An index theorem for the stability of periodic traveling waves of KdV type.

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Abstract

We consider the stability of periodic traveling wave solutions to a generalized Korteweg-deVries (KdV) equation, and prove an index theorem relating the number of unstable and potentially unstable eigenvalues to geometric information on the classical mechanics of the traveling wave ordinary differential equation. We illustrate this result with several examples including the integrable KdV and modified KdV equations, the L^2 critical KdV-4 equation that arises in the study of blowup, and the KdV-$\frac{1}{2}$ equation, which is an idealized model for plasmas.

1   Introduction

There has been a large amount of work aimed at understanding the stability of nonlinear dispersive equations that support solitary wave solutions [3, 6, 7, 20, 22, 33, 32, 34, 41, 43]. Much of this work relies on understanding detailed properties of the spectrum of the operator obtained by linearizing the flow around the solitary wave. These spectral properties, in turn, have important implications for the long-time behavior of solutions to the corresponding partial differential equation [5, 10, 13, 14, 17, 18, 27, 29, 30, 31, 37, 38, 39]- see the review paper of Soffer[40] for more details.

In this paper we consider periodic solutions to equations of Korteweg-de Vries type:

$$u_t + u_{xxx} + (f(u))_x = 0 \tag{1}$$

where f is assumed to be C^2. While the stability theory for periodic waves has received much recent attention [1, 2, 8, 9, 12, 15, 16, 24] the theory is much less developed than the analogous theory for solitary wave stability, and appears to be mathematically richer.

The main result of this paper is an index theorem giving an exact count of the number of unstable and potentially unstable eigenvalues of the linearized operator in terms of the number of zeros of the derivative of the traveling wave profile together with geometric information about a certain map between the constants of integration of the ordinary differential equation and the conserved quantities of the partial differential equation. This map encodes information about the kernel and generalized kernel of the linearized operator as well as some related self-adjoint operators, allowing us to establish the main result. This map is also closely connected with the classical mechanics of

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the underlying traveling wave ordinary differential equation, providing a link between PDE stability and ODE dynamics.

This index can be regarded as a generalization of both the Sturm oscillation theorem and the classical stability theory for solitary wave solutions for equations of Korteweg-de Vries type. In the case of a polynomial nonlinearity this index, together with a related one introduced earlier by Bronski and Johnson, can be expressed in terms of derivatives of period integrals on a Riemann surface. Since these period integrals satisfy a Picard-Fuchs equation these derivatives can be expressed in terms of the integrals themselves, leading to an expression in terms of various moments of the solution. We conclude with some illustrative examples.

Note: We will frequently consider the case where $f(u)$ is a power. We use the standard notation that KdV-p is

$$u_t + u_{xxx} \pm (u^{p+1})_x = 0. \tag{2}$$

For p odd the plus and minus signs are equivalent via $u \mapsto -u$. For p even the two signs are not equivalent. In the examples we will usually consider the focusing case (plus sign) since it tends to be the more interesting case, although the theory applies equally to either case.

2   Basic Results

We begin by writing down the periodic traveling waves to the gKdV equation and introducing some important notation. Assuming a traveling wave of the form $u(x,t) = u(x - ct)$ one is immediately led to the following nonlinear oscillator equation

$$\frac{u_x^2}{2} = E - V(u; a, c), \quad V(u; a, c) := -au - c\frac{u^2}{2} + F(u), \tag{3}$$

where F is the antiderivative of the nonlinearity f. As any traveling wave profile must satisfy (3), we refer to this as the traveling wave ODE corresponding to the gKdV equation (1). It follows the traveling wave profile u satisfies a Hamiltonian ODE with effective potential energy $V(u; a, c)$. Thus the periodic waves depend on three parameters E, a, and c together with a fourth constant of integration x_0 corresponding to spatial translations which can be quotiented out. Thus when we speak of a three parameter family of solutions we will be referring to a, E, c.

In many of the interesting cases the nonlinearity $f(u)$ is polynomial. In this case the zero set of the discriminant of the effective potential energy

$$\Gamma = \{(E, a, c) | \operatorname{disc}(E + au + c\frac{u^2}{2} - F(u)) = 0\}$$

gives a variety dividing the parameter space into open sets having a constant number of periodic solutions. The variety itself represents parameter values where the equation admits some combination of solitary wave solutions, constant solutions and periodic solutions. In particular, the origin $(a, E) = (0, 0)$ represents the solitary wave homoclinic to zero - the main case studied in the solitary wave papers cited above. In all of the examples worked out in this paper the wavespeed c can be scaled to $-1, 0, 1$, so the parameter space can be taken to be $\mathbb{R}^2$.

In order to ensure the existence of periodic orbits of (3) we assume that we are off of the discriminant Γ so that there exist simple roots of the equation $E = V(u; a, c)$, and that there are real roots $u_\pm$ satisfying $u_- < u_+$, and that $V(u; a, c) < E$ for $u \in (u_-, u_+)$ (see Figure 1). As a consequence, the roots $u_\pm$ are smooth functions of the traveling wave parameters (a, E, c). We
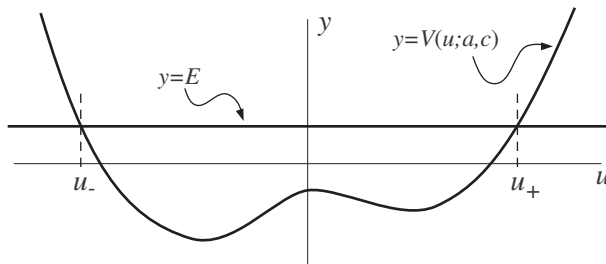
Figure 1: A pictorial representation of the effective potential energy $V(u; a, c)$ along with total energy E.

also break the translation invariance by assuming that $u(0) = u_-$. It follows that the period of the corresponding periodic solution of (3) can be expressed by the formula

$$T(a, E, c) = \sqrt{2} \int_{u_-}^{u_+} \frac{\mathrm{d}u}{\sqrt{E - V(u; a, c)}} = \frac{\sqrt{2}}{2} \oint_\gamma \frac{\mathrm{d}u}{\sqrt{E - V(u; a, c)}}, \tag{4}$$

where integration over γ represents a integration over an appropriate cycle in the complex plane.

In general, the gKdV equation (1) admits three conserved quantities which physically can be interpreted as a Hamiltonian energy, mass, and momentum. Given a T-periodic periodic traveling wave solution of (1), these quantities are defined by

$$H(a, E, c) = \int_0^T \left(\frac{u_x^2}{2} - F(u) \right) \mathrm{d}x = \frac{\sqrt{2}}{2} \oint_\gamma \frac{E - V(u; a, c) - F(u)}{\sqrt{E - V(u; a, c)}} \, \mathrm{d}u$$

$$M(a, E, c) = \int_0^T u(x) \, \mathrm{d}x = \frac{\sqrt{2}}{2} \oint_\gamma \frac{u \, \mathrm{d}u}{\sqrt{E - V(u; a, c)}} \tag{5}$$

$$P(a, E, c) = \int_0^T |u(x)|^2 \, \mathrm{d}x = \frac{\sqrt{2}}{2} \oint_\gamma \frac{u^2 \, \mathrm{d}u}{\sqrt{E - V(u; a, c)}} \tag{6}$$

respectively, where the integral over γ is defined as in (4). As above, it follows that each of these integrals can be regularized at their square root branch points and hence represent C^1 functions of the traveling wave parameters. As we will see, these quantities and their gradients will play a major role in our stability analysis.

In order to help with computations involving the gradients of the above conserved quantities, we also note the following useful identity. The classical action (in the sense of action-angle variables) for the traveling wave ordinary differential equation (3) is

$$K = \oint p \, \mathrm{d}q = \sqrt{2} \oint \sqrt{E - V(u; a, c)} \, \mathrm{d}u.$$

The classical action provides a generating function for the conserved quantities of the KdV equation evaluated on the traveling waves: specifically the classical action satisfies the relationships

$$T = \frac{\partial K}{\partial E}, \quad M = \frac{\partial K}{\partial a}, \quad P = 2\frac{\partial K}{\partial c}.$$

as well as

$$K = H + aM + \frac{c}{2}P + ET.$$

These relationships together imply the identity

$$E\nabla T + a\nabla M + \frac{c}{2}\nabla P + \nabla H = 0 \tag{7}$$

where $\nabla = \left(\frac{\partial}{\partial E}, \frac{\partial}{\partial a}, \frac{\partial}{\partial c}\right)$. It follows that so long as $E \neq 0$, gradients in the period can be expressed simply in terms of the gradients of the conserved quantities M, P, and H. Thus, while results in this paper will be stated in terms of the quantities T, M, and P, as these arise most naturally in our analysis, it is (generically) possible to re-express them completely in terms of the conserved quantities of the gKdV flow. Such an interpretation seems more natural from a physical point of view.

It is worth noting that the constants a and c admit a variational interpretation: using the above definitions we see the gKdV equation (1) can be written in a standard Hamiltonian form as

$$u_t = \frac{\partial}{\partial x}\frac{\delta H(u)}{\delta u}.$$

In this formulation, the traveling waves *of a fixed period* are realized as critical points of the augmented Hamiltonian functional $H + aM + cP/2$, i.e.

$$\frac{\delta}{\delta u}(H + aM + cP/2) = 0,$$

and thus represent critical points (in an appropriate space) of the Hamiltonian under the constraint of fixed mass and momentum, with the parameters a and c representing Lagrange multipliers enforcing the constraints of fixed mass and momentum. Variational techniques have been used extensively by many authors and form the backbone of much of the nonlinear stability analysis for solitary waves of nonlinear dispersive equations: in general, one finds conditions which guarantees the traveling wave profile is a constrained minimizer of the Hamiltonian.

In this paper, we are interested in both the spectral and orbital[1] (nonlinear) stability of spatially periodic traveling wave solutions of (1). The spectral stability problems has been recently considered [8, 11, 24] in which the authors considered stability to localized perturbations. In this case, given a T periodic solution u of (1) the corresponding linearized eigenvalue problem takes the form

$$\partial_x \mathcal{L} v = \mu v, \tag{8}$$

where $\mathcal{L} = -\partial_x^2 - f'(u) + c$ is a differential operator with periodic coefficients considered on the real Hilbert space $L^2(\mathbb{R})$, the skew-symmetric operator ∂_x gives the Hamiltonian structure and μ is the eigenvalue parameter. This is the standard form for the stability problem for solutions to equations with a Hamiltonian structure, although it must be emphasized that in the KdV case ∂_x has a non-trivial kernel (spanned by 1) which complicates matters somewhat. In order to consider the nonlinear stability of such solutions, however, the variational formulation outlined above requires us to restrict to a particular class of perturbations. Indeed, in order to make sense of variational computations involving integration by parts, we must consider periodic perturbations whose period is a multiple of that of the underlying wave, i.e. we must consider perturbations in the space $L^2_{\mathrm{per}}(\mathbb{T}_k)$ where $\mathbb{T}_k := \mathbb{R}/(kT\mathbb{Z})$ for some $k \in \mathbb{N}$. Thus, all operators considered throughout this paper will be considered on the Hilbert space of square integrable kT periodic functions for some $k \in \mathbb{N}$. Moreover, as noted in [12], the conservative form of (1) implies all nontrivial temporal

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[1]Due to the translation invariance of the gKdV equation (1), the best we can hope for is for nonlinear stability up to translation, i.e. orbital stability. See [1, 2, 6, 7, 15, 26] for more details.

evolution of a given solution occurs in the space of mean zero functions. Following the notation of Deconinck and Kapitula we denote this space as H_1:

$$H_1 = \{\phi \in L^2(\mathbb{T}_k) \mid \langle 1, \phi \rangle = 0\}$$

where $\langle f, g \rangle := \int_{\mathbb{T}_k} f(x)g(x)\,\mathrm{d}x$. Note that $H_1 = \ker(\partial_x)^\perp$.

