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Nonlinear Stability of Periodic Traveling-Wave Solutions of Viscous Conservation Laws in Dimensions One and Two*

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Abstract. Extending results of Oh and Zumbrun in dimensions $d \geq 3$, we establish nonlinear stability and asymptotic behavior of spatially periodic traveling-wave solutions of viscous systems of conservation laws in critical dimensions $d = 1, 2$, under a natural set of spectral stability assumptions introduced by Schneider in the setting of reaction diffusion equations. The key new steps in the analysis beyond that in dimensions $d \geq 3$ are a refined Green function estimate separating off translation as the slowest decaying linear mode and a novel scheme for detecting cancellation at the level of the nonlinear iteration in the Duhamel representation of a modulated periodic wave.

Key words. periodic traveling waves, Bloch decomposition, modulated waves

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1. Introduction.

Extending previous investigations of Oh and Zumbrun [17] in dimensions three and higher, we study stability of periodic traveling-wave solutions of systems of viscous conservation laws in the critical dimensions one and two. Our main result, generalizing those of [17] in dimensions $d \geq 3$, is to show that *strong spectral stability in the sense of Schneider* [19, 20, 21] *implies linearized and nonlinear* $L^1 \cap H^K \to L^\infty$ *bounded stability* for all dimensions $d \geq 1$, and *asymptotic stability* for dimensions $d \geq 2$.

More precisely, we show that small $L^1 \cap H^s$ perturbations of a planar periodic solution $u(x,t) \equiv \bar{u}(x_1)$ (without loss of generality taken stationary) converge at Gaussian rate in L^p, $p \geq 2$, to a modulation

$$\bar{u}(x_1 - \psi(x,t)) \tag{1.1}$$

of the unperturbed wave, where $x = (x_1, \tilde{x})$, $\tilde{x} = (x_2, \ldots, x_d)$, and ψ is a scalar function whose x- and t-gradients likewise decay at least at Gaussian rate in all L^p, $p \geq 2$, but which itself decays more slowly by a factor $t^{1/2}$; in particular, ψ is merely bounded in L^∞ for dimension $d = 1$.

The study of stability of spatially periodic traveling waves of systems of viscous conservation laws was initiated by Oh and Zumbrun [14] with a spectral stability analysis in one spatial dimension, carried out by direct Evans function computation under the restrictive assumption that wave speed be constant to first order along the manifold of nearby periodic

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solutions. This restriction was removed by Serre [22] by a quite different Evans function computation relating the linearized dispersion relation (the function $\lambda(\xi)$ relating spectra to the wave number of the linearized operator about the wave) near zero and the formal Whitham averaged system obtained by slow modulation, or WKB, approximation. This had the important further philosophical consequence of rigorously relating low-frequency stability to the usual physical definition derived through formal consistency considerations of modulational stability as hyperbolicity of the Whitham system; see [22, 16, 8] for further discussion.

In [16], this was extended to multiple dimensions, relating the linearized dispersion relation near zero to

$$\begin{aligned} &\partial_t M + \sum_j \partial_{x_j} F^j = 0, \\ &\partial_t(\Omega N) + \nabla_x(\Omega S) = 0, \end{aligned} \tag{1.2}$$

where $M \in \mathbb{R}^n$ denotes the average over one period, F^j the average of an associated flux, $\Omega = |\nabla_x \Psi| \in \mathbb{R}^1$ the frequency, $S = -\Psi_t/|\nabla_x \Psi| \in \mathbb{R}^1$ the speed s, and $N = \nabla_x \Psi/|\nabla_x \Psi| \in \mathbb{R}^d$ the normal ν associated with nearby periodic waves, with an additional constraint

$$\operatorname{curl}(\Omega N) = \operatorname{curl} \nabla_x \Psi \equiv 0. \tag{1.3}$$

As an immediate corollary, similarly to the one-dimensional case in [14, 22], this yielded as a necessary condition for multidimensional stability hyperbolicity of the averaged system (1.2)–(1.3).

The present study is informed by but does not directly rely on this observation relating Whitham averaging and spectral stability properties. Likewise, the Evans function techniques used in [22, 16] to establish this connection play no role in our analysis; indeed, the Evans function makes no appearance here. Rather, we rely on a direct Bloch-decomposition argument in the spirit of Schneider [19, 20, 21], combining sharp linearized estimates with subtle cancellation in nonlinear source terms arising from the modulated wave approximation.

The analytical techniques used to realize this program are somewhat different from those of [19, 20, 21], however, coming instead from the theory of stability of viscous shock fronts through a line of investigation carried out in [14, 15, 16, 17, 3]. In particular, the nonsmooth dispersion relation at $\xi = 0$ typical for convection-diffusion equations (see Remarks 1.1 and 2.4) requires different treatment in obtaining linear estimates from that of [19, 20, 21] in the reaction diffusion case; see [17] for further discussion. Moreover, we detect nonlinear cancellation in the physical x-t domain rather than the frequency domain as in [19, 20, 21]. This is important for our nonlinear analysis, which relies on direct estimates as in [17] rather than renormalization group techniques as in [19, 20, 21]; we note that the presence of multiple, distinct speeds of propagation in the asymptotic behavior of our system seems to preclude the use of renormalization in any obvious way. The main difference between the present analysis and that of [17] is the systematic incorporation of modulation approximation (1.1).

1.1. Equations and assumptions. Consider a parabolic system of conservation laws

$$u_t + \sum_j f^j(u)_{x_j} = \Delta_x u, \tag{1.4}$$

$u \in \mathcal{U}(\text{open}) \in \mathbb{R}^n$, $f^j \in \mathbb{R}^n$, $x \in \mathbb{R}^d$, $d \geq 1$, $t \in \mathbb{R}^+$, and a periodic traveling-wave solution

$$u = \bar{u}(x \cdot \nu - st), \tag{1.5}$$

of period X, satisfying the traveling-wave ODE $\bar{u}'' = (\sum_j \nu_j f^j(\bar{u}))' - s\bar{u}'$ with boundary conditions $\bar{u}(0) = \bar{u}(X) =: u_0$. Integrating, we obtain a first-order profile equation

$$\bar{u}' = \sum_j \nu_j f^j(\bar{u}) - s\bar{u} - q, \tag{1.6}$$

where $(u_0, q, s, \nu, X) \equiv \text{const}$. Without loss of generality, take $\nu = e_1$ and $s = 0$, so that $\bar{u} = \bar{u}(x_1)$ represents a stationary solution depending only on x_1.

Following [22, 16, 17], we assume the following:

(H1) $f^j \in C^{K+1}$, $K \geq [d/2] + 4$.

(H2) The map $H : \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times S^{d-1} \times \mathbb{R}^n \to \mathbb{R}^n$ taking $(X; a, s, \nu, q) \mapsto u(X; a, s, \nu, q) - a$ is full rank at point $(\bar{X}; \bar{u}(0), 0, e_1, \bar{q})$, where $u(\cdot; \cdot)$ is the solution operator for (1.6).

By the Implicit Function Theorem, conditions (H1)–(H2) imply that the set of periodic solutions in the vicinity of $\bar{u}$ form a smooth $(n + d + 1)$-dimensional manifold $\{\bar{u}^a(x \cdot \nu(a) - \alpha - s(a)t)\}$, with $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^{n+d}$.

