b,0}$ with algebraic multiplicity three and geometric multiplicity two. Indeed, notice that since (1.1) is invariant with respect to spatial translations it follows that

$$\mathcal{M}_{a,b,0}\partial_z P_{a,b}(z) = 0$$

so that $\lambda = 0$ is indeed an eigenvalue of $\mathcal{M}_{a,b,0}$. Furthermore, differentiating the profile equation (2.4) with respect to the parameters a and b yields

$$\begin{aligned} \mathcal{M}_{a,b,0}\partial_a P_{a,b}(z) &= -\partial_a\left(|k_{a,b}|^{\alpha}\right)\Lambda^{\alpha}\partial_z P_{a,b}, \\ \mathcal{M}_{a,b,0}\partial_b P_{a,b}(z) &= -\partial_b\left(|k_{a,b}|^{\alpha}\right)\Lambda^{\alpha}\partial_z P_{a,b} \end{aligned}$$

so that, in particular, we find

$$\mathcal{M}_{a,b,0}\left(\partial_b\left(|k_{a,b}|^{\alpha}\right)\partial_a P_{a,b}(z) - \partial_a\left(|k_{a,b}|^{\alpha}\right)\partial_b P_{a,b}(z)\right) = 0,$$

giving a second function in the kernel of $\mathcal{M}_{a,b,0}$. Finally, by a straightforward computation, using (2.4) and the fact that $\mathcal{L}_{a,b,0}\partial_z P_{a,b} = 0$, we find

$$\mathcal{M}_{a,b,0}P_{a,b}(z) = -p\left(|k_{a,b}|^{\alpha}\Lambda^{\alpha} + 1\right)\partial_z P_{a,b}$$

so that

$$\mathcal{M}_{a,b,0}\left(\partial_b\left(|k_{a,b}|^{\alpha}\right)P_{a,b} - p|k_{a,b}|^{\alpha}\partial_b P_{a,b}\right) = -p\partial_b\left(|k_{a,b}|^{\alpha}\right)\partial_z P_{a,b},$$

giving a 2π-periodic function in the generalized kernel of $\mathcal{M}_{a,b,0}$ ascending above the translation mode $\partial_z P_{a,b}$. In particular, this shows that for all values of a and b sufficiently small, the three eigenvalues $i\omega_{0,\xi}$ and $i\omega_{\pm 1,\xi}$ all vanish when $\xi = 0$, and that the eigenvalue $\lambda = 0$ indeed corresponds to an algebraically triple and geometrically double eigenvalue of the operator $\mathcal{M}_{a,b,0}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$.

Using the expansions in Lemma 2.1 we now obtain a basis for the critical eigenspace which is compatible with the basis found at $a = b = 0$. In particular, we take

$$\begin{aligned}
\eta_0(z;a,b,0) &= \frac{1}{p+1}\left(\partial_b\left(|k_{a,b}|^\alpha\right)\partial_a P_{a,b}(z) - \partial_a\left(|k_{a,b}|^\alpha\right)\partial_b P_{a,b}(z)\right)\\
&= \cos(z) - \frac{2+2^\alpha(p-1)}{4(2^\alpha-1)}a + \frac{p+1}{6}\cos(2z)a - \frac{2}{p}\cos(z)b + \mathcal{O}(a^2+b^2),\\
\eta_1(z;a,b,0) &= -\frac{1}{a}\partial_z P_{a,b} = \sin(z) + \frac{p+1}{2(2^\alpha-1)}\sin(2z)a + \mathcal{O}(a^2+b^2)\\
\eta_2(z;a,b,0) &= \partial_b\left(|k_{a,b}|^\alpha\right)P_{a,b} - p|k_{a,b}|^\alpha\partial_b P_{a,b}\\
&= 1 + (p+1)\cos(z)a - \frac{p+1}{p}b + \mathcal{O}(a^2+b^2).
\end{aligned}$$

In this basis, a straightforward calculation yields

$$I_{a,b,0} = \begin{pmatrix} 1 & 0 & \frac{2^\alpha(p+3)-2(p+2)}{2(2^\alpha-1)}a \\ 0 & 1 & 0 \\ \frac{2^\alpha(p+3)-2(p+2)}{4(2^\alpha-1)}a & 0 & 1 \end{pmatrix} + \mathcal{O}(a^2+b^2)$$

and

$$\mathcal{B}_{a,b,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{a,b} \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\sigma_{a,b} &= \frac{\langle \eta_1(z;a,b,0), \mathcal{M}_{a,b,0}\eta_2(z;a,b,0)\rangle}{\langle \eta_1(z;a,b,0), \eta_1(z;a,b,0)\rangle} = pa\partial_b\left(|k_{a,b}|^\alpha\right)\\
&= p(p+1)a - 2(p+1)ab + \mathcal{O}(|a|(a^2+b^2)).
\end{aligned}$$

Continuing, we note that the basis $\{\eta_j(z;a,b,0)\}_{j=0,1,2}$ defined above can be extended to a basis of the three-dimensional eigenspace bifurcating from the generalized kernel of $\mathcal{M}_{a,b,0}$ for sufficiently small a, b, and ξ. This provides us with an expansion of the form

$$\mathcal{B}_{a,b,\xi} = \mathcal{B}_{0,b,\xi} + \mathcal{B}_{a,b,0} + a\xi\mathcal{B}_1 + \mathcal{O}\left(|\xi|(a^2+b^2) + |\xi|^{\min(2,\alpha+1)}(a+b)\right). \tag{3.4}$$