In order to characterize the spectrum of the linearized operator $\partial_x \mathcal{L}$ we will consider several geometric quantities arising as Jacobians of maps from the parameter space (E, a, c), which we have chosen to parameterize the family of traveling wave solutions of (1), to the quantities (T, M, P) described above. For notational simplicity then, we introduce the following Poisson bracket style notation for two-by-two Jacobian determinants

$$\{F, G\}_{x,y} = \left| \begin{array}{cc} F_x & F_y \\ G_x & G_y \end{array} \right|$$

with the analogous notation for three-by-three Jacobian determinants:

$$\{F, G, H\}_{x,y,z} = \left| \begin{array}{ccc} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{array} \right|.$$

Finally, in order to count the number of unstable and "potentially" unstable eigenvalues of the linearized operator, we define the following eigenvalue counts.

Definition 1. *Given the linearized operator $\partial_x \mathcal{L}$ acting on $L^2_{\mathrm{per}}(\mathbb{T}_k)$ we define the Krein signature for purely imaginary eigenvalues as follows: if the eigenvalue $i\mu$ is algebraically simple with eigenfunction w then the Krein signature of $i\mu$ is given by the sign of $\langle w, \mathcal{L}w \rangle$. In the case where the eigenspace S is higher dimensional the number of eigenvalues of negative Krein signature is the number of negative eigenvalues of $\mathcal{L}|_S$.*

The Krein signature is an important geometric quantity associated with eigenvalue problems having a Hamiltonian structure, and is associated with the sense of transversality of the root of the eigenvalue relation. It is a fundamental result that if two eigenvalues of like Krein signature collide they will remain on the axis, while if two eigenvalues of opposite Krein signature collide they will (generically) leave the imaginary axis. See the text of Yakubovich and Starzhinskii [44], in particular, Ch 3 of Volume I, for more information.

Definition 2. *Given the linearized operator $\partial_x \mathcal{L}$ acting on $L^2_{\mathrm{per}}(\mathbb{T}_k)$ we define $k_{\mathbb{R}+}$ to be the number of eigenvalues of $\partial_x \mathcal{L}$ on the positive real axis, $k_{\mathbb{C}}$ to be the number of eigenvalues in the open first quadrant, and $k_{\mathbb{I}}^-$ to be the number of purely imaginary eigenvalues in the upper half-plane with negative Krein signature.*

It is worth making a few remarks on these definition. Firstly, notice that in the case in which the underlying periodic wave is spectrally stable in $L^2(\mathbb{T}_k)$ the quantities $k_{\mathbb{R}}$ and $k_{\mathbb{C}}$ vanish. We will show that there are only a finite number of imaginary eigenvalues of negative Krein signature - thus most eigenvalues of the problem have positive Krein signature. Thus $k_{\mathbb{I}}^-$ counts the number of "potential instabilities": the number of imaginary eigenvalues which could (under perturbation) leave the imaginary axis to become instabilities.

Given this background we now state the main result of this paper.

Theorem 1. *Suppose u is a periodic solution traveling wave solution to the generalized KdV equation (3). Let K be the classical action of this solution considered as a function of the parameters (a, E, c) defined in (2) and assume that the principal minors of the Hessian matrix of K are nonzero:*

$$K_{EE} = T_E \neq 0$$

$$\begin{vmatrix} K_{EE} & K_{aE} \\ K_{aE} & K_{aa} \end{vmatrix} = \{T, M\}_{Ea} \neq 0$$

$$\begin{vmatrix} K_{EE} & K_{aE} & K_{cE} \\ K_{aE} & K_{aa} & K_{ac} \\ K_{cE} & K_{ac} & K_{cc} \end{vmatrix} = \{T, M, P\}_{Eac} \neq 0$$

Moreover, let $k \in \mathbb{N}$ be fixed, let $p(\partial^2 K)$ denote the number of positive eigenvalues of $\partial^2 K$, the Hessian matrix of K with respect to (E, a, c), and let $k_\mathbb{R}, k_\mathbb{C}, k_\mathbb{I}^-$ denote the number of real eigenvalues, complex eigenvalues, and imaginary eigenvalues of negative Krein signature of the operator $\partial_x \mathcal{L}$ acting on $L^2(\mathbb{T}_k)$ Then the following equality holds:

$$k_\mathbb{R} + 2k_\mathbb{I}^- + 2k_\mathbb{C} = 2k - p(\partial^2 K). \tag{9}$$

Remark 1. *The vanishing of the right-hand side of the above equation is a sufficient condition for orbital stability, but this case obviously only occur for $k = 1$, corresponding to perturbations of the same period.*

Remark 2. *Evaluating the above result modulo 2 gives the following formula:*

$$k_{\mathbb{R}+} \equiv 0 \mod 2 \qquad \det(\partial^2 K) < 0$$
$$k_{\mathbb{R}+} \equiv 1 \mod 2 \qquad \det(\partial^2 K) > 0$$

This shows that positivity of the Hessian determinant is a sufficient condition for instability. It is this modulo two count that underlies the stability theory of the gKdV solitary wave: one consequence of a well-known result of Weinstein[42] is that in the solitary wave case one has

$$k_{\mathbb{R}+} \equiv 0 \mod 2. \qquad \frac{\partial P}{\partial c} > 0$$
$$k_{\mathbb{R}+} \equiv 1 \mod 2. \qquad \frac{\partial P}{\partial c} < 0$$

which can be recovered from the above by taking the long period limit. In the solitary wave case the creation of a pair of real eigenvalues (one positive, one negative) is essentially the only way in which instability can occur: the spectrum either consists of essential spectrum along the imaginary axis, or it consists of essential spectrum along the imaginary axis together with a pair of real eigenvalues placed symmetrically on the positive and negative real axes. In the solitary wave case positivity of $\frac{\partial P}{\partial c}$ was shown (in the aforementioned paper of Weinstein) to be a necessary and sufficient condition for stability. In the periodic problem, on the other hand, one can have bands of essential spectrum off of the imaginary axis with no real eigenvalues.

Remark 3. *In general the stability theory to periodic perturbations $(k = 1)$ is closely analogous to the solitary wave stability theory, while for $k \geq 2$ new phenomena occur. Note that in the case $k = 1$ the count on the right can equal zero, implying (spectral) stability, while this cannot occur for $k \geq 2$. The above count does not distinguish between complex eigenvalues (which lead to instability) and imaginary eigenvalues of negative Krein signature (which do not). Later we will introduce a second index which does distiguish between these cases, at least in a neighborhood of the origin.*

Remark 4. *Finally we note that the above result shows that the Hessian of the classical action*

$$K = \frac{\sqrt{2}}{2} \oint \sqrt{E - V(u; E, a, c)} \, du$$

cannot be positive definite. We are unaware of any independent way to prove this but it is supported by numerical experiments.

In the case where the nonlinearity F is polynomial the integrals (4), (5), and (6) are Abelian integrals on a Riemann surface and the above expressions can be greatly simplified. For instance for the case of the Korteweg-de Vries equation the quantities T_E, $\{T, M\}_{a,E}$, $\{T, M, P\}_{a,E,c}$ are homogeneous polynomials of degrees one, two and three respectively in T, M, while for the modified Korteweg-de Vries equation they are homogeneous polynomials in T and P. In general for a polynomial nonlinearity they are homogeneous polynomials of degree one, two and three in some finite number of moments of the solution

$$\mu_k := \oint \frac{u^k \, du}{\sqrt{E - V(u; a, c)}} = \int_0^T u^k(x) \, dx.$$

Thus, in the polynomial nonlinearity case Theorem 1 yields sufficient information for the stability of a periodic traveling wave solution in terms of a finite number of moments of the solution itself.

3 Proof of Main Results

The study of eigenvalues of operators of the form (8) has a long history (see [24] and the references therein for the most recent exposition). The basic observation is that, if $\mathcal{L}$ were positive definite the spectrum of $\partial_x \mathcal{L}$ would necessarily be purely imaginary, since this operator is skew-adjoint under the modified inner product $\langle\langle u, v \rangle\rangle = \langle \mathcal{L}^{1/2} u, \mathcal{L}^{1/2} v \rangle$. However, in the case of nonlinear dispersive waves $\mathcal{L}$ is never positive definite due to the presence of symmetries; consequently, it may be the case that there is spectra with nonzero real part. It turns out to be the case that one can count the number of possible eigenvalues off of the imaginary axis in terms of the dimensions of the kernel and the negative definite subspace of $\mathcal{L}$.

In the case of periodic solutions to the Korteweg-de Vries equation the best results of this type that we are aware of are due to Hărăguş and Kapitula [24] and Deconinck and Kapitula [12]. In particular, Kapitula and Deconinck give the following construction: consider the spectral problem (8) acting on the real Hilbert space $L^2(\mathbb{T}_k)$, and let $k_{\mathbb{R}}$, $k_{\mathbb{C}}$, and $k_{\mathbb{I}}^-$ be defined as before. Let $P : L^2(\mathbb{T}_k) \mapsto H_1$ represent the orthogonal projection, and define the operator $\mathcal{L}|_{H_1} : L^2(\mathbb{T}_k) \mapsto H_1$ by $\mathcal{L}|_{H_1} := P\mathcal{L}P$. Then one has the count

$$k_{\mathbb{I}}^- + k_{\mathbb{R}} + k_{\mathbb{C}} = \underbrace{n(\mathcal{L}) - n(\langle 1, \mathcal{L}^{-1}(1) \rangle)}_{n(\mathcal{L}|_{H_1})} - n(D). \tag{10}$$

where $n(\cdot)$ denotes the dimension of the negative definite subspace of the appropriate operator acting on $L^2(\mathbb{T}_k)$, and D is a symmetric matrix whose entries are given by

$$D_{i,j} = \langle y_i, \mathcal{L}|_{H_1} y_j \rangle$$

where $\{y_i\}$ is a basis for the generalized eigenspace of $\partial_x \mathcal{L}\big|_{H_1}$ in H_1 such that

$$\partial_x \mathcal{L}|_{H_1} \text{ span}\{y_i\} = \ker(\mathcal{L}).$$

The importance of this formula is the following: By using the results of [22], it is known that a sufficient condition for the orbital stability of a periodic traveling wave solution of (1) is given by $n(\mathcal{L}|_{H_1}) = n(D)$. However, in general very difficult to compute the quantity $n(\mathcal{L}|_{H_1})$. For example, in [12] it was necessary to either have complete knowledge of the eigenvalues and corresponding eigenfunctions of the operator $\mathcal{L}$ on $L^2(\mathbb{T}_k)$, or one could only look at the case of waves with small amplitude. While the complete knowledge of the spectra is certainly possible in special integrable cases, such a computational technique seems impractical in general. However, through the equality in (10) we see that if one is able to prove that

$$k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 0,$$

one can immediately conclude orbital stability in $L^2(\mathbb{T}_k)$. It follows that spectral stability can be upgraded to the orbital stability if there are no purely imaginary eigenvalues of negative Krein signature. However, also notice that Theorem 1 can only provide a positive nonlinear stability result in the case of $k = 1$, i.e. stability to co-periodic perturbations (as considered in [26]). Moreover, as noted in the introduction the count clearly gives information concerning the spectral stability of the underlying periodic wave. In particular, a necessary condition for the spectral stability of such a solution in $L^2(\mathbb{T}_k)$ is for the difference $n(\mathcal{L}|_{H_1}) - n(D)$ to be even. The main goal of this paper is to provide an alternative description of the quantity in the left hand side of (10) which is possibly more computable.