1.1.1. Linearized equations. Linearizing (1.4) about $\bar{u}(\cdot)$, we obtain

$$v_t = Lv := \Delta_x v - \sum (A^j v)_{x_j}, \tag{1.7}$$

where coefficients $A^j := Df^j(\bar{u})$ are now periodic functions of x_1. Taking the Fourier transform in the transverse coordinate $\tilde{x} = (x_2, \dots, x_d)$, we obtain

$$\hat{v}_t = L_{\tilde{\xi}} \hat{v} = \hat{v}_{x_1, x_1} - (A^1 \hat{v})_{x_1} - i \sum_{j \neq 1} A^j \xi_j \hat{v} - \sum_{j \neq 1} \xi_j^2 \hat{v}, \tag{1.8}$$

where $\tilde{\xi} = (\xi_2, \dots, \xi_d)$ is the transverse frequency vector.

1.1.2. Bloch–Fourier decomposition and stability conditions. Following [1, 19, 20, 21], we define the family of operators

$$L_\xi := e^{-i\xi_1 x_1} L_{\tilde{\xi}} e^{i\xi_1 x_1} \tag{1.9}$$

operating on the class of L^2 periodic functions on $[0, X]$. The (L^2) spectrum of $L_{\tilde{\xi}}$ is the union of the spectra of all L_ξ with ξ_1 real, with associated eigenfunctions

$$w(x_1, \xi_1, \tilde{\xi}, \lambda) := e^{i\xi_1 x_1} q(x_1, \xi_1, \tilde{\xi}, \lambda), \tag{1.10}$$

where q is a periodic eigenfunction of L_ξ on $[0, X]$.

Without loss of generality, taking $X = 1$, recall now the *Bloch–Fourier representation*

$$u(x) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi \cdot x} \hat{u}(\xi, x_1) d\xi_1 \, d\tilde{\xi} \tag{1.11}$$

of an L^2 function u, where $\hat{u}(\xi, x_1) := \sum_k e^{2\pi i k x_1} \hat{u}(\xi_1 + 2\pi k, \tilde{\xi})$ are periodic functions of period $X = 1$ and $\hat{u}(\xi)$ is the Fourier transform of u in the full variable x. By Parseval's identity, the Bloch–Fourier transform $u(x) \to \hat{u}(\xi, x_1)$ is an isometry in L^2:

$$\|u\|_{L^2(x)} = \|\hat{u}\|_{L^2(\xi; L^2(x_1))}, \tag{1.12}$$

where $L^2(x_1)$ is taken on $[0, 1]$ and $L^2(\xi)$ on $[-\pi, \pi] \times \mathbb{R}^{d-1}$. Moreover, a straightforward computation reveals that it diagonalizes the periodic-coefficient operator L, with diagonal part L_ξ, yielding the *inverse Bloch–Fourier transform representation*

$$e^{Lt} u_0 = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x} e^{L_\xi t} \hat{u}_0(\xi, x_1) d\xi_1 \, d\tilde{\xi}. \tag{1.13}$$

Following [17], we assume along with (H1)–(H2) the following *strong spectral stability* conditions:

(D1) $\sigma(L_\xi) \subset \{\mathrm{Re}\lambda < 0\}$ for $\xi \neq 0$.

(D2) $\mathrm{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}^d$ and $|\xi|$ sufficiently small.

(D3) $\lambda = 0$ is a semisimple eigenvalue of L_0 of multiplicity exactly $n + 1$.[1]

For each fixed angle $\hat{\xi} := \xi/|\xi|$, expand $L_\xi = L_0 + |\xi| L^1 + |\xi|^2 L^2$. By assumption (D3) and standard spectral perturbation theory, there exist $n + 1$ eigenvalues

$$\lambda_j(\xi) = -i a_j(\xi) + o(|\xi|), \tag{1.14}$$

smooth with respect to $|\xi|$, of L_ξ bifurcating from $\lambda = 0$ at $\xi = 0$, where $-ia_j$ are homogeneous degree one functions given by $|\xi|$ times the eigenvalues of $\Pi_0 L^1|_{\mathrm{Ker} L_0}$, with Π_0 the zero eigenprojection of L_0.

Conditions (D1)–(D3) are exactly the spectral assumptions of [19, 20, 21]. As in [17], we make the following nondegeneracy hypothesis:

(H3) The eigenvalues $\lambda = -ia_j(\xi)/|\xi|$ of $\Pi_0 L^1_{\mathrm{Ker} L_0}$ are simple.

The functions a_j may be seen to be the characteristics associated with the Whitham averaged system (1.2)–(1.3) linearized about the values of M, S, N, and Ω associated with the background wave $\bar{u}$; see [16, 17]. Thus, (D1) implies weak hyperbolicity of (1.2)–(1.3) (reality of a_j), while (H1) corresponds to strict hyperbolicity.

Remark 1.1. Note that we do not assert smoothness of $\lambda_j(\cdot)$ with respect to ξ, and the relation to the Whitham averaged system shows that in general this does not hold.

1.2. Main results. With these preliminaries, we can now state our main results. The first concerns the stability of periodic standing waves of (1.4) in dimension $d = 1$, and the second concerns the case $d = 2$.

Theorem 1.2. *Let $\bar{u}(x)$ be a periodic standing-wave solution of (1.4) in the case $d = 1$, and let $\tilde{u}(x, t)$ be any solution of (1.4) such that $\|\tilde{u} - \bar{u}\|_{L^1 \cap H^K}\big|_{t=0}$ is sufficiently small.*

[1]The zero eigenspace of L_0 is at least $(n + 1)$-dimensional by the linearized existence theory and (H2), and hence $n + 1$ is the minimal multiplicity; see [22, 16]. As noted in [14, 16], the minimal dimension of this zero eigenspace implies that $(M, N\Omega)$ of (1.2) gives a nonsingular coordinatization of the family of periodic traveling-wave solutions near $\bar{u}$.

Then assuming (H1)–(H3) *and* (D1)–(D3)*, there exist a constant* $C > 0$ *and a function* $\psi(\cdot, t) \in W^{K,\infty}(\mathbb{R})$ *such that for all* $t \geq 0$*,* $p \geq 2$*, and* $d = 1$ *we have the estimates*

$$\begin{aligned} \|\tilde{u} - \bar{u}(\cdot - \psi)\|_{L^p}(t) &\leq C(1+t)^{-\frac{d}{2}(1-1/p)}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \\ \|\tilde{u} - \bar{u}(\cdot - \psi)\|_{H^K}(t) &\leq C(1+t)^{-\frac{d}{4}}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \\ \|(\psi_t, \psi_x)\|_{H^K} &\leq C(1+t)^{-\frac{d}{4}}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \end{aligned} \tag{1.15}$$

and

$$\|\tilde{u} - \bar{u}\|_{L^p}(t),\ \|\psi(\cdot, t)\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})+\frac{1}{2}}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}. \tag{1.16}$$

In particular, $\bar{u}$ *is nonlinearly bounded* $L^1 \cap H^K \to L^\infty$ *stable for dimension* $d = 1$*.*