In order to determine the three eigenvalues of $\mathcal{B}_{a,b,\xi}$, we consider the characteristic polynomial

$$D(\lambda;a,b,\xi) = \det\left(\mathcal{B}_{a,b,\xi} - \lambda I_{a,b,\xi}\right) = c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,$$

where the coefficient functions $c_j = c_j(a,b,\xi)$, defined for $|(a,b,\xi)| \ll 1$, depend smoothly on the parameters a, b and are C^1 in ξ. Analyzing the dependence of the c_j on a, b, and ξ more carefully, notice that since the spectrum of $\mathcal{M}_{a,b,\xi}$ is symmetric with respect to the imaginary axis, considering the coefficients of D as symmetric functions of its roots, we see that the c_2 and c_0 must be purely imaginary, whereas c_1 and c_3 must be real. Furthermore, since

$$\sigma\left(\mathcal{M}_{a,b,\xi}\right)=\sigma\left(\mathcal{M}_{-a,b,\xi}\right), \quad \sigma\left(\mathcal{M}_{a,b,\xi}\right)=\overline{\sigma\left(\mathcal{M}_{a,b,-\xi}\right)},$$

it follows that the coefficients c_j must all be even in a and that c_2 and c_0 are odd[6] in ξ, while c_1 and c_3 are even in ξ. Also, notice that $c_3(a,b,\xi)=-\det\left(I_{a,b,\xi}\right)$ is nonzero for all $|(a,b,\xi)|\ll 1$. Together with the expansion of $\mathcal{B}_{a,b,\xi}$ above, these properties imply that the polynomial D is a cubic polynomial in the complex variables λ and ξ. It follows that the roots $\lambda=\lambda(a,b,\xi)$ of D can be written as $\lambda=ip\xi X$, where the complex numbers X are determined as the roots of the cubic polynomial

$$Q(X;a,b,\xi)=\det\left(\frac{1}{ip\xi}\mathcal{B}_{a,b,\xi}-XI_{a,b,\xi}\right)=d_3X^3+d_2X^2+d_1X+d_0,$$

where the coefficients d_j are real-valued and even in a and ξ. To determine whether the roots of Q, and hence the three critical eigenvalues, lie on the imaginary axis or not we consider the discriminant

$$\Delta_{a,b,\xi}=18d_3d_2d_1d_0+d_2^2d_1^2-4d_2^3d_0-4d_3d_1^3-27d_3^2d_0^2,$$

which here, by the symmetry properties of the coefficients d_j, can be expanded near for $|(a,b,\xi)|\ll 1$ as

$$\Delta_{a,b,\xi}=\Delta_{0,b,\xi}+\gamma a^2+\mathcal{O}\left((a^2\left(a^2+|b|+\xi^\delta\right)\right) \tag{3.5}$$

for some appropriate $\delta>0$, chosen independently of ξ, and some constant $\gamma\in\mathbb{R}$ still to be determined. In particular, the polynomial Q will have three real roots, corresponding to stability, when $\Delta_{a,b,\xi}>0$, while it will have one real and two complex-conjugate roots, corresponding to instability, when $\Delta_{a,b,\xi}<0$.

Now, using the above formulas for the matrices $\mathcal{B}_{0,b,\xi}$ and $I_{0,b,\xi}$ we can compute the quantity $\Delta_{0,b,\xi}$ explicitly as

$$\Delta_{0,b,\xi}=\frac{|k_{0,b}|^{6\alpha}}{p^6\xi^6}\left(A_1(\xi;\alpha)A_1(-\xi;\alpha)A_2(\xi;\alpha)\right)^2,$$

where, for $|\xi|\ll 1$, we have the expansions

$$\begin{aligned} A_1(\xi;\alpha)&=-1+(1+\xi)^{\alpha+1}-\xi|\xi|^\alpha=(\alpha+1)\xi+\mathcal{O}(|\xi|^{1+\varepsilon}),\\ A_2(\xi;\alpha)&=-2+(1-\xi)^{\alpha+1}+(1+\xi)^{\alpha+1}=\alpha(\alpha+1)\xi^2+\mathcal{O}(|\xi|^{2+\varepsilon})\end{aligned}$$

for some appropriate $\varepsilon>0$ chosen independently of ξ. Thus, for $|\xi|\ll 1$ we have

$$\Delta_{0,b,\xi}=\frac{|k_{0,b}|^{6\alpha}\alpha^2(\alpha+1)^6\xi^2}{p^6}+\mathcal{O}(|\xi|^{2+\varepsilon}),$$

which is clearly positive for ξ sufficiently small. It follows that the sign of the discriminant $\Delta_{a,b,\xi}$ in the asymptotic regime $|(a,b,\xi)|\ll 1$ is determined completely by the sign of the constant γ in (3.5). To determine γ, in turn, it sufficies to compute $\Delta_{a,0,0}$

[6]In fact, since $\lambda=0$ is an eigenvalue of $\mathcal{M}_{a,b,0}$ of algebraic multiplicity at least three it follows that $c_3=\mathcal{O}(|\xi|^3)$.

to order $\mathcal{O}(a^2)$ which, by a direct calculation, is possible by computing the matrix $\xi^{-1}\mathcal{B}_{a,0,\xi}$ to order one in the parameter a, i.e., it is sufficient to determine the matrix $\mathcal{B}_1$ in (3.4). To this end we must expand the Bloch operator $\mathcal{M}_{a,b,\xi}$ in the Bloch frequency ξ for $|\xi| \ll 1$, which is addressed in the following lemma.