In another paper, Bronski and Johnson [11] considered the analogous spectral stability problem to both co-periodic and localized perturbations from the view point of Whitham Modulation theory. When considering stability to co-periodic perturbations, it was proven using Evans function techniques that one has spectral instability if

$$\{T, M, P\}_{E,a,c} = \begin{vmatrix} K_{EE} & K_{Ea} & K_{Ec} \\ K_{Ea} & K_{aa} & K_{ac} \\ K_{Ec} & K_{ac} & K_{cc} \end{vmatrix} > 0$$

and spectral stability if $\{T, M, P\}_{E,a,c} < 0$ and $T_E = K_{EE} > 0$. Thus, it is no surprise that these quantities arise as key ingredients in Theorem 1. To see that the quantity $\{T, M\}_{a,E}$ must also play a roll, see comments below concerning the work on Johnson [26]. Concerning stability to localized perturbations, Bronski and Johnson gave a normal form calculation for the spectral problem in a neighborhood of the origin in the spectral plane, which amounts to studying the spectral stability of a periodic traveling wave solution of (1) to long-wavelength perturbations: so called modulational instability. It was found that the presence of such an instability could be detected by computing various Jacobians of maps from the conserved quantities of the gKdV flow to the parameter space (a, E, c) used to parameterize the periodic traveling waves. By deriving an asymptotic expansion of the periodic Evans function

$$D(\mu, e^{i\kappa}) = \det\left(\mathbf{M}(\mu) - e^{i\kappa}\mathbf{I}\right)$$

in a neighborhood of $(\mu, \kappa) = (0, 0)$, where $\mathbf{M}(\mu)$ is the monodromy matrix associated with third order ODE (8), $\mathbf{I}$ is the three-by-three identity matrix, and κ is the Floquet exponent, it was found that the structure of the spectrum of the operator $\partial_x \mathcal{L}$ in a neighborhood of the origin is determined by the modulational instability index

$$\Delta_{\mathrm{MI}} = \frac{1}{2}\left(\{T, P\}_{E,c} - 2\{M, P\}_{E,a}\right)^3 - 3\left(\frac{3}{2}\{T, M, P\}_{E,a,c}\right)^2. \tag{11}$$

In particular, it was found that if $\Delta_{\mathrm{MI}} > 0$ then the spectrum locally consists of a symmetric interval on the imaginary axis with multiplicity three (modulational stability), while if $\Delta_{\mathrm{MI}} < 0$ the spectrum locally consists of a symmetric interval of the imaginary axis with multiplicity one, along with two branches which, to leading order, bifurcate from the origin along straight lines with non-zero slope (modulational instability). Obviously modulational instability implies spectral instability in $L^2(\mathbb{T}_k)$ for $k \in \mathbb{N}$ sufficiently large - instability to long wavelength perturbations. Thus an understanding of both the index Δ_{MI} as well as the count $k_{\mathbb{I}}^- + k_{\mathbb{R}} + k_{\mathbb{C}}$ for a general $k \in \mathbb{N}$ yields a substantial amount of information concerning the stability of the underlying T-periodic wave.

A similar geometric construction was later found useful by Johnson [26] to prove orbitally stable in $L^2(\mathbb{T}_1)$, i.e. orbitally stable to co-periodic perturbations, provided that $T_E > 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} < 0$. In [26] the condition $\{T, M\}_{E,a} < 0$ was necessary for the proof of nonlinear stability: Theorem 1 implies in this case one still has nonlinear stability and removes this condition.

Results of this kind require a detailed understanding of the structure of the kernel and generalized kernels of the linear operators $\mathcal{L}, \mathcal{L}|_{H_1}, \partial_x\mathcal{L}$ and $\mathcal{L}\partial_x$ ascting on $\mathbb{T}_k$ - see, for example, the work of Gang and Weinstein[19]. A basic observation is that, because the underlying traveling wave ordinary differential equation is integrable one can explicitly generate the tangent space by computing the variations with respect to the integration parameters (E, a, c, x_0), and thus generate the kernels and generalized kernels of the relevant operators. This is the content of the next proposition. Since the operators under consideration are non-self-adjoint and the null-spaces typically have a non-trivial Jordan structure we will adopt the following notation: Given an operator $\mathcal{A}$ acting on $L^2(\mathbb{T}_k)$ for some $k \in \mathbb{N}$, we define the kth generalized kernel as

$$\mathrm{g\text{--}ker}_k(\mathcal{A}) = \ker(\mathcal{A}^{k+1})/\ker(\mathcal{A}^k).$$

Thus $\mathrm{g\text{--}ker}_0(\mathcal{A}) = \ker(\mathcal{A})$ is the usual kernel and $\mathcal{A} : \mathrm{g\text{--}ker}_{j+1}(\mathcal{A})) \to \mathrm{g\text{--}ker}_j(\mathcal{A})$. With this in mind, we begin by stating a preliminary lemma regarding the Jordan structure of the kernel of the linearized operators acting on $L^2(\mathbb{T}_k)$.

Proposition 1. *Given any* $k \in \mathbb{N}$, *one* generically *has* $\dim(\ker(\mathcal{L})) = 1$, $\dim(\ker(\partial_x\mathcal{L})) = 2$, $\dim(\mathrm{g\text{--}ker}_1(\partial_x\mathcal{L})) = 1$, *and* $\mathrm{g\text{--}ker}_j(\partial_x\mathcal{L}) = \emptyset$ *for* $j \geq 2$. *In particular, we have the following genericity conditions:*

- *If* $T_E \neq 0$ *then* $\ker(\mathcal{L}) = \mathrm{span}\{u_x\}$. *If* $T_E = 0$ *then* $\ker(\mathcal{L}) = \mathrm{span}\{u_x, u_E\}$.

- *If* T_E *and* T_a *do not simultaneously vanish then*

$$\ker(\partial_x\mathcal{L}) = \mathrm{span}\left\{u_x, \begin{vmatrix} u_E & T_E \\ u_a & T_a \end{vmatrix}\right\} \tag{12}$$

$$\ker(\mathcal{L}\partial_x) = \mathrm{span}\{1, u\} \tag{13}$$

- *If* T_E *and* T_a *simultaneously vanish then*

$$\ker(\partial_x\mathcal{L}) = \mathrm{span}\{u_x, u_a, u_E\} \tag{14}$$

$$\ker(\mathcal{L}\partial_x) = \mathrm{span}\left\{1, u, \int_0^x u_E \, dx\right\}. \tag{15}$$

> *Since the defining ordinary differential equation is third order the kernel cannot be more than three dimensional.*

- If $\{T, M\}_{E,a} \neq 0$ then

$$\ker(\mathcal{L}|_{H_1}) = \operatorname{span}\{u_x\}$$

- If $\{T, M\}_{E,a} \neq 0$ then

$$\text{g--ker}_1(\partial_x \mathcal{L}) = \operatorname{span}\left\{ \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix} \right\} \tag{16}$$

$$\text{g--ker}_1(\mathcal{L}\partial_x) = \operatorname{span}\left\{ \int_0^x \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix} \right\} \tag{17}$$

  thus the generalized kernel can be chosen such that $\text{g--ker}_1(\partial_x \mathcal{L}) \subset H_1$.

- The generalized kernels are one dimensional unless $\{T, M\}_{E,a}$ and $\{T, P\}_{E,a}$ vanish simultaneously.

- Assuming $\{T, M\}_{E,a} \neq 0$ the subsequent generalized kernels $\text{g--ker}_k(\partial_x \mathcal{L})$ and $\text{g--ker}_k(\mathcal{L}\partial_x)$ for $k \geq 2$ are empty as long as $\{T, M, P\}_{E,a,c} \neq 0$

Proof. This follows from the observation that the derivatives of the wave profile u with respect to the parameters a, E, c satisfy the following equations

$$\mathcal{L}u_x = 0, \quad \mathcal{L}u_E = 0, \quad \mathcal{L}u_a = -1 \left(= -\frac{\delta M}{\delta u} \right), \quad \mathcal{L}u_c = -u \left(= -\frac{\delta P}{\delta u} \right),$$

reflecting the fact that the constants (a, c) arise as Lagrange multipliers to enforce the mass and momentum constraints. In the above equality $\mathcal{L}$ denotes the formal operator without consideration for boundary conditions. In order to find elements of the kernel one must impose periodic boundary conditions. It is not hard to see that u_x is periodic while derivatives with respect to the quantities are not periodic. Since the period T depends on (E, a, c) "secular" terms (in the sense of multiple scale perturbation theory) arise; in particular, one sees that the change across a period is proportional to derivatives of the period:

$$\begin{pmatrix} u_E(T) \\ u_{xE}(T) \\ u_{xxE}(T) \\ \vdots \end{pmatrix} - \begin{pmatrix} u_E(0) \\ u_{xE}(0) \\ u_{xxE}(0) \\ \vdots \end{pmatrix} = T_E \begin{pmatrix} u_{xE}(0) \\ u_{xxE}(0) \\ u_{xxxE}(0) \\ \vdots \end{pmatrix}$$

with similar expressions for the change in the u_a, u_c across a period. Thus the quantity

$$\phi_1(x; a, c, e) = \begin{vmatrix} u_E & T_E \\ u_a & T_a \end{vmatrix}$$

is periodic and satisfies $\mathcal{L}\phi_1 = T_E$. Similarly the quantity

$$\phi_2(x; E, a, c) = \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix}$$

is by construction periodic and satisfies $\mathcal{L}\phi_2 = \{T, M\}_{E,c} - \{T, M\}_{E,a}u$, and thus $\partial_x \mathcal{L}\phi_2 = \{T, M\}_{E,a}u_x \in \ker(\partial_x \mathcal{L})$. Note that while ϕ_1 is essentially uniquely determined ϕ_2 is only determined up to an element of the kernel. Here we have chosen to make ϕ_2 have mean zero since

this is the convention required in the work of Deconinck and Kapitula. More will be said on this choice later.

The rest of the calculation follows in straightforward way from calculations of this sort. For instance the existence of a second element of the generalized kernel is equivalent to the solvability of

$$\partial_x \mathcal{L} = \begin{vmatrix} u_E & T_E \\ u_a & T_a \end{vmatrix}.$$

By the Fredholm alternative $\operatorname{ran}(\partial_x \mathcal{L}) = \ker(\mathcal{L}\partial_x)^\perp$ and thus the above is solvable if only if

$$\langle 1, \begin{vmatrix} u_E & T_E \\ u_a & T_a \end{vmatrix} \rangle = \{T, M\}_{a,E} = 0, \quad \langle u, \begin{vmatrix} u_E & T_E \\ u_a & T_a \end{vmatrix} \rangle = \{T, P\}_{a,E} = 0$$

The rest of the claims follow similarly. In the case that the genericity conditions do not hold we do not attempt to compute the Jordan form, but we do remark that the algebraic multiplicity of the zero eigenvalue must jump from three to at least five, and is necessarily odd. □

In essence the above proposition shows that the elements of the kernel of $\partial_x \mathcal{L}$ are given by elements of the tangent space to the (two-dimensional) manifold of solutions *of fixed period at fixed wavespeed*, while the element of the first generalized kernel is given by a vector in the tangent space to the (three-dimensional) manifold of solutions *of fixed period* with no restrictions on wavespeed. As one might expect all of the geometric information on independence in the above proposition can be expressed in terms of various Jacobians. The next fact we note is that the signs of certain of these quantities conveys geometric information about the various operators.

Lemma 1. *Let $n(\mathcal{L})$ be the dimension of the negative definite subspace of $\mathcal{L}$ as an operator on $L^2(\mathbb{T}_k)$ with periodic boundary conditions. Then*

$$n(\mathcal{L}) = \begin{cases} 2k - 1, & T_E \geq 0 \\ 2k, & T_E < 0 \end{cases}.$$

Proof. The basic observation here is that the vanishing of T_E signals a change in the dimension of the kernel $\ker(\mathcal{L})$. One always has that $u_x \in \ker(\mathcal{L})$ and $\ker(\mathcal{L}) = \operatorname{span}\{u_x\}$ as long as $T_E \neq 0$, and when T_E vanishes we have $\ker(\mathcal{L}) = \operatorname{span}\{u_x, u_E\}$. Thus T_E detects when an eigenvalue of $\mathcal{L}$ crosses from the positive to the negative half-line.