Theorem 1.3. *Let* $\bar{u}(x_1)$ *be a periodic standing-wave solution of* (1.4) *in the case* $d = 2$*, and let* $\tilde{u}(x, t)$ *be any solution of* (1.4) *such that* $\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}\big|_{t=0}$ *is sufficiently small. Then assuming* (H1)–(H3) *and* (D1)–(D3)*, for any* $\varepsilon > 0$ *there exist a constant* $C > 0$ *and a function* $\psi(\cdot, t) \in W^{K,\infty}(\mathbb{R}^2)$ *such that for all* $t \geq 0$*,* $p \geq 2$*, and* $d = 2$ *we have the estimates*

$$\begin{aligned} \|\tilde{u} - \bar{u}(\cdot - \psi)\|_{L^p}(t) &\leq C(1+t)^{-\frac{d}{2}(1-1/p)}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \\ \|\tilde{u} - \bar{u}(\cdot - \psi)\|_{H^K}(t) &\leq C(1+t)^{-\frac{d}{4}}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \\ \|(\psi_t, \psi_x)\|_{H^K} &\leq C(1+t)^{-\frac{d}{4}+\varepsilon-\frac{1}{2}}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} \|\tilde{u} - \bar{u}\|_{L^p}(t),\ \|\psi(\cdot, t)\|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})+\varepsilon}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}, \\ \|\tilde{u} - \bar{u}\|_{H^K}(t),\ \|\psi(\cdot, t)\|_{H^K} &\leq C(1+t)^{-\frac{d}{4}+\varepsilon}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}. \end{aligned} \tag{1.18}$$

In particular, $\bar{u}$ *is nonlinearly asymptotically* $L^1 \cap H^K \to H^K$ *stable for dimension* $d = 2$*.*

Remark 1.4. In Theorem 1.3, derivatives in $x \in \mathbb{R}^2$ refer to total derivatives. Moreover, unless specified by an appropriate index, throughout this paper derivatives in spatial variable x will always refer to the total derivative of the function.

In dimension one, Theorem 1.2 asserts only bounded $L^1 \cap H^K \to L^\infty$ stability, a very weak notion of stability. The absence of decay in perturbation $\tilde{u} - \bar{u}$ indicates the delicacy of the nonlinear analysis in this case. In particular, it is crucial to separate the nondecaying modulated behavior (1.1) from the remaining decaying part of the perturbed solution in order to close the nonlinear iteration argument.

Remark 1.5. In dimension $d = 1$, it is straightforward to show that the results of Theorem 1.2 extend to all $1 \leq p \leq \infty$ using the pointwise techniques of [15]; see Remark 3.4.

Remark 1.6. The slow decay of $\|\tilde{u} - \bar{u}\|_{L^p}(t) \sim \|\psi(\cdot, t)\|_{L^p}$ in (1.16) is due to nonlinear interactions; as shown in [15, 17], the linearized decay rate is faster by factor $(1 + t)^{-1/2}$ (Proposition 2.6). In [17], it was shown that for $d \geq 3$, where linear effects dominate behavior, (1.16) may be replaced by the stronger estimate

$$\|\tilde{u} - \bar{u}\|_{L^p}(t),\ \|\psi(\cdot, t)\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}\|\tilde{u} - \bar{u}\|_{L^1\cap H^K}|_{t=0}.$$

These distinctions reflect fine details of both the linearized estimates (section 3) and the nonlinear structure (sections 4.1–4.2) that are not immediately apparent from the formal Whitham approximation (1.2)–(1.3).

1.3. Discussion and open problems. Linearized stability under the same assumptions, with sharp rates of decay, was established for $d = 1$ [15] and for $d \geq 1$ in [17], along with nonlinear stability for $d \geq 3$. Theorem 1.2 completes this line of investigation by establishing nonlinear stability in the critical dimensions $d = 1, 2$, a fundamental open problem cited in [14, 17].

This gives a generalization of the work of [19, 20, 21] for reaction diffusion equations to the case of viscous conservation laws.[2] Recall that the analysis of [19, 20, 21] concerns also multiply periodic waves, i.e., waves that are either periodic or else constant in each coordinate direction. It is straightforward to verify that the methods of this paper apply essentially unchanged to this case to give a corresponding stability result under the analogue of (H1)–(H3), (D1)–(D3), as we intend to report further in a future work. Likewise, the extension from the semilinear parabolic case treated here to the general quasilinear case is straightforward, following the treatment of [17].[3]

On the other hand, as noted in [15], condition (D3) is in the conservation law setting nongeneric, corresponding mainly to the special "quasi-Hamiltonian" situation studied there; in particular, it implies that speed is to first order constant among the family of spatially periodic traveling-wave solutions nearby $\bar{u}$. In the generic case that (D3) is violated, behavior is essentially different [14, 15], and perturbations decay more slowly at the linearized level. Nonlinear stability remains an interesting open problem in this setting.

Our approach to stability in the critical dimensions $d = 1, 2$, as suggested in [17], is, loosely following the approach of [19, 20, 21], to subtract a more slowly decaying part of the solution described by an appropriate modulation equation and show that the residual decays sufficiently rapidly to close a nonlinear iteration. It is worth noting that the modulated approximation $\bar{u}(x_1 - \psi(x,t))$ of (1.1) is not the full Ansatz

$$\bar{u}^a(\Psi(x,t)), \tag{1.19}$$

$\Psi(x,t) := x_1 - \psi(x,t)$, associated with the Whitham averaged system (1.2)–(1.3), where $\bar{u}^a$ is the manifold of periodic solutions near $\bar{u}$ introduced below (H2), but only the translational part not involving perturbations a in the profile. (See [16] for the derivation of (1.2)–(1.3) and (1.19).) That is, we do not need to separate all variations along the manifold of periodic solutions, but only the special variations connected with translation invariance.

The technical reason is an asymmetry in the y-derivative estimates in the parts of the Green function associated with these various modes, something that is not apparent without a detailed study of linearized behavior as carried out here. This also makes sense formally if one considers that (1.2) indicates that variables a, $\nabla_x \Psi$ are roughly comparable, which would suggest, by the diffusive behavior $\Psi >> |\nabla_x \Psi|$, that a is negligible with respect to Ψ.

However, note that in the case that (D3) holds (hence wave speed is stationary along the manifold of periodic solutions), the final equation of (1.2) decouples to $(\Psi_x)_t = (\Omega N)_t = 0$ and could be written as $\Psi_t = 0$ in terms of Ψ alone. Hence, there is some ambiguity in this degenerate case, of which Ψ, Ψ_x is the primary variable, and in terms of linear behavior, the decay of variations a and Ψ are in fact comparable [17]; in the generic case, a and Ψ_x are

[2] In methods as well as results; see again Remarks 1.1 and 2.4.

[3] See also the preprint [9] completed after this work, treating the quasilinear partially parabolic case.

comparable at the linearized level [15]. It would be very interesting to better understand the connection between the Whitham averaged system (or a suitable higher-order correction) and behavior at the nonlinear level, as explored at the linear level in [16, 17, 5, 8].