LEMMA 3.3. *Let $\alpha > 0$ and $f \in H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ be fixed. Then for all $|\xi| \ll 1$, we have*

$$\begin{aligned} e^{-i\xi x}\partial_x\Lambda^\alpha e^{i\xi x}f(x) = \partial_x\Lambda^\alpha f(x) + i\xi|\xi|^\alpha \hat{f}(0) \\ + \left(\sum_{l=1}^{\infty} i\binom{\alpha+1}{2l-1}\Lambda^{\alpha-2(l-1)}\xi^{2l-1}\right)\mathcal{P}f(x) \\ + \left(\sum_{r=1}^{\infty} \binom{\alpha+1}{2r}\partial_x\Lambda^{\alpha-2r}\xi^{2r}\right)\mathcal{P}f(x), \end{aligned}$$

where $\binom{m}{n} := \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}$ denotes the generalized Binomial coefficient, defined for $m \in \mathbb{C}$ and $n \in \mathbb{N}$, and $\mathcal{P}$ denotes the orthogonal projection of $H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ onto the subspace of mean-zero functions.

Proof. Using the Fourier series representation of f we find

$$e^{-i\xi x}\partial_x\Lambda^\alpha e^{i\xi x}f(x) = \sum_{k\in\mathbb{Z}} i(k+\xi)|k+\xi|^\alpha e^{ikx}\hat{f}(k).$$

Since $|\xi| \ll 1$, we have $|k+\xi| = |k| + \operatorname{sgn}(k)\xi$ so that, for $k \neq 0$, the term $|k+\xi|^\alpha$ may be expanded using Newton's binomial series as

$$|k+\xi|^\alpha = \sum_{m=0}^{\infty}\binom{\alpha}{m}|k|^{\alpha-m}\operatorname{sgn}(k)^m\xi^m,$$

and hence

$$(k+\xi)|k+\xi|^\alpha = |k|^\alpha k + \sum_{m=1}^{\infty}\operatorname{sgn}(k)^{m-1}\left[\binom{\alpha}{m} + \binom{\alpha}{m-1}\right]|k|^{\alpha-(m-1)}\xi^m,$$

valid for $k \neq 0$. If $m = 2l-1$ for some $l \in \mathbb{N}$, then

$$\operatorname{sgn}(k)^{m-1}|k|^{\alpha-(m-1)} = |k|^{\alpha-2(l-1)},$$

while if $m = 2r$ for some $r \in \mathbb{N}$, we have

$$\operatorname{sgn}(k)^{m-1}|k|^{\alpha-(m-1)} = k|k|^{\alpha-2r}.$$

Using Pascal's rule $\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$, valid for all $m \in \mathbb{C}$ and $n \in \mathbb{N}$, we find for $k \neq 0$

$$\begin{aligned} (k+\xi)|k+\xi|^\alpha = k|k|^\alpha + \sum_{l=1}^{\infty}\binom{\alpha+1}{2l-1}|k|^{\alpha-2(l-1)}\xi^{2l-1} \\ + \sum_{r=1}^{\infty}\binom{\alpha+1}{2r}k|k|^{\alpha-2r}\xi^{2r}, \end{aligned}$$

from which the claimed expansion follows. ∎

Remark 3.1. Notice that for a given $\alpha > 0$ the smoothness of the operator $e^{-i\xi x}\partial_x \Lambda^\alpha e^{i\xi x}$ in ξ near $\xi = 0$ is determined by the smoothness of the term $\xi|\xi|^\alpha$ near the origin. In particular, for $\alpha \in 2\mathbb{N}$ one obtains an analytic expansion, while for positive $\alpha \notin 2\mathbb{N}$ one obtains only a $C^{\lfloor \alpha \rfloor + 1}$ expansion. It is expected then that for every $\alpha > 0$ the spectrum of $\mathcal{M}_{a,b,\xi}$ bifurcating from the $(\lambda, \xi) = (0,0)$ will be at least C^1 in a neighborhood of $\xi = 0$.

From Lemma 3.1, we can expand for $|(a,b,\xi)| \ll 1$ the operator $\mathcal{M}_{a,b,\xi}$ to any desired order in ξ. For our purposes, it is sufficient to identify the term of first order in ξ, which is

$$\frac{\partial}{\partial \xi}\mathcal{M}_{a,b,\xi}\Big|_{\xi=0} = i\left((\alpha+1)k_{a,b}^\alpha \Lambda^\alpha + 1 - (p+1)P_{a,b}^p\right). \tag{3.6}$$

Using (3.6), we find that the matrix $\mathcal{B}_1$ in (3.4) is given explicitly by[7]

$$\mathcal{B}_1 = i\begin{pmatrix} 0 & 0 & (\alpha-1)p(p+1) + \frac{p(2+2^\alpha(p-1)}{2(2^\alpha-1)} \\ 0 & 0 & 0 \\ \frac{(\alpha-1)p(p+1)}{2} + \frac{p(2+2^\alpha(p-1)}{4(2^\alpha-1)} & 0 & 0 \end{pmatrix};$$

see [10] for related calculations. Using Mathematica [31], it follows that

$$\Delta_{a,0,0} = \left(\frac{(p+1)\alpha(1+\alpha)^4\left[2^\alpha\left(4-(p-1)(\alpha-1)\right)-4-2(\alpha+p)\right]}{2(2^\alpha-1)}\right)a^2 + \mathcal{O}(a^4).$$

Defining

$$p^*(\alpha) := \frac{2^\alpha(3+\alpha)-4-2\alpha}{2+2^\alpha(\alpha-1)}, \tag{3.7}$$

it follows that for $p > \max(1, p^*(\alpha))$ we have $\Delta_{a,0,0} < 0$ for $|a| \ll 1$, corresponding to instability. In particular, noting that $p^*(\alpha) < 1$ for $\alpha \in (1/2, 1)$, we find instability for all $p \geq 1$ when $\alpha \in (1/2, 1)$. On the other hand, for $\alpha > 1$ and $1 \leq p < p^*(\alpha)$, we find that $\Delta_{a,0,0} > 0$ for $|a| \ll 1$ corresponding to stability for sufficiently small a Together, the above analysis establishes our main result.