Given this intuition the result follows in a relatively straightforward manner from standard results in Floquet theory, and it is sufficient to prove it for the case $k = 1$. From the Sturm oscillation theorem and the fact that u_x has 2 roots in a period it is clear that either $n(\mathcal{L}) = 1$ on $L^2(\mathbb{T}_1)$ (if zero is an upper band-edge) or $n(\mathcal{L}) = 2$ if zero is a lower band-edge. The spectrum of the eigenvalue problem $\mathcal{L}v = \mu v$ is characterized by the Floquet discriminant $k(\mu)$. The spectrum of $\mathcal{L}$ (on $L^2(\mathbb{R})$) is characterized as the set of values for which the Floquet discriminant $k(\mu)$ is between -2 and 2 :

$$\operatorname{spec}(\mathcal{L}) = \{\mu | k(\mu) \in [-2, 2]\}$$

with periodic eigenvalues corresponding to points where $k(\mu) = +2$ and anti-periodic eigenvalues corresponding to points where $k(\mu) = -2$. The Floquet discriminant thus has positive slope at an upper band-edge and negative slope at a lower band-edge (and vanishes at a double point), and thus serves to distinguish the two cases: if the sign of the derivative is positive then there is a single periodic eigenvalue below zero, and if the sign is positive then there are two periodic eigenvalues below zero. It can be shown (see [11] or [26]) that the sign of $k'(\mu)$ is equal to the sign of T_E and thus the result follows. See Figure 2 for an illustration. □
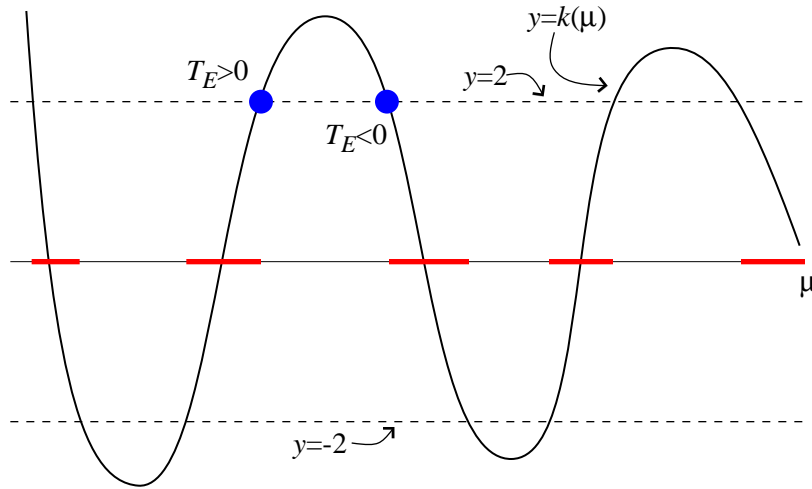
Figure 2: (color online) The Floquet discriminant $k(\mu)$. The thick (red) bands correspond to spec($\mathcal{L}$). The sign of T_E determines whether the origin is a lower band-edge or an upper band-edge.

This lemma implies that the vanishing of T_E signals an eigenvalue of $\mathcal{L}$ passing through the origin and a change in the dimension of $n(\mathcal{L})$, the number of negative eigenvalues of the the second variation of the energy. Next, we would like to give a similar interpretation for $n(\mathcal{L}|_{H_1})$, the number of negative eigenvalues of the restriction of the second variation to the subspace of mean zero functions. To this end, we state a preliminary lemma.

Lemma 1. *Suppose $H(s)$ is a C^1 family of operators and $\phi(s)$ is a C^1 family of functions with $H(0)\phi(0) = 0$ is a simple eigenfunction. Then we have that, in a neighborhood of $s = 0$*

$$\langle \phi(s), H(s)\phi(s) \rangle = \lambda(s)\|\phi(0)\|^2 + \mathcal{O}(s^2)$$

where $\lambda(s)$ is the corresponding eigenvalue bifurcating from $\lambda(0) = 0$.

Proof. Simply notice that the simplicity of the eigenvalue implies the function $\lambda(s)$ is analytic in a neighborhood of $s = 0$ and satisfies that

$$\lambda(s) = s\frac{\langle \phi(0), H'(0)\phi(0) \rangle}{\langle \phi(0), \phi(0) \rangle} + \mathcal{O}(s^2).$$

Upon noticing that

$$\langle \phi(s), H(s)\phi(s) \rangle = 2s \langle \phi'(0), H(0)\phi(0) \rangle + s \langle \phi(0), H'(0)\phi(0) \rangle + \mathcal{O}(s^2)$$
$$= s \langle \phi(0), H'(0)\phi(0) \rangle + \mathcal{O}(s^2),$$

the proof is now complete. $\qquad\square$

With this elementary result in mind, we now present a result which relates the dimension $n(\mathcal{L}|_{H_1})$, which recall is in general very difficult to compute, to the dimension $n(\mathcal{L})$, which was just computed above.

Proposition 1. *Assume that T_E and $\{T, M\}_{a,E}$ never vanish simultaneously. Then we have the equality*

$$n(\mathcal{L}|_{H_1}) = n(\mathcal{L}) - n(T_E\{T, M\}_{a,E}).$$

Proof. Since we are restricting the operator $\mathcal{L}$ to a codimension one subspace, the Courant minimax principle immediately implies that we have either $n(\mathcal{L}|_{H_1}) = n(\mathcal{L})$ or $n(\mathcal{L}|_{H_1}) = n(\mathcal{L}) - 1$. To determine which case occurs, we begin by finding a necessary and sufficient condition for $\mathcal{L}|_{H_1}$ to have an extra element in the kernel. Then, we perform a local perturbation analysis to determine the direction in which the corresponding eigenvalue bifurcates from the origin.

To begin, notice that the function u_x belongs to H_1 and satisfies $\mathcal{L}|_{H_1}u_x = 0$. Thus, the kernel of the operator $\mathcal{L}|_{H_1}$ is always at least one dimensional. Moreover the function $\{u, T\}_{a,E}$ satisfies

$$\mathcal{L}\{u, T\}_{a,E} = -T_E$$

and hence, defining the projection $Q : L^2(\mathbb{T}_k) \to H_1$ it follows that

$$Q\mathcal{L}\{u, T\}_{a,E} = 0.$$

Thus, $\{u, T\}_{a,E}$ corresponds to an element of the kernel of $\mathcal{L}|_{H_1} = Q\mathcal{L}Q$ provided that $Q\{u, T\}_{a,E} = \{u, T\}_{a,E}$, i.e. if $\langle 1, \{u, T\}_{a,E}\rangle = \{T, M\}_{a,E} = 0$. It follows that the vanishing of $\{T, M\}_{a,E}$ signals a change in the dimension of $\ker(\mathcal{L}|_{H_1})$. Applying Lemma 1, we see[2] near a zero of $\{T, M\}_{a,E}$ that there is an eigenvalue of $\mathcal{L}|_{H_1}$ which is given by

$$\lambda = -\frac{T_E\{T, M\}_{a,E}}{\|\phi_1\|^2} + o(\{T, M\}_{a,E}).$$

Thus, the desired equality holds in a neighborhood of a point in parameter space where $\{T, M\}_{a,E} = 0$. To extend this to all parameter values, we note that the difference $n(\mathcal{L}) - n(\mathcal{L}|_{H_1})$ gives the number of negative eigenvalues of $\mathcal{L}|_{H_1}$ relative to $\mathcal{L}$. Since this quantity is locally constant in parameter space, we need only check where these quantities change. Since $n(\mathcal{L})$ changes if and only if T_E changes sign by Lemma 1 and the above relative count does not change, it follows the desired equality holds so long as there exists a point in parameter space at which the quantity $\{T, M\}_{a,E}$ vanishes.

To complete the proof note that the results of Bronski and Johnson [11] imply the following identity:

$$k_{\mathbb{R}} = \begin{cases} 0 \pmod 2, & n(\{T, M, P\}_{a,E,c}) = 0 \\ 1 \pmod 2, & n(\{T, M, P\}_{a,E,c}) = 1. \end{cases}$$

Since $k_{\mathbb{I}}^-$ and $k_{\mathbb{C}}$ are even they do not change the count modulo two. In the case $\{T, M\}_{a,E}$ is non-vanishing the count is determined to within one, and is thus the count is exact if one knows the parity. Applying the result of Bronski and Johnson thus justifies the desired equality in general. $\square$

Remark 5. *Using a functional analytic proof it was shown in [12] that*

$$n(\mathcal{L}|_{H_1}) = n(\mathcal{L}) - n(\langle \mathcal{L}^{-1}(1), 1\rangle)$$

(also see equation (10)). Using the above calculation it is clear that

$$\langle \mathcal{L}^{-1}(1), 1\rangle = \frac{\{T, M\}_{E,a}}{T_E} = \frac{\begin{vmatrix} K_{EE} & K_{Ea} \\ K_{Ea} & K_{aa} \end{vmatrix}}{K_{EE}}$$

showing the equivalence of the two formulae.

---

[2]Notice that while 0 is not actually a simple eigenvalue of $\mathcal{L}|_{H_1}$, the function u_x is odd while $\{u, T\}_{a,E}$ is even. Thus, the eigenspaces split and one is essentially doing simple perturbation theory.

Finally, to conclude the proof of Theorem 1, we must calculate $n(D)$. This is the content of the following lemma.

Lemma 2. *Under the assumptions of Theorem 1, one has that $D \in \mathbb{R}$ with*

$$D = -\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}.$$

Thus, $n(D)$ is either 0 or 1 depending if $\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$ is negative or positive, respectively.

Proof. Under the assumptions of Theorem 1, we know that $\ker(\mathcal{L}) = \operatorname{span}\{u_x\}$ and

$$\partial_x \mathcal{L}|_{H_1} \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix} = -\{T, M\}_{a,E} u_x$$

from Proposition 1. It follows that the matrix D is a real number in this case, with value equal to

$$D = \left\langle \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix}, \mathcal{L} \begin{vmatrix} u_E & T_E & M_E \\ u_a & T_a & M_a \\ u_c & T_c & M_c \end{vmatrix} \right\rangle = -\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$$

as claimed. $\qquad\square$

Remark 6. *The last three results show that geometric quantities associated to the classical mechanics of the traveling waves contain information about changes in the nature of the spectrum of the linearized problem. Specifically:*

- *Vanishing of $T_E = K_{EE}$ signals a jump in the dimension of $\ker(\partial_x \mathcal{L})$ and $\ker(\mathcal{L})$ corresponding to an eigenvalue of $\mathcal{L}$ crossing from the negative to the positive half-line (or vice-versa).*

- *Vanishing of $\{T, M\}_{a,E} = -\begin{vmatrix} K_{EE} & K_{Ea} \\ K_{aE} & K_{aa} \end{vmatrix}$ or, equivalently vanishing of $\langle \mathcal{L}^{-1}(1), 1 \rangle$ signals a jump in the dimension of $\ker(\partial_x \mathcal{L})$ and $\ker(\mathcal{L}|_{H_1})$ corresponding to an eigenvalue of $\mathcal{L}|_{H_1}$ crossing from the negative to the positive half-line (or vice-versa).*

- *Vanishing of $\{T, M, P\}_{a,E,c}$ signals a change in the length of the Jordan chain of $\partial_x \mathcal{L}$.*

Remark 7. *It should be noted that the quantity D computed in Lemma 2 also arose naturally in [26] when considering orbital stability of periodic traveling wave solutions of (1) to perturbations with the same periodic structure, i.e. orbital stability in $L^2(\mathbb{T}_1)$. There, the negativity of D was sufficient in order to ensure the quadratic form induced by $\mathcal{L}$ acting on $L^2(\mathbb{T}_1)$ was positive definite on an appropriate subspace. As the methods therein are based on classical energy functional calculations, such a requirement was necessary to classify the periodic traveling wave as a local minimizer of the Hamiltonian subject to the momentum and mass constraints.*

Remark 8. *In [12] it was shown that D had the functional formulation*

$$D = \frac{\begin{vmatrix} \langle \mathcal{L}^{-1}(u), u \rangle & \langle \mathcal{L}^{-1}(u), 1 \rangle \\ \langle \mathcal{L}^{-1}(u), 1 \rangle & \langle \mathcal{L}^{-1}(1), 1 \rangle \end{vmatrix}}{\langle \mathcal{L}^{-1}(1), 1 \rangle}.$$

Proof 1 (Proof of Main Theorem). *The proof of Theorem 1 is essentially complete. If we define the function*

$$n(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0. \end{cases}$$

for $x \in \mathbb{R}/0$ then the results of lemmas 1 and 1 and proposition 1 we have the following count:

$$k_{\mathbb{I}}^- + k_{\mathbb{R}} + k_{\mathbb{C}} = n(\mathcal{L}|_{\mathrm{ran}(\partial_x)}) - n(D)$$
$$= 2k - 1 + n(T_E) - n(T_E\{T,M\}_{a,E}) + n(\{T,M\}_{a,E}\{T,M,P\}_{a,E,c}).$$

From this the main theorem follows from the fact that $K_E = T, K_a = M, K_c = P/2$, and thus that $T_E, \{T,M\}_{aE}, \{T,M,P\}_{E,a,c}$ are (to within a multiplicative constant in the last case) the principle minors of the Hessian of K. From the Jacobi-Sturm rule, which states that the number of negative eigenvalues of a symmetric matrix is equal to the number of sign changes in the sequence of principle minors, we find the main result.