Note. Since the completion of this analysis, there have been rapid further developments extending the techniques introduced here. In particular, by a refined linear analysis suggested by more careful consideration of the structure of the Whitham expansion, we have answered the main open question cited above, showing that spectral stability implies nonlinear stability also in the generic case that (D3) is violated; i.e., wave speed is not stationary along the manifold of periodic solutions [6].[4] An interesting point missed in the original discussion is that the nonlinear analysis of [19] in the reaction diffusion case, based on renormalization group methods, *also applies only in the special case of stationary wave speed*, even though (D3) is not violated in that case; see [7] for further discussion. We remove this restriction, too, in [7], by a modification of the methods used here. Finally, we mention the extension in [9] of our results to periodic roll-wave solutions of the St. Venant equations of shallow water flow, which are equations of quasilinear, partially parabolic form.

2. Basic linearized stability estimates.

We begin by recalling the basic linearized stability estimates derived in [17], repeating in their entirety both statements and proofs (the latter both for completeness and for later reference). We will sharpen these afterward in section 3.

By standard spectral perturbation theory [10], the total eigenprojection $P(\xi)$ onto the eigenspace of L_ξ associated with the eigenvalues $\lambda_j(\xi)$, $j = 1, \dots, n+1$, described in the introduction is well defined and analytic in ξ for ξ sufficiently small, since these (by discreteness of the spectra of L_ξ) are separated at $\xi = 0$ from the rest of the spectrum of L_0. Introducing a smooth cut-off function $\phi(\xi)$ that is identically one for $|\xi| \le \varepsilon$ and identically zero for $|\xi| \ge 2\varepsilon$, $\varepsilon > 0$ sufficiently small, we split the solution operator $S(t) := e^{Lt}$ into low- and high-frequency parts

$$S^I(t)u_0 := \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi(\xi)P(\xi)e^{L_\xi t}\hat{u}_0(\xi, x_1)d\xi_1\, d\tilde{\xi} \tag{2.1}$$

and

$$S^{II}(t)u_0 := \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}(I - \phi P(\xi))e^{L_\xi t}\hat{u}_0(\xi, x_1)d\xi_1\, d\tilde{\xi}. \tag{2.2}$$

2.1. High-frequency bounds.

By standard sectorial bounds [2, 18] and spectral separation of $\lambda_j(\xi)$ from the remaining spectra of L_ξ, we have trivially the exponential decay bounds

$$\begin{aligned} \|e^{L_\xi t}(I - \phi P(\xi))f\|_{L^2([0,X])} &\le Ce^{-\theta t}\|f\|_{L^2([0,X])}, \\ \|e^{L_\xi t}(I - \phi P(\xi))\partial_{x_1}^l f\|_{L^2([0,X])} &\le Ct^{-\frac{l}{2}}e^{-\theta t}\|f\|_{L^2([0,X])}, \\ \|\partial_{x_1}^l e^{L_\xi t}(I - \phi P(\xi))f\|_{L^2([0,X])} &\le Ct^{-\frac{l}{2}}e^{-\theta t}\|f\|_{L^2([0,X])} \end{aligned} \tag{2.3}$$

[4] To prevent possible confusion, we note that the degenerate case treated here is distinct from the generic case treated in [6] and requires a separate analysis. In particular, the proof of the key Proposition 3.4 in [6] (linearized bounds) refers to the present paper for the proof in the degenerate case.

for θ, $C > 0$, and $0 \le m \le K$ (K as in (H1)). Together with (1.12), these give immediately the following estimates.

Proposition 2.1 (see [17]). *Under assumptions* (H1)–(H3) *and* (D1)–(D2), *for some* θ, $C > 0$, *and all* $t > 0$, $2 \le p \le \infty$, $0 \le l \le K+1$, $0 \le m \le K$,

$$\begin{aligned} \|\partial_x^l S^{II}(t) f\|_{L^2(x)},\ \|S^{II}(t)\partial_x^l f\|_{L^2(x)} &\le C t^{-\frac{l}{2}} e^{-\theta t}\|f\|_{L^2(x)}, \\ \|\partial_x^m S^{II}(t) f\|_{L^p(x)},\ \|S^{II}(t)\partial_x^m f\|_{L^p(x)} &\le C t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{m}{2}} e^{-\theta t}\|f\|_{L^2(x)}. \end{aligned} \tag{2.4}$$

Proof (following [17]). The first inequalities follow immediately by (1.12). The second follow for $p = \infty$, $m = 0$ by Sobolev embedding from

$$\|S^{II}(t) f\|_{L^\infty(\tilde{x}; L^2(x_1))} \le C t^{-\frac{d-1}{4}} e^{-\theta t}\|f\|_{L^2([0,X])}$$

and

$$\|\partial_{x_1} S^{II}(t) f\|_{L^\infty(\tilde{x}; L^2(x_1))} \le C t^{-\frac{d-1}{4}-\frac{1}{2}} e^{-\theta t}\|f\|_{L^2([0,X])},$$

which follow by an application of (1.12) in the x_1 variable and the Hausdorff–Young inequality $\|f\|_{L^\infty(\tilde{x})} \le \|\hat{f}\|_{L^1(\tilde{\xi})}$ in the variable $\tilde{x}$. The result for derivatives in x_1 and general $2 \le p \le \infty$ then follows by L^p interpolation. Finally, the result for derivatives in $\tilde{x}$ follows from the inverse Fourier transform, (2.2), and the large $|\xi|$ bound

$$\|e^{Lt} f\|_{L^2(x_1)} \le e^{-\theta|\tilde{\xi}|^2 t}\|f\|_{L^2(x_1)}, \quad |\xi| \text{ sufficiently large,}$$

which easily follows from Parseval and the fact that L_ξ is a relatively compact perturbation of $\partial_x^2 - |\xi|^2$. Thus, by the above estimate we have

$$\begin{aligned} \|e^{Lt}\partial_{\tilde{x}} f\|_{L^2(x)} &\le C\|e^{L_\xi t}|\tilde{\xi}|\hat{f}\|_{L^2(x_1,\xi)} \\ &\le C \sup\left(e^{-\theta|\tilde{\xi}|^2 t}|\xi|\right)\|\hat{f}\|_{L^2(x_1,\xi)} \\ &\le C t^{-1/2}\|f\|_{L^2(x)}. \end{aligned}$$

A similar argument applies for $1 \le m \le K$.[5] ■

2.2. Low-frequency bounds.

Denote by

$$G^I(x,t;y) := S^I(t)\delta_y(x) \tag{2.5}$$

the Green kernel associated with S^I and by

$$[G_\xi^I(x_1,t;y_1)] := \phi(\xi)P(\xi)e^{L_\xi t}[\delta_{y_1}(x_1)] \tag{2.6}$$

the corresponding kernel appearing within the Bloch–Fourier representation of G^I, where the brackets around $[G_\xi]$ and $[\delta_y]$ denote the periodic extensions of these functions onto the whole line. Then, we have the following descriptions of G^I, $[G_\xi^I]$, reminiscent of those obtained for constant-coefficient operators by Fourier transform.

[5] Here, the separate treatment of $\tilde{x}$-derivatives repairs a minor omission in [17].