THEOREM 3.4. *Let $\alpha > \frac{1}{2}$ and $p \geq 1$ be fixed such that either $p \in \mathbb{N}$ or $p = \frac{m}{n}$, where m and n are even and odd integers, respectively. Then the small amplitude periodic traveling wave solutions $u_{a,b} = P_{a,b}(k_{a,b}\cdot)$ of* (1.1) *constructed in Theorem* A.1 *for $|(a,b)| \ll 1$ are spectrally stable if $\alpha > 1$ and $p < p^*(\alpha)$ and are spectrally unstable if $\alpha \in (1/2, 1)$ or if $\alpha > 1$ and $p > p^*(\alpha)$, where here $p^*(\alpha)$ is defined in* (3.7).

Remark 3.2. As mentioned in the introduction, the spectral instability detected in Theorem 3.4 is of sideband type and is hence not detectable if one restricts to perturbations with the same period as the underlying wave.

Remark 3.3. For the classical Benjamin–Ono equation, corresponding to $\alpha = p = 1$, we find from $p^*(1) = 1$ that $\Delta_{a,0,0} = \mathcal{O}(a^4)$ for $|a| \ll 1$. In this case, the above analysis is insufficient to determine the stability/instability of the small periodic traveling wave solutions, and one must hence carry the above calculations to higher orders. We do not attempt this here but consider this an interesting open question.

[7]Notice that explicit forms of the variations $\partial_\xi \eta_j|_{\xi=0}$ are not needed at this order in the calculation.

As far as the author is aware, Theorem 3.4 is the first rigorous spectral stability/instability result in the periodic context for nonlocal KdV like equations of form (1.1). A plot of $p^*(\alpha)$ for $\alpha \geq 1$ is provided in Figure 1.1. Notice that when $\alpha = 2$, corresponding to the classic gKdV equation (1.2), Theorem 3.4 recovers the result of Haragus and Kapitula in [21], yielding spectral stability for $1 \leq p < 2$ and spectral instability for $p > 2$. As one begins to decrease α from this classical case, the critical power $p^*(\alpha)$ decreases nonlinearly, in contrast to the solitary wave case where the critical power decreases linearly as α is decreased; see [7]. This illustrates an interesting, but not necessarily unexpected, difference between the solitary and periodic theories. It is also interesting to note that $p^*(\alpha) < 2\alpha$ for all $\alpha > \frac{1}{2}$, indicative of the fact that periodic traveling waves are generally *less* stable than their solitary wave counterparts. This seems intuitively clear since the class of periodic traveling wave solutions of (1.1) generally has a richer structure than that for solitary waves. Also, the admissible classes of perturbations for periodic waves is considerably larger: indeed, in the periodic setting one can consider perturbations with twice the fundamental period of the underlying wave, a situation that is not possible in the solitary wave theory, which more closely resembles a co-periodic (i.e., zero-Bloch frequency) stability analysis.

What seems striking about Theorem 3.4 is the fact that the function $p^*(\alpha)$ attains a global maximum of approximately 2.19 on $(1/2, \infty)$ at a critical $\alpha^* \approx 2.7486$ and that for $\alpha > \alpha^*$ the function $p^*(\alpha)$ monotonically decreases with limiting behavior

$$\lim_{\alpha \to \infty} p^*(\alpha) = 1.$$

This suggests that for $p \in (1, p^*(\alpha^*))$, there exists a finite range of α such that the associated model equation (1.1) admits spectrally stable small periodic traveling waves. Indeed, for $p = 1$ this indicates that for all $\alpha > 1$ the small amplitude periodic traveling wave solutions of (1.1), as constructed in Theorem A.1, are spectrally stable to localized perturbations on the line. For $p = 2$, on the other hand, we see that (1.1) admits spectrally stable small periodic traveling wave solutions provided $\alpha \in (2, 4)$, and that for $\alpha \notin [2, 4]$ no such spectrally stable waves exist. Furthermore, for $p > p^*(\alpha^*)$, the model equation (1.1) does not admit spectrally stable small amplitude periodic traveling waves for any $\alpha > 1/2$. It is also important to note that $p^*(1) = 1$, so that our analysis is unable to determine the spectral stability of the small amplitude periodic traveling waves constructed in Theorem A.1 in the case $\alpha = p = 1$, i.e., our analysis is inconclusive regarding the stability of such waves in the classical Benjamin–Ono equation. It does, however, indicate that all such small periodic waves are spectrally unstable in the generalized Benjamin–Ono equation, corresponding to (1.1) with $\alpha = 1$ and $p > 1$.

Finally, recall that the solitary wave solutions of (1.1) are known to be spectrally (and nonlinearly) stable provided $p < 2\alpha$ [7]. Hence, for any $p \geq 1$ it is possible to find a stable solitary wave solution of (1.1) so long as α is sufficiently large. In contrast, the analysis presented in this paper not only provides an upper bound $p^*(\alpha^*)$ on p for which models of the form (1.1) can admit spectrally stable small periodic traveling waves but also provides for each $p \in (1, p^*(\alpha^*))$ lower *and* upper bounds on the dispersion parameter α for such stable waves to exist. It seems possible that this striking difference between the solitary and periodic wave cases is fundamentally due to the nature of dispersion on the line versus "dispersion" on the circle (corresponding to periodic boundary conditions). Indeed, notice that the sideband type instability detected by our analysis can be seen, via the Bloch transform, as a *periodic* instability in the space $L^2(\mathbb{R}/2\pi n\mathbb{Z})$ for some $n \gg 1$. At this time, however, we do not attempt to provide a more detailed or rigorous account for the striking differences between the periodic and solitary wave cases and instead leave this as an interesting open problem.