Notice that Theorem 1 gives a sufficient requirement for a spatially periodic traveling wave of (1) to be orbitally stable in $L^2(\mathbb{T}_k)$ for any $k \in \mathbb{N}$ and any sufficiently smooth nonlinearity f. In the next two sections, we analyze Theorem 1 in the case of a power-nonlinearity by using complex analytic methods to reduce the expression for the Jacobians involved in (9) in terms of moments of the underlying wave itself. This has the obvious advantage of being more amenable to numerical experiments as one no longer has to numerically differentiate, a procedure that always involves a loss of accuracy. We will also discuss the computation of Δ_{MI} for power-law nonlinearities. In particular, we will prove a new theorem in the case of the focusing and defocusing MKdV which relates the modulational stability of a spatially periodic traveling wave to the number of distinct families of periodic solutions existing for the given parameter values.

4   Polynomial Nonlinearities and the Picard-Fuchs System

In the previous section we derived the formula

$$k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - p(\partial^2 K)$$

relating the number of unstable and potentially unstable eigenvalues on $L^2(\mathbb{T}_k)$ to the Hessian of the classical action K. In earlier work Bronski and Johnson derived the modulational instability index

$$\Delta_{\mathrm{MI}} = \frac{1}{2}\left(\{T,P\}_{E,c} - 2\{M,P\}_{E,a}\right)^3 - 3\left(\frac{3}{2}\{T,M,P\}_{E,a,c}\right)^2.$$

which detects the nature of the spectrum at the origin. If this quantity is positive the spectrum in a neighborhood of the origin lies on the imaginary axis with multiplicity three, while if this quantity is negative the spectrum in a neighborhood of the origin consists of three curves through the origin, one along the imaginary axis and two going off in a complex directions. One major simplification of this theory occurs when the nonlinearity $f(u)$ is polynomial. In this case the fundamental quantities (T, M, P) are given by Abelian integrals of the first, second or third kind on a Riemann surface. On a Riemann surface of genus g there are g integrals of the first kind, g integrals of the second kind, and 1 integral of the third kind. Since derivatives of these quantities with resepect to the coefficients are again Abelian integrals they must necessarily be expressible as a linear combination of the original $2g + 1$ integrals, a fact known as the Picard-Fuchs relation.

While we cannot give a detailed exposition of this theory here the basics are very straightforward. Suppose that $P(u) = a_0 + a_1 u + \ldots a_n u^n$ is a polynomial of degree n. If the polynomial is of degree $2g+2$ or $2g+1$ then the quantity $u^k\,\mathrm{d}u/\sqrt{P(u)}$ is an Abelian differential on a Riemann surface of genus g. If we define the kth moment μ_k of the solution $u(x)$ as follows:

$$\mu_k = \int_0^T u^k(x)\,\mathrm{d}x = \oint_\gamma \frac{u^k\,\mathrm{d}u}{\sqrt{P(u)}}$$

then one obviously has

$$\frac{\mathrm{d}\mu_k}{\mathrm{d}a_j} = \frac{\mathrm{d}\mu_j}{\mathrm{d}a_k} = -\frac{1}{2}\oint_\gamma \frac{u^{k+j}\,\mathrm{d}u}{P^{\frac{3}{2}}(u)} =: I_{k+j}$$

for any loop γ in the correct homotopy class. For our purposes we are interested in branch cuts on the real axis though none of what will be said in this section assumes this. In the context of the stability problem one only needs $I_0 \ldots I_4$, since $T_E, T_a, \ldots P_c$ can all be expressed in terms of these five quantities, but the theory requires that one consider all such moments. The main observation is that the above integrals $\{I_k\}$ are again Abelian integrals and thus can be expressed in terms of $\{\mu_k\}$.

In practice the simplest way to do this is to use the identities

$$\mu_m = \oint \frac{u^m P(u)\,\mathrm{d}u}{P^{\frac{3}{2}}(u)} = \sum_{j=0}^n a_j I_{j+m}$$

for $m \in \{0..n-1\}$ and

$$\oint \frac{u^m P'(u)\,\mathrm{d}u}{P^{\frac{3}{2}}(u)} = 2m \oint \frac{u^{m-1}\,\mathrm{d}u}{\sqrt{P(u)}} = 2n\mu_{m-1} \tag{18}$$

$$\sum_{j=0}^n j a_j I_{j+m-1} = 2m\mu_{m-1} \tag{19}$$

for $m \in \{0, 1, ..., n\}$. This gives a linear system of $2n-1$ equations in $2n-1$ unknowns $\{I_k\}_{k=0}^{2n-2}$:

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n & 0 & 0 & \ldots \\ 0 & a_0 & a_1 & \ldots & a_n & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & a_0 & a_1 & \ldots & a_n \\ a_1 & 2a_2 & \ldots & na_n & 0 & 0 & \ldots \\ 0 & a_1 & 2a_2 & \ldots & na_n & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & a_1 & 2a_2 & \ldots & na_n \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ \vdots \\ \vdots \\ I_{2n-3} \\ I_{2n-2} \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{n-2} \\ 0 \\ 2\mu_0 \\ \vdots \\ 2(n-1)\mu_{n-2} \end{pmatrix}$$

The matrix which arises in the above linear systems is the Sylvester matrix of $P(u)$ and $P'(u)$. It is a standard result of commutative algebra that the Sylvester matrix of $P(u)$ and $Q(u)$ is singular if and only if the polynomials P and Q have a common root. In our case $P(u)$ and $P'(u)$ having a common root is equivalent to $P(u)$ having a root of higher multiplicity. In the case where P has a multiple root the a pair of branch points degenerate to a pole and the genus of the surface decreases by one. We will later work an example where this occurs.

For a given polynomial it is rather straightforward to work these out, particularly with the aid of computer algebra systems. In this paper we did some of the more laborious calculations with Mathematica[25]. Some examples are presented in the next section.

5 Examples

5.1 The Korteweg -de Vries Equation (KdV-1)

The Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + (u^2)_x = 0$$

is, of course, completely integrable and the spectrum of the linearized flow can in principle be understood by the machinery of the inverse scattering transform. We note in particular the construction via Baker-Akheizer functions detailed in the text of Belokolos, Bobenko, Enolskii, Its and Matveev[4]. Nevertheless this problem provides a good test for our methods, which we believe to be considerably simpler and easier to calculate than the algebro-geometric approach.

In the notation of (3) the effective potential is given by

$$V(u) = -au - c\frac{u^2}{2} + \frac{u^3}{3}$$

and the solutions are associated with the genus-1 curve $y^2 = E + au + cu^2/2 - u^3/3$. The discriminant is given by

$$\mathrm{disc}(E - V(u)) = \frac{1}{12}\left(16a^3 + 3a^2c^2 - 36Eac - 6Ec^3 - 36E^2\right)$$

and the variety defined by the vanishing of the discriminant (for $c = 1$) is explicitly parameterized by

$$a = s^2 - s$$

$$E = \frac{s^2}{2} - \frac{2s^3}{3}.$$

On the zero set of the discriminant the solutions are (up to a Galilean boost) the solitary wave and constant solution. The KdV equation has (non-constant) periodic solutions if and only if $\mathrm{disc}(E - V(u))$ is positive. Moreover, by scaling (and possibly a map $u \mapsto -u$) the wave speed c can be assumed to be $c = +1$.

The Picard-Fuchs system for the curve associated to KdV-1 is the following set of five linear equations:

$$\begin{pmatrix} E & a & c/2 & 1/3 & 0 \\ 0 & E & a & c/2 & 1/3 \\ a & c & -1 & 0 & 0 \\ 0 & a & c & -1 & 0 \\ 0 & 0 & a & c & -1 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} T \\ M \\ 0 \\ 2T \\ 4M \end{pmatrix},$$

where

$$\mu_k = \oint \frac{u^k \, du}{\sqrt{2R(u; a, E, c)}}, \quad I_k = \oint \frac{u^k \, du}{(2R(u; a, E, c))^{3/2}}.$$

After some algebra the various Jacobians arising in Theorem 1 and the modulational stability index (4) can be expressed in terms of the period T and the mass M as follows:

$$T_E = \frac{\left(4a + c^2\right) M + (6E + ac)\, T}{12 \operatorname{disc}(R(u,a,E,c))}$$

$$\{T, M\}_{a,E} = -\frac{T^2 V'(\frac{M}{T})}{12 \operatorname{disc}(R(u,a,E,c))}$$

$$\{T, M, P\}_{a,E,c} = \frac{T^3 (E - V(\frac{M}{T}))}{2 \operatorname{disc}(R(u,a,E,c))}$$

$$2\Delta_{MI} = \frac{\left(\alpha_{3,0} T^3 + \alpha_{2,1} T^2 M + \alpha_{1,2} T M^2 + \alpha_{0,3} M^3\right)^2}{2^{10} 3^7 \operatorname{disc}^3(R(u,a,E,c))}$$

where

$$\alpha_{3,0} = 36E + 18aEc - 8a^3$$
$$\alpha_{2,1} = 18Ec^2 - 6a^2 c + 36aE$$
$$\alpha_{1,2} = -18cE + 24a^2 + 3ac^2$$
$$\alpha_{0,3} = c^3 + 6ac + 12E$$

These quantities are all positive. The positivity of T_E follows from a result of Schaaf[35]. The non-negativity of Δ_{MI} is clear: in principle the cubic polynomial in the numerator could vanish but numerics shows that it does not in the region where $\operatorname{disc}(R(u,a,E,c)) > 0$. The positivity of $\{T, M, P\}_{a,E,c}$ is clear. Finally $\{T, M\}_{a,E}$ is positive from Jensen's inequality since

$$\oint \frac{V'(u)\, \mathrm{d}u}{\sqrt{R(u; a, E, c)}} = 0.$$

Remark 9. *Recall from [11] that the Jacobian $\{T, M, P\}_{a,E,c}$ arises naturally as an orientation index for the gKdV linearized spectral problem for a sufficiently smooth nonlinearity. Indeed, one has that $\{T, M, P\}_{a,E,c} < 0$ is sufficient to imply the existence of a non-zero real periodic eigenvalue of the linearized operator $\partial_x \mathcal{L}$, i.e. an unstable real eigenvalue in $L^2(\mathbb{T}_1)$. Moreover, from [26] it follows that if $T_E > 0$, then such an eigenvalue can not exist if $\{T, M, P\}_{a,E,c}$ is positive: however, no such claim can be made in the case where $T_E < 0$.*

Theorem 1 now implies the following index result: if one considers the linearized operator acting on $L^2(\mathbb{T}_k)$ for $k \in \mathbb{N}$ then u_x has $2k$ roots in $\mathbb{T}_k$ and the number of real eigenvalues, complex eigenvalues, and imaginary eigenvalues of negative Krein signature satisfy

$$k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2(k - 1) \tag{20}$$

In particular when $k = 1$, so one is considering stability to perturbations of the same period, the only eigenvalues lie on the imaginary axis and have positive Krein signature thus proving orbital stability of such solutions in $L^2(\mathbb{T}_1)$. Furthermore, considered as an operator on $L^2[-\infty, \infty]$ the spectrum in a neighborhood of the origin in the spectral domain consists of the imaginary axis with multiplicity three, thus implying modulational stability of the periodic traveling wave solutions of the KdV equation.