Proposition 2.2 (see [17]). *Under assumptions* (H1)–(H3) *and* (D1)–(D3),

$$(2.7)\qquad \begin{aligned} [G^I_\xi(x_1,t;y_1)] &= \phi(\xi)\sum_{j=1}^{n+1} e^{\lambda_j(\xi)t} q_j(\xi,x_1)\tilde q_j(\xi,y_1)^*,\\ G^I(x,t;y) &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}[G^I_\xi(x_1,t;y_1)]d\xi\\ &= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j=1}^{n+1} e^{\lambda_j(\xi)t} q_j(\xi,x_1)\tilde q_j(\xi,y_1)^* d\xi, \end{aligned}$$

where $*$ denotes matrix adjoint, or the complex conjugate of the matrix transpose, $q_j(\xi,\cdot)$ and $\tilde q_j(\xi,\cdot)$ are right and left eigenfunctions of L_ξ associated with eigenvalues $\lambda_j(\xi)$ defined in (1.14)*, normalized so that $\langle \tilde q_j, q_j\rangle \equiv 1$, where $\lambda_j/|\xi|$ is a smooth function of $|\xi|$ and $\hat\xi := \xi/|\xi|$, and q_j and $\tilde q_j$ are smooth functions of $|\xi|$, $\hat\xi := \xi/|\xi|$, and x_1 or y_1, with* $\mathrm{Re}\lambda_j(\xi) \le -\theta|\xi|^2$.

Proof (following [17]). Smooth dependence of λ_j and of q, $\tilde q$ as functions in $L^2[0,X]$ follows from standard spectral perturbation theory [10] using the fact that λ_j splits to first order in $|\xi|$ as ξ is varied along rays through the origin, and that L_ξ varies smoothly with angle $\hat\xi$. Smoothness of q_j, $\tilde q_j$ in x_1, y_1 then follows from the fact that they satisfy the eigenvalue equation for L_ξ, which has smooth, periodic coefficients. Likewise, (2.7) is immediate from the spectral decomposition of elliptic operators on finite domains. Substituting (2.5) into (2.1) and computing

$$(2.8)\qquad \widehat{\delta_y}(\xi,x_1) = \sum_k e^{2\pi i k x_1}\widehat{\delta_y}(\xi+2\pi k e_1) = \sum_k e^{2\pi i k x_1} e^{-i\xi\cdot y - 2\pi i k y_1} = e^{-i\xi\cdot y}[\delta_{y_1}(x_1)],$$

where the second and third equalities follow from the fact that the Fourier transform either continuous or discrete of the delta-function is unity, we obtain

$$\begin{aligned} G^I(x,t;y) &= \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi P(\xi) e^{L_\xi t}\widehat{\delta_y}(\xi,x_1)d\xi\\ &= \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}\phi P(\xi) e^{L_\xi t}[\delta_{y_1}(x_1)]d\xi, \end{aligned}$$

yielding (2.7)(ii) by (2.6)(i) and the fact that ϕ is supported on $[-\pi,\pi]$. ∎

Proposition 2.3 (see [17]). *Under assumptions* (H1)–(H3) *and* (D1)–(D3),

$$(2.9)\qquad \sup_y \|G^I(\cdot,t,;y)\|_{L^p(x)},\ \sup_y \|\partial_{x,y}G^I(\cdot,t,;y)\|_{L^p(x)} \le C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}$$

for all $2\le p\le\infty$, $t\ge 0$, where $C>0$ is independent of p.

Proof (following [17]). From representation (2.7)(ii) and $\Re\lambda_j(\xi) \le -\theta|\xi|^2$, we obtain by the triangle inequality

$$(2.10)\qquad \|G^I\|_{L^\infty(x,y)} \le C\|e^{-\theta|\xi|^2 t}\phi(\xi)\|_{L^1(\xi)} \le C(1+t)^{-\frac{d}{2}},$$

verifying the bounds for $p = \infty$. Derivative bounds follow similarly, since derivatives falling on q_j or $\tilde{q}_j$ are harmless, whereas derivatives falling on $e^{i\xi\cdot(x-y)}$ bring down a factor of ξ, which is again harmless because of the cut-off function ϕ.

To obtain bounds for $p = 2$, we note that (2.7)(ii) may itself be viewed as a Bloch–Fourier decomposition with respect to variable $z := x - y$, with y appearing as a parameter. Recalling (1.12), we may thus estimate

$$
\begin{aligned}
\sup_y \|G^I(x,t;y)\|_{L^2(x)} &= \sum_j \sup_y \|\phi(\xi)e^{\lambda_j(\xi)t}q_j(\cdot,z_1)\tilde{q}_j^*(\cdot,y_1)\|_{L^2(\xi;L^2(z_1\in[0,X]))} \\
&\le C\sum_j \sup_y \|\phi(\xi)e^{-\theta|\xi|^2 t}\|_{L^2(\xi)}\|q_j\|_{L^2(0,X)}\|\tilde{q}_j\|_{L^\infty(0,X)} \\
&\le C(1+t)^{-\frac{d}{4}},
\end{aligned}
\tag{2.11}
$$

where we have used in a crucial way the boundedness of $\tilde{q}_j$; derivative bounds follow similarly. Finally, bounds for $2 \le p \le \infty$ follow by L^p-interpolation. ■

Remark 2.4. As noted in [17], we have made essential use of the periodic structure of q_j, $\tilde{q}_j$ in obtaining the key L^2 estimates above by what is essentially a direct analogue of the simple Fourier transform argument typically used to treat the constant-coefficient case [3]. Viewed as a general pseudodifferential operator, G^I does not have sufficient smoothness (i.e., blow up in ξ derivatives at less than the critical rate $|\xi|^{-1}$) to apply the standard $L^2 \to L^2$ bounds of Hörmander [4]. Nor do the weighted energy estimate techniques used in [19, 20, 21] apply here, as these rely on C^k smoothness of λ_j, q_j, $\tilde{q}_j$ with respect to ξ at the origin $\xi = 0$, $k \ge 1$. The lack of smoothness of the linearized dispersion relation at the origin is an essential technical difference separating the conservation law from the reaction diffusion case [17].

Corollary 2.5 (see [17]). *Under assumptions* (H1)–(H3) *and* (D1)–(D3), *for all $p \ge 2$, $t \ge 0$,*

$$
\|S^I(t)f\|_{L^p},\ \|\partial_x S^I(t)f\|_{L^p},\ \|S^I(t)\partial_x f\|_{L^p} \le \quad C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}\|f\|_{L^1}. \tag{2.12}
$$

Proof. The proof is immediate from (2.9) and the triangle inequality, as, for example,

$$
\|S^I(t)f(\cdot)\|_{L^p} = \left\|\int_{\mathbb{R}^d} G^I(x,t;y)f(y)dy\right\|_{L^p(x)} \le \int_{\mathbb{R}^d}\sup_y \|G^I(\cdot,t;y)\|_{L^p}|f(y)|dy. \qquad ■
$$

Proposition 2.6 (see [17]). *Assuming* (H1)–(H3), (D1)–(D3) *for some $C > 0$, all $t \ge 0$, $p \ge 2$, $0 \le l \le K+1$, we have*

$$
\|S(t)\partial_x^l u_0\|_{L^p} \le Ct^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{l}{2}}(1+t)^{-\frac{d}{4}+\frac{l}{2}}\|u_0\|_{L^1\cap L^2}. \tag{2.13}
$$