Appendix A. Existence of small periodic traveling waves. In this appendix, we utilize a Lyapunov–Schmidt reduction to establish the existence and basic regularity properties of small amplitude periodic traveling wave solutions of the nonlocal profile equation (2.2). The method of proof parallels that given for Theorem 1 in [22] in the context of the fifth order Kawahara equation.

THEOREM A.1. *Let $\alpha > \frac{1}{2}$ and $p \geq 1$ be fixed such that either $p \in \mathbb{N}$ or else $p = \frac{m}{n}$, where m and n are even and odd integers, respectively. Then there exist constants $a_0, b_0 \in \mathbb{R}$ such that for any fixed $b \in (-b_0, b_0)$ the nonlocal profile equation (2.2) admits a one-parameter family of even, periodic solutions $\{u_{a,b}\}_{a\in(-a_0,a_0)}$ of the form*

$$u_{a,b}(x) = P_{a,b}(k_{a,b}x),$$

where $P_{a,b}$ is 2π-periodic and smooth in its argument. Moreover, the following properties hold:

(i) *The map $k : (-a_0, a_0) \times (-b_0, b_0) \to \mathbb{R}$ is analytic, even in a, and satisfies*

$$k_{a,b}^{\alpha} = k^*(b) + \tilde{k}(a, b),$$

where $k^*(b) = (p+1)Q_b^p - 1$ and

$$\tilde{k}(a,b) = \sum_{n\geq 1} \tilde{k}_{2n}(b)a^{2n}, \quad \left|\tilde{k}_{2n}(b)\right| \leq \frac{K_0}{\rho_0^{2n}},$$

for any $|(a, b)| \ll 1$ and some positive constants K_0 and $\rho_0 > a_0$.

(ii) *The map $(-a_0, a_0) \times (-b_0, b_0) \ni (a, b) \mapsto P_{a,b} \in H^{\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ is analytic and can be expanded as*

$$P_{a,b}(z) = Q_b + a\cos(z) + \sum_{\substack{n,m\neq 0,\ n+m\geq 2\\ n-m\neq\pm 1}} \tilde{p}_{n,m}(b)e^{i(n-m)z}a^{n+m},$$

where $\tilde{p}_{n,m} \in \mathbb{R}$ are such that $\tilde{p}_{n,m}(b) = \tilde{p}_{m,n}(b)$ with

$$|\tilde{p}_{n,m}(b)| \leq \frac{C_0}{\rho_0^{n+m}}$$

for any $|b| \ll 1$ and some $C_0 > 0$.

(iii) *The Fourier coefficients $\hat{p}_n(a, b)$ of the 2π-periodic function $P_{a,b}$,*

$$P_{a,b}(z) = \sum_{n\in\mathbb{Z}} \hat{p}_n(a,b)e^{inz},$$

are real and satisfy $\hat{p}_0(a, 0) = 1 + \mathcal{O}(a^2)$ and $\hat{p}_n(a, 0) = \mathcal{O}(|a|^n)$ for $n \neq 0$ as $a \to 0$. Moreover, the map $a \mapsto \hat{p}_n(a, b)$ is even (resp., odd) for even (resp., odd) values of n.

Remark A.1. The restriction to $\alpha > \frac{1}{2}$ in Theorem A.1 is made to ensure that the Sobolev space $H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ is embedded in $L^\infty(\mathbb{R}/2\pi\mathbb{Z})$, which ensures smoothness of the map F defined below. Also, the restriction on the form of p is made to ensure that the mapping $\mathbb{R} \ni x \to x^p \in \mathbb{R}$ is well defined, thus ensuring that we don't have to restrict our search a priori to positive solutions of (2.2).

Proof. Renormalizing the period to 2π, let $k \in \mathbb{R}^+$ denote a wave number and set $z = kx$ so that the rescaled profile equation becomes

$$-k^\alpha \Lambda^\alpha u - u + u^{p+1} = b. \tag{A.1}$$

We seek 2π-periodic solutions of (A.1). To this end, first notice that since $\alpha > \frac{1}{2}$ one can show by arguments identical to that in Appendix B in [17] that any $u \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ satisfying (A.1) automatically lies in $H^{2\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$. Iterating now the identity

$$u = [k^\alpha(\Lambda^\alpha + 1)]^{-1} \left(u^{p+1} + (k^\alpha - 1)u - b\right)$$

implies that $u \in H^\infty(\mathbb{R}/2\pi\mathbb{Z})$, so that any 2π-periodic H^α solution of (A.1) is automatically a smooth function of z. Thus, it is sufficient to seek solutions of (A.1) which lie in $H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$.

Set $X^\alpha := H^\alpha(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^+ \times \mathbb{R}$, considered to be equipped with the natural graph norm, and define the map $F : X^\alpha \to L^2(\mathbb{R}/2\pi\mathbb{Z})$ by

$$F(v, k, b) := -k^\alpha \Lambda^\alpha v - v + v^{p+1} - b,$$

noting that F is well defined by Sobolev embedding. Clearly then, the zeros of F correspond to 2π-periodic solutions of (A.1), which can be taken to be even functions of z by applying an appropriate spatial translation. First, we claim that F is C^1 on X^α. Indeed, given any $(v, k, b) \in X^\alpha$ we find that

$$\frac{\partial F}{\partial v} = -k^\alpha \Lambda^\alpha - 1 + (p+1)v^p, \quad \frac{\partial F}{\partial k} = -\alpha k^{\alpha-1}\Lambda^\alpha,$$

and $\frac{\partial F}{\partial b} = -1$. Clearly $\frac{\partial F}{\partial k}$ and $\frac{\partial F}{\partial b}$ depend continuously on $(v, k, b) \in X^\alpha$, and $\frac{\partial F}{\partial v}$ depends continuously on $(k, b) \in \mathbb{R}^+ \times \mathbb{R}$. To see that $\frac{\partial F}{\partial v}$ depends continuously on $v \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$, notice that by Sobolev embedding we have for all $v_1, v_2 \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$