This example provides an independent check of the result: in the KdV case Kapitula and Deconinck have explicitly computed the spectrum of the linearized KdV, and shown that the spectrum consists of the imaginary axis, a symmetric interval of which is of multiplicity three and the

remainder of which has multiplicity one. They further show that in the interval of multiplicity three there are two eigenvalues of positive Krein signature and one of negative Krein signature, and that the region of spetral multiplicity one has only positive Krein signature eigenvalues. By extending their analysis one can actually count the number of eigenvalues of negative Krein signature of the operator on $\mathbb{T}_k$ and one finds that it is $2(k-1)$, in agreement with the above calculation.

5.2 Example: Modified Korteweg- de Vries (KdV-2)

The MKdV equation

$$u_t + u_{xxx} \pm (u^3)_x = 0$$

arises as a model for wave propagation in plasmas and as a model for the propagation of interfacial waves in a stratified medium. It is also integrable and the same caveats apply as for the KdV regarding the algebro-geometric construction of the spectrum of the linearized operator. The MKdV is invariant under the scaling $x \mapsto \alpha x, t \mapsto \alpha^3 t, u \mapsto \alpha^{-\frac{2}{3}} u$, and thus the wavespeed c can be scaled to be $c = 0, \pm 1$. The most physically and mathematically interesting case is the focusing MKdV (the plus sign above) with right-moving waves where c can be scaled to $+1$. In this case the (genus 1) curve is given by

$$y^2 = (u_x)^2 = E + au + \frac{1}{2}u^2 - \frac{u^4}{4}$$

For the focusing MKdV the zero set of the discriminant is the familiar swallowtail curve defined implicitly by the equation

$$\mathrm{disc}(E + au + u^2/2 - u^4/4) = 2a^2 - 27a^4 - 4E + 72a^2 E - 32E^2 - 64E^3 = 0$$

or by the polynomial parametric representation

$$a = s - s^3$$

$$E = \frac{s^2}{2} - \frac{3s^4}{4}$$

- see Figure 3) for an illustration.

On the discriminant the torus "pinches off" and degenerates to a cylinder, and all of the elliptic integrals can be evaluated in terms of elementary functions. Unlike the KdV case, where the only periodic solutions on the curve of vanishing discriminant are constant, the MKdV admits non-trivial periodic solutions on the swallowtail curve. At every point on the discriminant there is a constant solution $u(x) = -s$. Along the upper (dashed) branch ($s \in [-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$) there is a solitary wave homoclinic to $u = s$ defined by

$$\frac{1}{\frac{\sqrt{2(1-s^2)}}{1-3s^2}\cosh(\sqrt{1-3s^2}\eta) - \frac{2s}{1-3s^2}} - s$$

with $\eta = x - x_0 - t$. The soliton solution corresponds to $s = 0$. Along the lower (dotted) branch $s \in [-1, -\frac{\sqrt{3}}{3}] \cup [\frac{\sqrt{3}}{3}, 1]$ in addition to the constant solution there is a non-constant periodic solution.

$$u(\eta) = \frac{1}{\frac{s}{3s^2-1} + \frac{\sqrt{2(1-s^2)}}{6s^2-2}\sin(\sqrt{3s^2-1}\eta)} - s$$

Along the remaining portions of the curve there are no non-constant solutions.
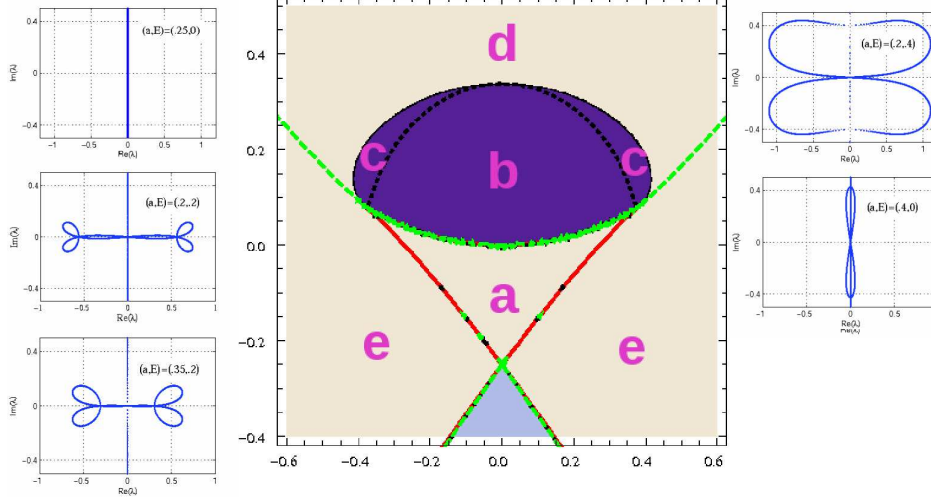
Figure 3: The configuration space for focusing MKdV with $c = +1$. The swallowtail figure divides the plane into regions containing $0, 1,$ and 2 periodic solutions. The domain is colored according to the sign of $\{T, M, P\}_{a,E,c}$. The spectral pictures correspond to various regions in parameter space. For example, the picture in the bottom left corner was numerically derived for the parameter values $(a, E) = (.35, .2)$, which corresponds to region (c).

The modulational instability index turns out, in this case, to be particularly simple. After solving the Picard-Fuchs system

$$
\begin{pmatrix}
E & a & \frac{1}{2} & 0 & -\frac{1}{4} & 0 & 0 \\
0 & E & a & \frac{1}{2} & 0 & -\frac{1}{4} & 0 \\
0 & 0 & E & a & \frac{1}{2} & 0 & -\frac{1}{4} \\
a & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & a & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & a & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & a & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6
\end{pmatrix}
=
\begin{pmatrix}
T \\ M \\ P \\ 0 \\ 2T \\ 4M \\ 6P
\end{pmatrix}
$$

one finds the following expressions for the various Jacobians:

$$
T_E = -\frac{(3a^2 - 16E^2 - 4E)T + (9a^2 - 4E - 1)P}{16 \ \mathrm{disc}(E - V(u))}
$$

$$
\{T, M\}_{a,E} = -\frac{(3a^2 - 4E)T^2 + (4E - 1)PT + P^2}{16 \ \mathrm{disc}(E - V(u))}
$$

$$
\{T, M, P\}_{a,E,c} = -\frac{(2a^2 - 4E)T^3 + 4EPT^2 - TP^2 + P^3}{32 \ \mathrm{disc}(E - V(u))}
$$

$$
\Delta_{MI} = \frac{\left(\alpha_{0,3}T^3 + \alpha_{1,2}T^2P + \alpha_{2,1}TP^2 + \alpha_{3,0}P^3\right)^2}{4194304 \ \mathrm{disc}(E - V(u))^3}
$$

where

$$
\mathrm{disc}(E - V(u)) := \mathrm{disc}(E + au + u^2/2 - u^4/4) = 27a^4 - 72Ea^2 - 2a^2 + 64E^3 + 32E^2 + 4E
$$

and

$$\alpha_{3,0} = 1 + 36E - 27a^2$$
$$\alpha_{2,1} = 27a^2 + 144E^2 - 60E$$
$$\alpha_{1,2} = 36E - 240E^2 - 18a^2 + 108E$$
$$\alpha_{0,3} = 54a^4 - 180a^2E + 144E^2 + 64E.$$

We note a few things. First notice that while the Picard-Fuchs system involves T, M, and P the resulting Jacobians only involve T and P. While this is not obvious from the point of view of linear algebra there is a clear complex analytic reason why this must be so: the Abelian differentials defining T and P have zero residue about the point at infinity, as do $T_E, T_a \ldots P_c$, while M has a non-vanishing residue at infinity. Thus $T_E, T_a \ldots P_c$ must be expressible in terms of only T and P.

Secondly we note that while there are two distinct families of solutions inside the swallowtail they have the same orientation index $\{T, M, P\}_{a,E,c}$ and modulational instability index Δ_{MI}. This is special to the genus one case since the integrals over one cycle can be simply related to the integrals over the other cycle via

$$\oint_{\gamma_{\text{left}}} \frac{\mathrm{d}u}{\sqrt{E - V(u)}} = \oint_{\gamma_{\text{right}}} \frac{\mathrm{d}u}{\sqrt{E - V(u)}} \tag{21}$$

$$\oint_{\gamma_{\text{left}}} \frac{u \, \mathrm{d}u}{\sqrt{E - V(u)}} = \oint_{\gamma_{\text{right}}} \frac{u \, \mathrm{d}u}{\sqrt{E - V(u)}} + 2\sqrt{2}\pi \tag{22}$$

$$\oint_{\gamma_{\text{left}}} \frac{u^2 \, \mathrm{d}u}{\sqrt{E - V(u)}} = \oint_{\gamma_{\text{right}}} \frac{u^2 \, \mathrm{d}u}{\sqrt{E - V(u)}} \tag{23}$$

by deforming the contour onto the other cycle and picking up the contribution from the residue at infinity. Since the orientation and modulational instability indices are built of derivatives of the above quantities these indices must be the same for both families of solutions.

The above observation extends the calculation of Haragus and Kapitula [24] for the zero amplitude waves to the periodic waves on the swallowtail curve defined in equation (5.2). Haragus and Kapitula utilize a perturbation argument to calculate the stability properties of the periodic waves in a small neighborhood of the bifurcation point - in other words in a small neighborhood of the discriminant, when one of the cycles has almost pinched off. The family of (large amplitude) periodic waves in (5.2) represents the solutions associated to the other cycle, which the above shows to have the same stability indices.

As in the KdV case the modulational instability index, which is a homogeneous polynomial of degree 6 in T and P, can be expressed as the square of a homogeneous polynomial of degree 3 over an odd power of the discriminant of the polynomial $E - V(u)$. A similar expression holds in the defocusing case, as well as for general values of c. The sign of this quantity is obviously the same as of the sign of the discriminant of the quartic, which is in turn positive if the quartic has no real roots or 4 real roots, and negative if the quartic has two only real roots. Thus we establish the following surprising fact:

Theorem 2. *The traveling wave solutions to the MKdV equation*

$$u_t + u_{xxx} \pm (u^3)_x = 0$$

are modulationally unstable for a given set of parameter values if the polynomial

$$E + au + cu^2/2 \pm u^4/4$$

has two real roots, and are modulationally stable if it has four real roots.