Proof. The proof is immediate from (2.4) and (2.12). ■

2.3. Additional estimates.

Lemma 2.7. *Assuming* (H1)–(H3), (D1)–(D3) *for all $t \ge 0$, $0 \le l \le K$, we have*

$$
\|\partial_x^l S^I(t)f\|_{L^p(x)},\ \|S^I(t)\partial_x^l f\|_{L^p(x)} \le C(1+t)^{-\frac{d}{2}(1/2-1/p)}\|f\|_{L^2(x)}. \tag{2.14}
$$

Proof. From boundedness of the spectral projections $P_j(\xi) = q_j\langle \tilde q_j, \cdot\rangle$ in $L^2[0,X]$ and their derivatives, another consequence of first-order splitting of eigenvalues $\lambda_j(\xi)$ at the origin, we obtain boundedness of $\phi(\xi)P(\xi)e^{L_\xi t}$ and, thus, by (1.12), the global bounds

$$\|\partial_x^l S^I(t)f\|_{L^2(x)},\ \|S^I(t)\partial_x^l f\|_{L^2(x)} \le C\|f\|_{L^2(x)} \tag{2.15}$$

for all $t \ge 0$, yielding the result for $p = 2$. Moreover, by boundedness of $\tilde q$, q in all $L^p(x_1)$, we have

$$\|\phi(\xi)P(\xi)e^{L_\xi t}\hat f(\xi,\cdot)\|_{L^\infty(x_1)} \le Ce^{-\theta|\xi|^2 t}\|P(\xi)\hat f(\xi,\cdot)\|_{L^\infty(x_1)} \le Ce^{-\theta|\xi|^2 t}\|\hat f(\xi,\cdot)\|_{L^2(x_1)},$$

C, $\theta > 0$, yielding by $S^I f = (\frac{1}{2\pi})^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi(\xi)P(\xi)e^{L_\xi t}\hat f(\xi, x_1)d\xi_1\, d\tilde\xi$ the bound

$$\begin{aligned}
\|S^I(t)f\|_{L^\infty(x)} &\le \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} \|\phi(\xi)P(\xi)e^{L_\xi t}\hat f(\xi,\cdot)\|_{L^\infty(x_1)} d\xi_1\, d\tilde\xi \\
&\le \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} C\phi(\xi)e^{-\theta|\xi|^2 t}\|\hat f(\xi,\cdot)\|_{L^2(x_1)} d\xi_1\, d\tilde\xi \\
&\le C\|\phi(\xi)e^{-\theta|\xi|^2 t}\|_{L^2(\xi)}\|\hat f\|_{L^2(\xi,x_1)} \\
&= C(1+t)^{-\frac{d}{4}}\|f\|_{L^2([0,X])},
\end{aligned} \tag{2.16}$$

yielding the result for $p = \infty$, $l = 0$. The result for $p = \infty$, $1 \le l \le K+1$ follows by a similar argument. The result for general $2 \le p \le \infty$ then follows by L^p-interpolation between $p = 2$ and $p = \infty$. ∎

By Riesz–Thorin interpolation between (2.14) and (2.12), we obtain the following, apparently sharp, bounds between various L^q and L^p.[6]

Corollary 2.8 (see [17]). *Assuming* (H1)–(H3) *and* (D1)–(D3) *for all* $1 \le q \le 2 \le p$, $t \ge 0$, $0 \le l \le K+1$, *we have*

$$\|\partial_x^l S^I(t)f\|_{L^p},\ \|S^I(t)\partial_x^l f\|_{L^p} \le C(1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q}. \tag{2.17}$$

Proposition 2.9. *Assuming* (H1)–(H3), (D1)–(D3) *for some* $C > 0$, *all* $t \ge 0$, $1 \le q \le 2 \le p$, *and* $0 \le l \le K+1$, *we have*

$$\|S(t)\partial_x^l u_0\|_{L^p} \le Ct^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{l}{2}}(1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{2})+\frac{l}{2}}\|u_0\|_{L^q\cap L^2}. \tag{2.18}$$

Proof. The proof is immediate from (2.4) and (2.17). ∎

3. Refined linearized estimates. The bounds of Proposition 2.6 are sufficient to establish nonlinear stability and asymptotic behavior in dimensions $d \ge 3$, as shown in [17]. However, they are not sufficient in the critical dimensions $d = 1, 2$; see Remark 1, section 7, of [17]. Comparison with standard diffusive stability arguments as in [25] shows that this is due to the fact that the full solution operator $|S(t)\partial_x|$ decays no faster than $S(t)$, or, equivalently, G_y no faster than G.

Following the basic strategy introduced in [26, 27, 23, 11, 13] in the context of viscous shock waves, we now perform a refined linearized estimate separating slower-decaying translational

[6] The inclusion of general $p \ge 2$ in Lemma 2.7 repairs an omission in [17], where the bounds (2.17) were stated but not used.

modes from a faster-decaying "good" part of the solution operator. This will be used in section 4 in combination with certain nonlinear cancellation estimates to show convergence to the modulated approximation (1.1) at a faster rate sufficient to close the nonlinear iteration.

The key to this decomposition is the following observation.

Lemma 3.1. *Assuming* (H1)–(H3), (D1), (D3), *let* $\lambda_j(\xi/|\xi|,\xi)$, $q_j(\xi/|\xi|,\xi,\cdot)$, $\tilde q_j(\xi/|\xi|,\xi,\cdot)$ *denote the eigenvalues and associated right and left eigenfunctions of* L_ξ, *with* q_j, $\tilde q_j$ *smooth functions of* $\xi/|\xi|$ *and* $|\xi|$ *as noted in Proposition* 2.2. *Then, without loss of generality,* $q_1(\omega,0,\cdot)\equiv\bar u'$, *while* $\tilde q_j(\omega,0,\cdot)$ *for* $j\neq 1$ *are constant functions depending only on angle* $\omega=\xi/|\xi|$.

Proof. Expanding $L_\xi = L_0+|\xi|L^1_{\xi/|\xi|}+|\xi|^2L^2_{\xi/|\xi|}$ as in the introduction, consider the continuous family of spectral perturbation problems in $|\xi|$ indexed by angle $\omega=\xi/|\xi|$. Then, both facts follow by standard perturbation theory [10] using the observations that $\bar u'$ is in the right kernel of L_0 and constant functions c are in the left kernel of L_0, with

$$\langle c, L^1\bar u'\rangle = \left\langle c, \left(\omega_1(2\partial_{x_1}-A_1)-\sum_{j\neq 1}\omega_jA_j\right)\bar u'\right\rangle = \left\langle c, \omega_1\partial_{x_1}^2\bar u-\sum_{j\neq 1}\omega_j\partial_{x_1}f^j(\bar u)\right\rangle\equiv 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^2(x_1)$ inner product on the interval $x_1\in[0,X]$, that the dimension of $\ker L_0$ by assumption is $(n+1)$, so that the orthogonal complement of $\bar u'$ in $\mathrm{Ker}L_0$ is dimension n, so exactly the set of constant functions, and that by (H3) the functions $q_j(\omega,0,\cdot)$ and $\tilde q_j(\omega,0)$ are right and left eigenfunctions of $\Pi_0L^1|_{\ker L_0}$ (Π_0 as before denoting the zero eigenprojection associated with L_0). ■

Remark 3.2. The key observation of Lemma 3.1 can be motivated by the form of the Whitham averaged system (1.2). For, recalling (section 1.3) that (D3) implies that speed s is stationary to first order at $\bar u$ along the manifold of nearby periodic solutions, we find that the last equation of (1.2) reduces to $(\nabla_x\Psi)_t=0$; i.e., the equation for the translational variation Ψ decouples from the equations for variations in other modes. This corresponds heuristically to the fact derived above that the translational mode $\bar u'(x_1)$ decouples in the first-order eigenfunction expansion.