$$\begin{aligned}\left\| v_1^{p+1} - v_2^{p+1} \right\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} &\le C \|v_1 - v_2\|_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} \\ &\quad \times \left(\|v_1\|^p_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} + \|v_2\|^p_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} \right)\end{aligned}$$

for some constant $C > 0$, from which we find

$$\begin{aligned}\left\| \frac{\partial F}{\partial v}(v_1, k, b)f - \frac{\partial F}{\partial v}(v_2, k, b)f \right\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})} &\le C\|v_1 - v_2\|_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} \\ &\quad \times \left(\|v_1\|^p_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} + \|v_2\|^p_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})} \right) \\ &\quad \times \|f\|_{H^\alpha(\mathbb{R}/2\pi\mathbb{Z})}\end{aligned}$$

for all $f \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$, which establishes the continuity on v, as claimed. Together, the above arguments verify that F is C^1 on X^α.

Continuing, by inspection we see that $F(Q_b, k, b) = 0$ for $|b| \ll 1$ and any $k \in \mathbb{R}^+$. Furthermore, for fixed $b \ll 1$ and $k > 0$ we have

$$\frac{\partial F}{\partial v}(Q_b, k, b)) = -k^\alpha \Lambda^\alpha - 1 + (p+1)Q_b^p,$$

which, by Fourier analysis, has a trivial kernel in $L^2(\mathbb{R}/2\pi\mathbb{Z})$ provided $k^\alpha \neq k^*(b)$, where $k^*(b)$ is given in the statement of the theorem. When $k^\alpha \neq k^*(b)$ then, the implicit function theorem implies that the root (Q_b, k, b) of $F(v, k, b)$ continues uniquely for $|k - k_0| \ll 1$ and $|b - b_0| \ll 1$. By inspection, this must correspond to the nearby equilibrium solutions Q_b. To find nonconstant solutions then, we must consider the case $k^\alpha = k^*(b)$, in which case the kernel of the above linear operator is two-dimensional and is spanned by $e^{\pm iz}$. We now construct periodic solutions v to the equation $F(v, k, b) = 0$ by using a Lyapunov–Schmidt reduction for $|b| \ll 1$ and $|k^\alpha - k^*(b)| \ll 1$.

We set $k^\alpha = k^*(b) + \tilde{k}$ and

$$v(z) = Q_b + \frac{1}{2}Ae^{iz} + \frac{1}{2}\bar{A}e^{-iz} + h(z), \tag{A.2}$$

where $A \in \mathbb{C}$ and $h \in H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ satisfies

$$\int_0^{2\pi} h(z)e^{\pm iz}dz = 0.$$

Substituting these expressions into the equation $F(v, k, b) = 0$ leads to an equation of the form

$$L_b h = \mathcal{N}(h, A, \bar{A}, \tilde{k}, b), \tag{A.3}$$

where

$$L_b := \frac{\partial F}{\partial v}(Q_b, k^*(b), b)$$

and $\mathcal{N}(0, 0, 0, \tilde{k}, b) = 0$ for any $\tilde{k}, b \in \mathbb{R}$. Again using Fourier analysis, we see that the kernel of L_b is two-dimensional and is spanned by $e^{\pm iz}$. We denote by $P : L^2(\mathbb{R}/2\pi\mathbb{Z}) \to \ker(L_b)$ the spectral projection onto the kernel of L_b, defined for any $u \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ by

$$Pu(z) = \hat{u}(1)e^{iz} + \hat{u}(-1)e^{-iz}.$$

Since $Ph = 0$ then, the perturbation equation (A.3) is equivalent to the system

$$\begin{cases} L_b h = (\mathrm{I} - P)\mathcal{N}(h, A, \bar{A}, \tilde{k}, b), \\ \quad 0 = P\mathcal{N}(h, A, \bar{A}, \tilde{k}, b), \end{cases} \tag{A.4}$$

which is valid for any $|b| \ll 1$.

Notice the restriction of L_b to $\mathcal{Z} := (\mathrm{Id} - P)H^\alpha(\mathbb{R}/2\pi\mathbb{Z})$ has a bounded inverse given by

$$\left(L_b|_{\mathcal{Z}}^{-1} \right) v = \sum_{n \neq \pm 1} \frac{\hat{v}(n)}{k^*(b)^\alpha (1 - |n|^\alpha)}$$

for any $v \in \mathcal{Z}$. In particular, this formula shows that the operators $\left(L_b\big|_{\mathcal{Z}}\right)^{-1}$ form a family of bounded linear operators depending analytically on b. Therefore, the system (A.4) is equivalent to

$$\begin{cases} h = L_b^{-1}\,(\mathrm{Id} - P)\mathcal{N}(h, A, \bar{A}, \tilde{k}, b), \\ 0 = P\mathcal{N}(h, A, \bar{A}, \tilde{k}, b). \end{cases} \tag{A.5}$$

Using the implicit function theorem [12, Theorem 2.3], we can solve the first equation in (A.5) to find a unique solution $h = H_*(A, \bar{A}, \tilde{k}, b) \in \mathcal{Z}$ that depends analytically on $(A, \bar{A}, \tilde{k}, b)$ in a neighborhood of $(0, 0, 0, b_0)$ in the space $\mathrm{diag}(\mathbb{C}^2) \times \mathbb{R}^2$, where $\mathrm{diag}(\mathbb{C}^2) := \{(z, \bar{z}) : z \in \mathbb{C})$. In particular, notice the uniqueness of this solution implies that

$$H_*(0, 0, \tilde{k}, b) = 0 \tag{A.6}$$

for all $|b| \ll 1$. Furthermore, the invariance of (2.2) with respect to spatial translations $z \mapsto z + z_0$ and the reflection $z \mapsto -z$ implies the relations

$$\begin{cases} H_*(A, \bar{A}, \tilde{k}, b)(z + z_0) = H_*(Ae^{iz_0}, \bar{A}e^{-iz_0}, \tilde{k}, b)(z), \\ \quad H_*(A, \bar{A}, \tilde{k}, b)(-z) = H_*(\bar{A}, A, \tilde{k}, b)(z). \end{cases} \tag{A.7}$$