Remark 10. *In the case of focusing MKdV, Theorem 2 implies that if the parameter values give rise to one periodic solution then this solution is unstable to perturbations of sufficiently long wavelength. If there are two periodic solutions then the spectrum of the linearization about one of these solutions in the neighborhood of the origin consists of the imaginary axis with multiplicity three. For the case of defocusing MKdV the situation is slightly different: there only exists a periodic solution when the polynomial has positive discriminant, in which case this solution is modulationally stable - there is no spectrum off of the real axis in a neighborhood of the origin.*

Note that while this problem is in principle completely solvable using algebro-geometric techniques, Theorem 2 is new. While explicit the classical algebro-geometric calculations are sufficiently complicated that they are exceedingly tedious to do in general. For examples of this sort of calculation see the original text of Belokolos et. al.[4] as well as the papers of Bottman and Deconinck[8] and Deconinck and Kapitula[12].

We now summarize the more interesting situation of the focusing MKdV in Figure 3 and below:

(a) There are two families of solutions in this region. For both of these solutions the modulational instability index is positive and thus in a neighborhood of the origin the imaginary axis is in the spectrum with multiplicity three. Solutions in this region have $T_E > 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} < 0$ implying $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2(k-1)$.

The solutions in the remaining regions have a modulational instability index that is negative showing that they are always unstable to perturbations of sufficiently long wavelength.

(b) In this region $T_E < 0$, $\{T, M\}_{E,a} > 0$, and $\{T, M, P\}_{E,a,c} > 0$ implying $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 1$.

(c) In this region $T_E < 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} > 0$ implying $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 1$. As one crosses between regions b and c the indices $n(\mathcal{L}|_{H^1})$ and $n(D)$ both increase (resp. decrease) by one, leaving the total count the same.

(d) In this region $T_E < 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} < 0$ implying $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2(k-1)$.

(e) In this region $T_E > 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} < 0$ implying $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2(k-1)$.

In regions (a), (d), and (e) when considering periodic perturbations ($k = 1$) one finds that $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 0$ implying both spectral and orbital stability in $L^2(\mathbb{T}_1)$. However, as mentioned above, in regions (d) and (e) the solution is spectrally unstable in $L^2(\mathbb{T}_k)$ for $k \in \mathbb{N}$ sufficiently large. Moreover, in regions (b) and (c) there always exists a non-zero real periodic eigenvalue, i.e. the linearized operator $\partial_x \mathcal{L}$ acting on $L^2(\mathbb{T}_k)$ always has a non-zero real eigenvalue and hence such solutions are always spectrally unstable.

Remark 11. *It should be noted that the above counts are consistent with the calculations of Deconinck and Kapitula [12] in which they consider stability of the cnoidal wave solutions*

$$U(x, t; \kappa) = \sqrt{2}\mu \, \mathrm{cn} \left(\mu x - \mu^2 (2 - \kappa^2) t; k \right)$$

of the focusing MKdV equation, where $\mu > 0$ and $\kappa \in [0, 1)$. Such solutions always correspond to regions (b) and (d) along with the constraint $a = 0$. There, the authors find numerically that there is a critical elliptic-modulus $\kappa^* \approx 0.909$ such that solutions with $0 \leq \kappa < \kappa^*$, corresponding to region (b) are orbitally stable in $L^2(\mathbb{T}_1)$ while solutions with $\kappa^* < \kappa < 1$, corresponding to region (d) are spectrally unstable in $L^2(\mathbb{T}_k)$ for all $k \in \mathbb{N}$ due to the presence of a non-zero real eigenvalue of the linearized operator.

5.3 Example: L^2 critical Korteweg-de Vries (KdV-4)

Finally we consider the equation

$$u_t + u_{xxx} + (u^5)_x = 0.$$

This equation is not a physical model for any system that we are aware of but is mathematically interesting for a number of reasons. This is the power where the solitary waves first go unstable. Equivalently this is the L^2 critical case, where one has the scaling $u \mapsto \sqrt{\gamma}u(\gamma x)$ preserving the L^2 norm and the relative contributions of the kinetic and potential energy to the Hamiltonian. Again we focus on the focusing case, which is the more interesting, and we scale everything so that $c = +1$. In this case the curve of genus 2 is given by

$$y^2 = (u_x)^2 = 2(E + au + \frac{1}{2}u^2 - \frac{u^6}{6})$$

As is always the case for $KdV - 2n$ the parameter space is divided by a swallowtail curve (the discriminant) into regions containing no periodic solution, one periodic solution, and two periodic solutions. The implicit representation is given by

$$\Gamma = \{(a, E) | -48a^2 + 3125a^6 + 96E - 11250a^4E + 10800a^2E^2 - 1728E^3 + 7776E^5 = 0\}$$

or parametrically by

$$a = s^5 - s, \quad E = \frac{s^2}{2} - \frac{5s^6}{6}$$

(see Figure 4). Again the picture is qualitatively similar to the MKdV case: the portion of the swallowtail parameterized by $s \in (-5^{-\frac{1}{4}}, 5^{-\frac{1}{4}})$ represents parameter values for which there are two solutions: one constant and one homoclinic to a constant, with the origin representing the soliton solution (the solution homoclinic to zero) and the zero solution. The portions of the curves parameterized by $s \in (-\frac{1}{\sqrt[4]{5}}, -1) \cup (\frac{1}{\sqrt[4]{5}}, 1)$ represent parameter values for which there are two solutions: a constant and a periodic solution. The remainder of the curve represents parameter values for which there is only the constant solution.

The Picard-Fuchs system is following set of eleven equations:

$$
\begin{pmatrix}
E & a & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & E & a & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & E & a & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & E & a & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & E & a & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} \\
a & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & a & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 1 & 0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \\ I_9 \\ I_{10}
\end{pmatrix}
=
\begin{pmatrix}
\mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ 0 \\ 2\mu_0 \\ 4\mu_1 \\ 6\mu_2 \\ 8\mu_3 \\ 10\mu_4
\end{pmatrix}.
$$

These quantities are homogeneous polynomials in $\mu_0 (= T)$, $\mu_1 (= M)$, μ_3, and μ_4 but are independent of $P = \mu_2$ since the differential corresponding to momentum has a non-trivial residue at infinity, similar to the case of the MKdV. We have explicit expressions for the various Jacobians arising in the theory, but they are cumbersome - the quantity $\{T, M, P\}_{E,a,c}$ for instance has $\binom{4+3-1}{3} = 20$ terms, all of which are non-zero. However they are quite well-suited to symbolic
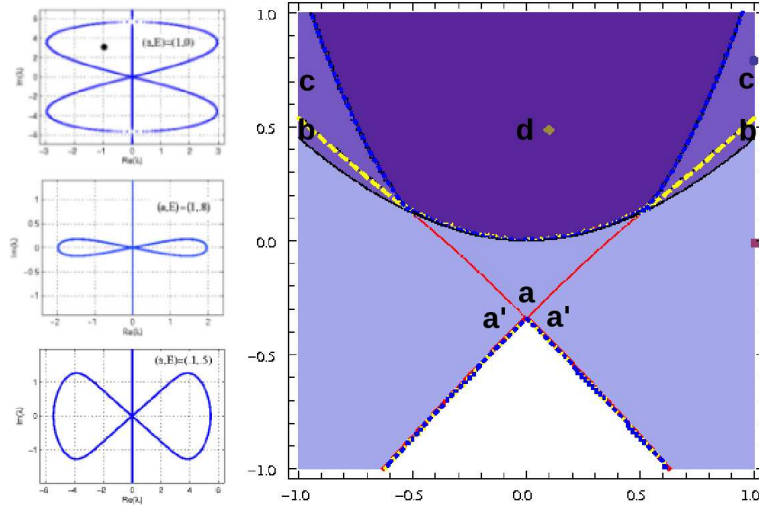
Figure 4: The configuration space for focusing KdV-4 with $c = +1$. The swallowtail figure divides the plane into regions containing $0, 1,$ and 2 periodic solutions. The spectral pictures on the left correspond to various regions in parameter space. For example, the picture in the upper left corner was numerically derived for the parameter values $(a, E) = (1, 0)$, which corresponds to region (a').

manipulation. Below we present some numerics. In the numerics that follow we used the analytical expressions for the various Jacobians and computed the moments T, M, μ_3, μ_4 via numerical integration. This is numerically quite easy and quite stable compared with trying to numerically differentiate T, M, P.

The stability diagram of these solutions is depicted in Figure 4. Numerics indicate that the modulational instability index is always negative, indicating that solutions are *always modulationally unstable*. This is a physically very interesting observation, and seems to be connected with the fact that KdV-4 is the L^2 critical scaling case. The modulational instability of the periodic waves suggests that these waves are unstable to collapse.

There are three curves emerging from the cusps of the swallowtail. The lowest of these is the curve on which the orientation index $\{T, M, P\}_{a,E,c}$ vanishes, the middle (dashed) where T_E vanishes and the upper (dotted) where $\{T, M\}_{a,E}$ vanishes.

The behavior in the various regions is summarized as follows:

(a) There are two solution families in this region, both of which satisfy $T_E > 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} < 0$. This implies that Hessian has two positive eigenvalues, the linearized operator has no real periodic eigenvalues and that $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2(k-1)$

(a') There is only one solution family in this region, otherwise the behavior is the same as in region (a)

(b) The family of solutions in this region has $T_E > 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} > 0$. This implies that $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 1$

(c) The family of solutions in this region has $T_E < 0$, $\{T, M\}_{E,a} < 0$, and $\{T, M, P\}_{E,a,c} > 0$. This implies that $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 1$.

(d) The family of solutions in this region has $T_E < 0$, $\{T, M\}_{E,a} > 0$, and $\{T, M, P\}_{E,a,c} > 0$. This implies that $k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 1$.

It follows that solutions in region (a) are orbitally stable in $L^2(\mathbb{T}_1)$ and spectrally unstable in $L^2(\mathbb{T}_k)$ for $k \in \mathbb{N}$ sufficiently large. Moreover, solutions in the remaining regions are spectrally unstable in $L^2(\mathbb{T}_k)$ for any $k \in \mathbb{N}$ due to the presence of a non-zero real periodic eigenvalue.

It is interesting that all periodic solutions to the L^2 critical KdV are unstable to perturbations of sufficiently long wavelength (or, equivalently, unstable to peturbations in $L^2(\mathbb{R})$). Presumably this is due to the criticality: the periodic solutions are modulationally unstable to collapse and blow-up. In contrast the (sub-critical) KdV-3 exhibits some parameter regimes which are modulationally stable. It would be interesting to understand this phenomenon better.

5.4   A model arising in plasma physics (KdV-$\frac{1}{2}$)

The following variant of the Korteweg-de Vries equation

$$u_t + u_{xxx} + \frac{5}{2}(u^{\frac{3}{2}})_x = 0$$

has been studied as a model for plasmas. The quantity u, representing a density, must be a positive quantity. The traveling waves are defined implicitly by

$$\int \frac{du}{\sqrt{2(E + au + \frac{c}{2}u^2 - u^{\frac{5}{2}})}} = x - ct.$$

The obvious change of variable $v^2 = u$ shows that the travelling wave solutions to this equation are associated with the genus two curve $y^2 = E + av^2 + cv^4/2 - v^5$. The period, mass, and momentum of the travelling wave are given by

$$T = \oint \frac{du}{\sqrt{2(E + au + \frac{c}{2}u^2 - u^{\frac{5}{2}})}} = \oint \frac{2vdv}{\sqrt{2(E + av^2 + \frac{c}{2}v^4 - v^5)}} \tag{24}$$

$$M = \oint \frac{udu}{\sqrt{2(E + au + \frac{c}{2}u^2 - u^{\frac{5}{2}})}} = \oint \frac{2v^3dv}{\sqrt{2(E + av^2 + \frac{c}{2}v^4 - v^5)}} \tag{25}$$

$$P = \oint \frac{u^2du}{\sqrt{2(E + au + \frac{c}{2}u^2 - u^{\frac{5}{2}})}} = \oint \frac{2v^5dv}{\sqrt{2(E + av^2 + \frac{c}{2}v^4 - v^5)}}. \tag{26}$$

Scaling the wavespeed to $c = 1$ the zero set of the discriminant $\Gamma = \{(a, E)|\text{disc}(E + av^2 + \frac{c}{2}v^4 - v^5) = 0\}$ is given by

$$\Gamma = \{E = 0\} \cup \{(5/2s^3 - s^2, -\frac{3}{2}s^5 + \frac{1}{2}s^4)|s \in (-\infty, \infty)\}.$$

The physically admissable parameter regime is that for which the polynomial has a bounded interval in which it is non-negative corresponding to a non-negative periodic solution. This is depicted in Figure (5).