Corollary 3.3. *Under assumptions* (H1)–(H3), (D1)–(D3), *the Green function* $G(x,t;y)$ *of* (1.7) *decomposes as* $G=E+\tilde G$,

$$E=\bar u'(x)e(x,t;y), \tag{3.1}$$

where, for some $C>0$, *all* $t>0$, $1\le q\le 2\le p\le\infty$, $0\le j,k,l$, $j+l\le K+1$, $1\le r\le 2$,

$$\begin{aligned}
\left\|\int_{-\infty}^{+\infty}\tilde G(x,t;y)f(y)dy\right\|_{L^p(x)} &\le Ct^{-\frac d2(1/2-1/p)}(1+t)^{-\frac d2(1/q-1/2)}\|f\|_{L^q\cap L^2},\\
\left\|\int_{-\infty}^{+\infty}\partial_y^r\tilde G(x,t;y)f(y)dy\right\|_{L^p(x)} &\le Ct^{-\frac d2(1/2-1/p)-\frac r2}\\
&\quad\times(1+t)^{-\frac d2(1/q-1/2)-\frac12+\frac r2}\|f\|_{L^q\cap L^2},\\
\left\|\int_{-\infty}^{+\infty}\partial_t^r\tilde G(x,t;y)f(y)dy\right\|_{L^p(x)} &\le Ct^{-\frac d2(1/2-1/p)-r}\\
&\quad\times(1+t)^{-\frac d2(1/q-1/2)-\frac12+r}\|f\|_{L^q\cap L^2},
\end{aligned}\tag{3.2}$$

$$\left\| \int_{-\infty}^{+\infty} \partial_x^j \partial_t^k \partial_y^l e(x,t;y) f(y)dy \right\|_{L^p} \le (1+t)^{-\frac{d}{2}(1/q-1/p)-\frac{(j+k)}{2}} \|f\|_{L^q}. \tag{3.3}$$

Moreover, $e(x,t;y) \equiv 0$ *for* $t \le 1$.

Proof. We first treat the simpler case $q = 1$. Recalling that

$$G^I(x,t;y) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \phi(\xi) \sum_{j=1}^{n+1} e^{\lambda_j(\xi)t} q_j(\xi, x_1) \tilde{q}_j(\xi, y_1)^* d\xi, \tag{3.4}$$

define

$$\tilde{e}(x,t;y) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \phi(\xi) e^{\lambda_1(\xi)t} \tilde{q}_1(\xi, y_1)^* d\xi, \tag{3.5}$$

so that

$$\begin{aligned} &G^I(x,t;y) - \bar{u}'(x_1)\tilde{e}(x,t;y) \\ &\quad = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \phi(\xi) \sum_{j=2}^{n+1} e^{\lambda_j(\xi)t} q_j(\xi/|\xi|, 0, x_1) \tilde{q}_j(\xi, y_1)^* d\xi \\ &\qquad + \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \sum_{j=1}^{n+1} e^{i\xi\cdot(x-y)} \phi(\xi) e^{\lambda_j(\xi)t} O(|\xi|) d\xi. \end{aligned} \tag{3.6}$$

Noting, by Lemma 3.1, that $\partial_y \tilde{q}(\omega, 0, y) \equiv \text{const}$ for $j \ne 1$, we have therefore

$$\partial_y^r (G^I(x,t;y) - \bar{u}'(x_1)\tilde{e}(x,t;y)) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} \phi(\xi) \sum_{j=1}^{n+1} e^{\lambda_j(\xi)t} O(|\xi|) d\xi, \tag{3.7}$$

which readily gives

$$\|\partial_y^r (G^I(x,t;y) - \bar{u}'(x_1)\tilde{e}(x,t;y))\|_{L^p} \le C(1+t)^{-\frac{d}{2}(1-1/p)-\frac{1}{2}}, \tag{3.8}$$

$p \ge 2$, by the same argument used to prove (2.9), and similarly

$$\|\partial_t^r (G^I(x,t;y) - \bar{u}'(x_1)\tilde{e}(x,t;y))\|_{L^p} \le c(1+t)^{-\frac{d}{2}(1-1/p)-\frac{1}{2}}. \tag{3.9}$$

These yield (3.2) by the triangle inequality.

Defining $e(x,t;y) := \chi(t)\tilde{e}(x,t;y)$, where χ is a smooth cut-off function such that $\chi(t) \equiv 1$ for $t \ge 2$ and $\chi(t) \equiv 0$ for $t \le 1$, and setting $\tilde{G} := G - \bar{u}'(x_1)e(x,t;y)$, we readily obtain the estimates (3.2) by combining (3.9) with bound (2.4) on G^{II}. Bounds (3.3) follow from (3.5) by the argument used to prove (2.9), together with the observation that x- or t-derivatives bring down factors of $|\xi|$, followed again by an application of the triangle inequality.

The cases $1 \le q \le 2$ follow similarly, by the arguments used to prove (2.14) and (2.17). ■

Remark 3.4. Despite their apparent complexity, the above bounds may be recognized as essentially just the standard diffusive bounds satisfied for the heat equation [25]. For dimension

$d = 1$, it may be shown using pointwise techniques as in [15] that the bounds of Corollary 3.3 extend to all $1 \le q \le p \le \infty$.

Note the strong analogy between the Green function decomposition of Corollary 3.3 and that of [12, 24] in the viscous shock case. We pursue this analogy further in the nonlinear analysis of the following sections, combining the "instantaneous tracking" strategy of [26, 23, 24, 25, 11, 13] with a type of cancellation estimate introduced in [3].

4. Nonlinear stability in dimension one. For clarity, we carry out the nonlinear stability analysis in detail in the most difficult, one-dimensional, case, indicating afterward by a few brief remarks the extension to $d = 2$. Hereafter, take $x \in \mathbb{R}^1$, dropping the indices on f^j and x_j and writing $u_t + f(u)_x = u_{xx}$.

4.1. Nonlinear perturbation equations. Given a solution $\tilde{u}(x,t)$ of (1.4), define the nonlinear perturbation variable

$$v = u - \bar{u} = \tilde{u}(x + \psi(x,t), t) - \bar{u}(x), \tag{4.1}$$

where

$$u(x,t) := \tilde{u}(x + \psi(x,t), t) \tag{4.2}$$

and $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is to be chosen later.