Substituting the above into $P\mathcal{N}(h, A, \bar{A}, \tilde{k}, b) = 0$ now yields the equation

$$P\mathcal{N}(H_*(A, \bar{A}, \tilde{k}, b)(z), A, \bar{A}, \tilde{k}, b) = 0,$$

which must be solved. Using the explicit form of the projection P, this equation has solutions provided the two orthogonality conditions

$$J_\pm(A, \bar{A}, \tilde{k}, b) = \int_0^{2\pi} \frac{Ae^{iz} \pm \bar{A}e^{-iz}}{2} \mathcal{N}(H_*(A, \bar{A}, \tilde{k}, b)(z), A, \bar{A}, \tilde{k}, b)dz = 0$$

are satisfied. Notice that the relations (A.7) imply that the functions $J_\pm$ satisfy

$$\begin{aligned} J_+(Ae^{iz_0}, \bar{A}e^{-iz_0}, \tilde{k}, b) &= J_+(A, \bar{A}, \tilde{k}, b) = J_+(\bar{A}, A, \tilde{k}, b), \\ J_-(Ae^{iz_0}, \bar{A}e^{-iz_0}, \tilde{k}, b) &= J_-(A, \bar{A}, \tilde{k}, b) = -J_-(\bar{A}, A, \tilde{k}, b). \end{aligned} \tag{A.8}$$

In particular, taking $z_0 = -2\arg(A)$ in the equalities for J_- implies that

$$J_-(\bar{A}, A, \tilde{k}, b) = J_-(A, \bar{A}, \tilde{k}, b) = -J_-(\bar{A}, A, \tilde{k}, b),$$

from which we see that the condition $J_-(A, \bar{A}, \tilde{k}, b) = 0$ is always satisfied.

As for the solvability condition $J_+ = 0$, taking $z_0 = -\arg(A)$ in (A.8) we find $J_+(A, \bar{A}, \tilde{k}, b) = J_+(|A|, |A|, \tilde{k}, b)$ so that the associated solvability condition becomes $J_+(a, a, \tilde{k}, b) = 0$, where $a \in \mathbb{R}$ belongs to a sufficiently small neighborhood of the origin. Noting that (A.6) implies that $a^{-1}H_*(a, a, \tilde{k}, b)$ is analytic in a near $a = 0$, it follows from the explicit form of the function $\mathcal{N}$ that

$$\begin{aligned} J_+(a, a, \tilde{k}, b) &= \int_0^{2\pi} a\cos(z)\mathcal{N}(H_*(A, \bar{A}, \tilde{k}, b)(z), A, \bar{A}, \tilde{k}, b)dz \\ &= a^2\left(\tilde{k} + \widetilde{J}(a, \tilde{k}, b)\right), \end{aligned}$$

where $\widetilde{J}$ is analytic in its argument, even with respect to the parameter a, and satisfies $\widetilde{J}(0,0,b) = \partial_{\tilde{k}}\widetilde{J}(0,0,b) = 0$. By the implicit function theorem [12] again, for $0 < |a| \ll 1$ we obtain a solution $\tilde{k}(a,b)$ of $\widetilde{J}(a,\tilde{k},b) = -\tilde{k}$, and hence of $J_+(a,a,\tilde{k},b) = 0$, defined for sufficiently small $a,b \in \mathbb{R}$. Furthermore, it follows that $\tilde{k}$ is even in a and analytic in a sufficiently small neighborhood of the origin in $\mathbb{R}^2$, so that the function $k(a,b) = \big(k^*(b) + \tilde{k}(a,b)\big)^{1/\alpha}$ satisfies the properties discussed in (i).

From the above considerations, it follows that the system (A.5) has a unique solution

$$(h,\tilde{k}) = \Big(H_*(A,\bar{A},\tilde{k}(|A|,b),b), \tilde{k}(|A|,b)\Big),$$

defined for any sufficiently small $A \in \mathbb{C}$ and $|b| \ll 1$. Substituting $h = H_*(A,\bar{A},\tilde{k}(|A|,b),b)$ into (A.2) thus yields a 2π-periodic solution of (2.2). The periodic solutions $P_{a,b}$ described in the theorem are now found by restricting to $A \in \mathbb{R}$, so that

$$P_{a,b}(z) = Q_b + a\cos(z) + v_{a,b}(z), \qquad v_{a,b}(z) = H_*(a,a,\tilde{k}(a,b),b)(z).$$

The properties of $P_{a,b}$ described in (ii) are now easily deduced from analyticity and the symmetries of the function $H_*(A,\bar{A},\tilde{k}(|A|,b),b)$, while (iii) follows directly from the expansion of $P_{a,b}$ given in (ii). $\square$

Appendix B. Proof of Proposition 3.1. In this appendix, we establish a nonlocal type of Floquet–Bloch theory that is suitable for the stability analysis presented in this paper.

Proof of Proposition 3.1. Clearly (iii) holds for some $\xi \in [-1/2,1/2)$ if and only if the kernel of the operator $\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$ is nontrivial. Furthermore, the operator $\mathcal{M}_{a,b,\xi}$ acting on $L^2(\mathbb{R}/2\pi\mathbb{Z})$ has a densely defined and compactly embedded domain $H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ and hence has a compact resolvent. As a result, the spectrum of $\mathcal{M}_{a,b,\xi}$ consists of isolated eigenvalues of finite multiplicity and, in particular, we see that $\lambda \in \sigma(\mathcal{M}_{a,b,\xi})$ if and only if the operator $\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I}$ has a nontrivial kernel. This establishes that (ii) $\Longleftrightarrow$ (iii).