The Picard-Fuchs system is a set of nine equations

$$\begin{pmatrix} E & 0 & a & 0 & \frac{1}{2} & -\frac{1}{5} & 0 & 0 & 0 \\ 0 & E & 0 & a & 0 & \frac{1}{2} & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & E & 0 & a & 0 & \frac{1}{2} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & E & 0 & a & 0 & \frac{1}{2} & -\frac{1}{5} \\ 0 & 2a & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2a & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2a & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2a & 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \\ I_8 \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \\ 2\mu_0 \\ 4\mu_1 \\ 6\mu_2 \\ 8\mu_3 \end{pmatrix}$$
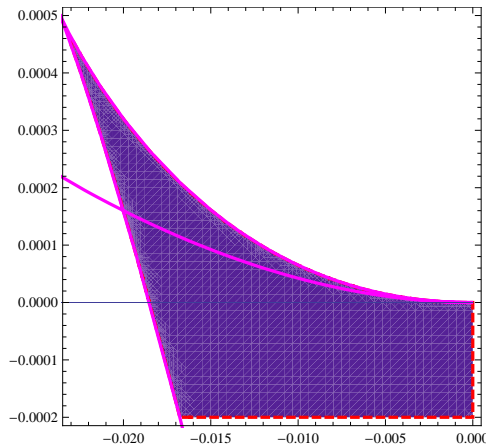
Figure 5: The parameter space for KdV-$\frac{1}{2}$ scaled so that $c = 1$. The dark region admits positive periodic traveling waves. Numerical calculation of the instability indices shows that the traveling waves do not have a real instability or a modulational instability.

Again the stability indices can be reduced to homogeneous polynomials of degree three and six in the quantities

$$\mu_j = \oint \frac{2v^j dv}{\sqrt{2(E + av^2 + \frac{c}{2}v^4 - v^5)}} \qquad\qquad j \in \{0\dots3\}$$

The expressions are a little large to write out, so we do not reproduce them here, but they are well-suited to numerical computations. Numerical evaluation of the analytic formulae suggests that the Hessian of the classical action always has one negative eigenvalue and two positive eigenvalues, leading to a count of

$$k_{\mathbb{R}} + 2k_{\mathbb{I}}^- + 2k_{\mathbb{C}} = 2k - 2$$

while the modulational instability index is always positive, indicating that in a neighborhood of the origin the imaginary axis is in the spectrum, with multiplicity three. Direct numerical simulations of the linearized eigenvalue problem support this conclusion.

6   Conclusions

We have proven an index theorem for the linearization of Korteweg-de Vries type flows around a traveling wave solution and shown that the number of eigenvalues in the right half-plane plus the number of purely imaginary eigenvalues of negative Krein signature given be expressed in terms of the Hessian of the classical action of the traveling wave ordinary differential equation or (equivalently) in terms of the Jacobian of the map from the Lagrange multipliers to the conserved quantities. In the case of polynomial nonlinearity these quantities can be expressed in terms of homogeneous polynomials in Abelian integrals on a finite genus Riemann surface.

The main drawback of the result is that it does not really distinguish between the eigenvalues in the right half-plane, which lead to an instability, and the imaginary eiegnvalues of negative Krein signature, which are generally not expected to lead to an instability. The index does distinguish between real eigenvalues and imaginary eigenvalues of negative Krein signature, but only modulo two. This is sufficient to deal with the solitary wave case, where the only possible instability mechanism is the emergence of a single real eigenvalue from the origin. However in the periodic

problem, where the behavior of the spectral problem is much richer, it would be preferable to have more information.

The modulational instability index gives some additional information about the stability of solutions. Roughly this quantity allows one to distinguish between imaginary eigenvalues of negative Krein signature and complex eigenvalues *in a neighborhood of the origin.* However by the nature of the way it was derived it does not allow one to conclude global information. We believe that a stronger result is possible: namely that there is spectrum off of the imaginary axis if and only if the modulational instability index is negative. In numerical experiments that we have conducted this has always been true: if the solution is unstable then the modulational instability index is negative. While we have some ideas of how one might attempt to prove this using Krein signature arguments we currently do not have a proof.

It is worth noting that there is a large literature devoted to estimating the number of zeroes of period integrals in connection with the so-called infinitesimal sixteenth Hilbert problem or Arnold-Hilbert problem (see problem 7 in the survey of Smale [36]). This problem is obviously closely connected with the one of determining the sign of such period integrals, and techniques from the former problem might be useful in analyzing the stability of periodic waves. In fact there have been a few papers in this direction already[23, 21, 9]. Also Hessians of conservation laws arise in the Whitham theory of *integrable* systems[28], and some of the techniques developed there may be of use in the current problem.

References

[1] J. Angulo Pava. Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations. *J. Differential Equations*, 235(1):1–30, 2007.

[2] J. Angulo Pava, J. L. Bona, and M. Scialom. Stability of cnoidal waves. *Adv. Differential Equations*, 11(12):1321–1374, 2006.

[3] T. B. Benjamin. The stability of solitary waves. *Proc. Roy. Soc. (London) Ser. A*, 328:153–183, 1972.

[4] E.D. Beolokolos, A.I. Bobenko, V.Z. Enol'skii, A.R. I ts, and V.B Matveev. *Algebro-geometric approach to nonlinear integrable equations.* Springer-Verlag, Berlin, 1994.

[5] A. L. Bertozzi and M. C. Pugh. Long-wave instabilities and saturation in thin film equations. *Comm. Pure Appl. Math.*, 51(6):625–661, 1998.

[6] J. Bona. On the stability theory of solitary waves. *Proc. Roy. Soc. London Ser. A*, 344(1638):363–374, 1975.

[7] J. L. Bona, P. E. Souganidis, and W. A. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London Ser. A*, 411(1841):395–412, 1987.

[8] N. Bottman and B. Deconinck. Kdv cnoidal waves are linearly stable. preprint.

[9] T.J. Bridges and G. Rowlands. Instability of spatially quasi-periodic states of the Ginzburg-Landau equation. *Proc. Roy. Soc. London Ser. A*, 444(1921):347–362, 1994.

[10] J. C. Bronski and R. L. Jerrard. Soliton dynamics in a potential. *Math. Res. Lett.*, 7(2-3):329–342, 2000.

[11] J. C. Bronski and M. Johnson. The modulational instability for a generalized KdV equation. To Appear: *Archive for Rational Mechanics and Analysis DOI: 10.1007/s00205-009-0270-5* .

[12] B. Deconinck and T. Kapitula. On the orbital (in)stability of spatially periodic stationary solutions of generalized korteweg-de vries equations. *submitted*, 2009.

[13] L. Demanet and W. Schlag. Numerical verification of a gap condition for a linearized nonlinear Schrödinger equation. *Nonlinearity*, 19(4):829–852, 2006.

[14] G. Fibich, F. Merle, and P. Raphaël. Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation. *Phys. D*, 220(1):1–13, 2006.

[15] T. Gallay and M. Hǎrǎguş. Orbital stability of periodic waves for the nonlinear Schrödinger equation. *J. Dynam. Differential Equations*, 19(4):825–865, 2007.

[16] T. Gallay and M. Hǎrǎguş. Stability of small periodic waves for the nonlinear Schrödinger equation. *J. Differential Equations*, 234(2):544–581, 2007.

[17] Z. Gang and I. M. Sigal. On soliton dynamics in nonlinear Schrödinger equations. *Geom. Funct. Anal.*, 16(6):1377–1390, 2006.

[18] Z. Gang and I. M. Sigal. Relaxation of solitons in nonlinear Schrödinger equations with potential. *Adv. Math.*, 216(2):443–490, 2007.

[19] Z. Gang and M.I. Weinstein. Dynamics of nonlinear Schrödinger / Gross-Pitaevskii equation; Mass transfer in systems with solitons and degenerate neutral modes,. *Analysis and PDE,*, 1(3):267–322., 2008.

[20] R. A. Gardner. Spectral analysis of long wavelength periodic waves and applications. *J. Reine Angew. Math.*, 491:149–181, 1997.

[21] Lubomir Gavrilov. The infinitesimal 16th Hilbert problem in the quadratic case. *Invent. Math.*, 143(3):449–497, 2001.

[22] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I,II. *J. Funct. Anal.*, 74(1):160–197,308–348, 1987.

[23] Sevdzhan Hakkaev, Iliya D. Iliev, and Kiril Kirchev. Stability of periodic travelling shallow-water waves determined by Newton's equation. *J. Phys. A*, 41(8):085203, 31, 2008.

[24] M. Hǎrǎguş and T. Kapitula. On the spectra of periodic waves for infinite-dimensional Hamiltonian sytems. *To appear: Physica D*.

[25] Wolfram Research Inc. Mathematica, version 7.0, Wolfram Research inc., Champaign, Illinois (2009).

[26] M. Johnson. Nonlinear stability of periodic traveling wave solutions of the generalized korteweg-de vries equation. preprint.

[27] E. Kirr and A. Zarnescu. On the asymptotic stability of bound states in 2D cubic Schrödinger equation. *Comm. Math. Phys.*, 272(2):443–468, 2007.

[28] I. M. Krichever. The "Hessian" of integrals of the Korteweg-de Vries equation and perturbations of finite-gap solutions. *Dokl. Akad. Nauk SSSR*, 270(6):1312–1317, 1983.

[29] J. Krieger and W. Schlag. Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension. *J. Amer. Math. Soc.*, 19(4):815–920 (electronic), 2006.

[30] Y. Martel and F. Merle. Asymptotic stability of solitons of the gKdV equations with general nonlinearity. *Math. Ann.*, 341(2):391–427, 2008.

[31] Y. Martel and F. Merle. Stability of two soliton collision for nonintegrable gKdV equations. *Comm. Math. Phys.*, 286(1):39–79, 2009.

[32] R. L. Pego and M.I. Weinstein. Asymptotic stability of solitary waves. *Comm. Math. Phys.*, 164(2):305–349, 1994.

[33] R.L. Pego and M.I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A*, 340(1656):47–94, 1992.

[34] G. Rowlands. On the stability of solutions of the non-linear schrödinger equation. *J. Inst. Maths Applics*, 13:367–377, 1974.

[35] R. Schaaf. A class of hamiltonian systems with increasing periods. *J. Reine Agnew. Math*, 363:96–109, 1985.

[36] Steve Smale. Dynamics retrospective: great problems, attempts that failed. *Phys. D*, 51(1-3):267–273, 1991. Nonlinear science: the next decade (Los Alamos, NM, 1990).

[37] A. Soffer. Soliton dynamics and scattering. In *International Congress of Mathematicians. Vol. III*, pages 459–471. Eur. Math. Soc., Zürich, 2006.

[38] A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, 133(1):119–146, 1990.

[39] A. Soffer and M. I. Weinstein. Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. *Invent. Math.*, 136(1):9–74, 1999.

[40] Avy Soffer. Soliton dynamics and scattering. In *International Congress of Mathematicians. Vol. III*, pages 459–471. Eur. Math. Soc., Zürich, 2006.

[41] M.I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

[42] M.I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive equations. *Comm. Pure and Appl. Math*, 39:51–68, 1986.

[43] M.I. Weinstein. Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation. *Comm. Partial Differential Equations*, 12(10):1133–1173, 1987.

[44] V.A. Yakubovich and V.M. Starzhinskii. *Linear Differential Equations with Periodic Coefficients I,II*. Wiley, 1975.