Now, assume that (ii) does not hold, i.e., that the operator $\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I}$ is boundedly invertible on $L^2(\mathbb{R}/2\pi\mathbb{Z})$ for all $[-1/2,1/2)$. Then there exists a constant $C > 0$ such that

$$\|(\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I})\,v\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \geq C\|v\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})}$$

for all $v \in H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ and $\xi \in [-1/2,1/2)$. Using (3.1) then, we find for all $w \in H^{\alpha+1}(\mathbb{R})$ that

$$\begin{aligned}
\|(\mathcal{M}_{a,b} - \lambda\mathbf{I})\,w\|^2_{L^2(\mathbb{R})} &= 2\pi\int_{-1/2}^{1/2} \|(\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I})\,\check{w}(\xi,\cdot)\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})}\,d\xi \\
&\geq 2\pi C^2 \int_{-1/2}^{1/2} \|\check{w}(\xi,\cdot)\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})}\,d\xi \\
&= C^2\|w\|_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows then that $\mathcal{M}_{a,b} - \lambda\mathbf{I}$ is boundedly invertible as an operator acting on $L^2(\mathbb{R})$. This establishes that (i)$\Rightarrow$(ii).

Finally, assume that (ii) holds, i.e., that for some $\xi_0 \in [-1/2, 1/2)$ the operator $\mathcal{M}_{a,b,\xi_0} - \lambda\mathbf{I}$ is not boundedly invertible on $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Then by the above considerations there exists a $v \in H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ such that $(\mathcal{M}_{a,b,\xi_0} - \lambda\mathbf{I})\, v = 0$. For each $0 < \varepsilon < 1$, let $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi_\varepsilon(\xi) = \begin{cases} \varepsilon^{-1/2} & \text{if } |\xi| < \varepsilon/2, \\ 0 & \text{if } |\xi| \geq \varepsilon/2, \end{cases}$$

and note that $\|\phi_\varepsilon\|_{L^2(\mathbb{R})} = 1$. Given a fixed $\varepsilon \in (0,1)$ then, notice that the function

$$\mathbb{R}^2 \ni (\xi, z) \mapsto v(z)\phi_\varepsilon(\xi - \xi_0) \in \mathbb{C}$$

belongs to $L^2([-1/2, 1/2); L^2(\mathbb{R}/2\pi\mathbb{Z}))$ and can hence be viewed as the Bloch-transform of some function $v_\varepsilon \in L^2(\mathbb{R})$ with

$$\|v_\varepsilon\|^2_{L^2(\mathbb{R})} = \int_{-1/2}^{1/2} \phi_\varepsilon(\xi - \xi_0)^2 \|v\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})} d\xi = \|v\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})}$$

for all $\varepsilon > 0$ sufficiently small. Then for each $0 < \varepsilon \ll 1$, we find using (3.1) that

$$\begin{aligned} \|(\mathcal{M}_{a,b} - \lambda\mathbf{I})\, v_\varepsilon\|^2_{L^2(\mathbb{R})} &= \int_{-1/2}^{1/2} \phi_\varepsilon(\xi - \xi_0)^2 \, \|(\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I})\, v(\cdot)\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \, d\xi \\ &= \frac{1}{\varepsilon} \int_{|\xi - \xi_0| < \varepsilon/2} \|(\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I})\, v(\cdot)\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \, d\xi. \end{aligned}$$

Next, we want to show the above quantity tends to zero as $\varepsilon \to 0^+$.

To this end, notice that Lemma 3.3 in section 3.3 implies that the mapping $\xi \to \mathcal{M}_{a,b,\xi}$ is continuous in the operator norm from $H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ to $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Indeed, for a given $w \in H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})$ and $\xi_1, \xi_2 \in [-1/2, 1/2)$ we have the estimate

$$\|(\mathcal{M}_{a,b,\xi_1} - \mathcal{M}_{a,b,\xi_2})\, w\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \lesssim |\xi_1 - \xi_2| \|w\|_{H^{\alpha+1}(\mathbb{R}/2\pi\mathbb{Z})},$$

yielding the desired continuity. It now follows that

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{|\xi - \xi_0| < \varepsilon/2} & \|(\mathcal{M}_{a,b,\xi} - \lambda\mathbf{I})\, v(\cdot)\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})} \, d\xi \\ &= \|(\mathcal{M}_{a,b,\xi_0} - \lambda\mathbf{I})\, v\|^2_{L^2(\mathbb{R}/2\pi\mathbb{Z})} = 0, \end{aligned}$$

so that, in particular, given any $n \gg 1$ there exists an $\varepsilon_n > 0$ such that

$$\|(\mathcal{M}_{a,b} - \lambda\mathbf{I})\, v_{\varepsilon_n}\|^2_{L^2(\mathbb{R})} < \frac{1}{n}.$$

Recalling that $\|v_\varepsilon\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R}/2\pi\mathbb{Z})}$ for all $0 < \varepsilon \ll 1$, it follows that the operator $\mathcal{M}_{a,b} - \lambda\mathbf{I}$ is not boundedly invertible. This establishes (ii)⇒(i), which completes the proof. ∎

Acknowledgments. The author is grateful to Vera Mikyoung Hur, Atanas Stefonov, Mariana Haragus, and L. Miguel Rodrigues for valuable discussions at various stages of this project. Finally, the author thanks the referees for their careful reading of the manuscript and for their many useful suggestions and references.

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