Lemma 4.1. *For v, u as in (4.1), (4.2),*

$$u_t + f(u)_x - u_{xx} = (\partial_t - L)\, \bar{u}'(x)\psi(x,t) + \partial_x R + (\partial_t + \partial_x^2) S, \tag{4.3}$$

where

$$R := v\psi_t + v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x} = O\left(|v|(|\psi_t| + |\psi_{xx}|) + \left(\frac{|\bar{u}_x| + |v_x|}{1 - |\psi_x|} \right) |\psi_x|^2 \right)$$

and

$$S := -v\psi_x = O(|v||\psi_x|).$$

Proof. To begin, notice from the definition of u in (4.2) that we have by a straightforward computation

$$\begin{aligned} u_t(x,t) &= \tilde{u}_x(x + \psi(x,t), t)\psi_t(x,t) + \tilde{u}_t(x + \psi, t), \\ f(u(x,t))_x &= df(\tilde{u}(x + \psi(x,t), t))\tilde{u}_x(x + \psi, t) \cdot (1 + \psi_x(x,t)) \end{aligned}$$

and

$$\begin{aligned} u_{xx}(x,t) &= (\tilde{u}_x(x + \psi(x,t), t) \cdot (1 + \psi_x(x,t)))_x \\ &= \tilde{u}_{xx}(x + \psi(x,t), t) \cdot (1 + \psi_x(x,t)) + (\tilde{u}_x(x + \psi(x,t), t) \cdot \psi_x(x,t))_x . \end{aligned}$$

Using the fact that $\tilde{u}_t + df(\tilde{u})\tilde{u}_x - \tilde{u}_{xx} = 0$, it follows that

$$\begin{aligned} u_t + f(u)_x - u_{xx} &= \tilde{u}_x\psi_t + df(\tilde{u})\tilde{u}_x\psi_x - \tilde{u}_{xx}\psi_x - (\tilde{u}_x\psi_x)_x \\ &= \tilde{u}_x\psi_t - \tilde{u}_t\psi_x - (\tilde{u}_x\psi_x)_x, \end{aligned} \tag{4.4}$$

where it is understood that derivatives of $\tilde{u}$ appearing on the right-hand side are evaluated at $(x+\psi(x,t),t)$. Moreover, by another direct calculation, using the fact that $L(\bar{u}'(x))=0$ by translation invariance, we have

$$(\partial_t - L)\,\bar{u}'(x)\psi = \bar{u}_x\psi_t - \bar{u}_t\psi_x - (\bar{u}_x\psi_x)_x.$$

Subtracting and using the facts that, by differentiation of $(\bar{u}+v)(x,t)=\tilde{u}(x+\psi,t)$,

$$\begin{aligned} \bar{u}_x + v_x &= \tilde{u}_x(1+\psi_x),\\ \bar{u}_t + v_t &= \tilde{u}_t + \tilde{u}_x\psi_t, \end{aligned} \tag{4.5}$$

so that

$$\begin{aligned} \tilde{u}_x - \bar{u}_x - v_x &= -(\bar{u}_x + v_x)\frac{\psi_x}{1+\psi_x},\\ \tilde{u}_t - \bar{u}_t - v_t &= -(\bar{u}_x + v_x)\frac{\psi_t}{1+\psi_x}, \end{aligned} \tag{4.6}$$

we obtain

$$u_t + f(u)_x - u_{xx} = (\partial_t - L)\bar{u}'(x)\psi + v_x\psi_t - v_t\psi_x - (v_x\psi_x)_x + \left((\bar{u}_x+v_x)\frac{\psi_x^2}{1+\psi_x}\right)_x,$$

yielding (4.3) by $v_x\psi_t - v_t\psi_x = (v\psi_t)_x - (v\psi_x)_t$ and $(v_x\psi_x)_x = (v\psi_x)_{xx} - (v\psi_{xx})_x$. ■

Corollary 4.2. *The nonlinear residual v defined in (4.1) satisfies*

$$v_t - Lv = (\partial_t - L)\,\bar{u}'(x_1)\psi - Q_x + R_x + (\partial_t + \partial_x^2)S, \tag{4.7}$$

where

$$Q := f(\tilde{u}(x+\psi(x,t),t)) - f(\bar{u}(x)) - df(\bar{u}(x))v = \mathcal{O}(|v|^2), \tag{4.8}$$

$$R := v\psi_t + v\psi_{xx} + (\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x}, \tag{4.9}$$

and

$$S := -v\psi_x = O(|v||\psi_x|). \tag{4.10}$$

Proof. Subtracting the relation $\bar{u}_t + f(\bar{u})_x - \bar{u}_{xx} = 0$ from (4.3) yields

$$v_t + \left(f(u) - f(\bar{u})\right)_x - v_{xx} = (\partial_t - L)\,\bar{u}'(x) + \partial_x R + \left(\partial_t + \partial_x^2\right)S.$$

The result follows by comparison of this equation with (1.7) and noting that

$$f(u) - f(\bar{u}) - df(\bar{u})v = \mathcal{O}(|v|^2)$$

by Taylor's theorem. ■

4.2. Cancellation estimate. Our strategy in writing (4.7) is motivated by the following basic cancellation principle.

Proposition 4.3 (see [3]). *For any $f(y,s) \in L^p \cap C^2$ with $f(y,0) \equiv 0$, there holds*

$$\int_0^t \int G(x,t-s;y)(\partial_s - L_y)f(y,s)dy\,ds = f(x,t). \tag{4.11}$$

Proof. Integrating the left-hand side by parts, we obtain

$$\int G(x,0;y)f(y,t)dy - \int G(x,t;y)f(y,0)dy + \int_0^t \int (\partial_t - L_y)^* G(x,t-s;y)f(y,s)dy\,ds. \tag{4.12}$$

Noting, by duality, that

$$(\partial_t - L_y)^* G(x,t-s;y) = \delta(x-y)\delta(t-s),$$

$\delta(\cdot)$ here denoting the Dirac delta-distribution, we find that the third term on the right-hand side vanishes in (4.12), while, because $G(x,0;y) = \delta(x-y)$, the first term is simply $f(x,t)$. The second term vanishes by $f(y,0) \equiv 0$. ∎

Remark 4.4. For $\psi = \psi(t)$, term $(\partial_t - L)\bar{u}'\psi$ in (4.7) reduces to the term $\dot{\psi}(t)\bar{u}'(x)$ appearing in the shock wave case [26, 23, 24, 25, 11, 13].

4.3. Nonlinear damping estimate.

Proposition 4.5. *Let $v_0 \in H^K$ (K as in* (H1)*), and suppose that for $0 \le t \le T$, the H^K norm of v and the H^{K+1} norms of $\psi_t(\cdot,t)$ and $\psi_x(\cdot,t)$ remain bounded by a sufficiently small constant. There are then constants $\theta_{1,2} > 0$ so that, for all $0 \le t \le T$,*

$$\|v(\cdot,t)\|_{H^K}^2 \le Ce^{-\theta_1 t}\|v(0)\|_{H^K}^2 + C\int_0^t e^{-\theta_2(t-s)}(\|v\|_{L^2}^2 + \|(\psi_t,\psi_x)\|_{H^K}^2)(s)\,ds. \tag{4.13}$$

Proof. Subtracting from (4.4) the equation for $\bar{u}$, we may write the nonlinear perturbation equation as

$$v_t + (df(\bar{u})v)_x - v_{xx} = Q(v)_x + \tilde{u}_x\psi_t - \tilde{u}_t\psi_x - (\tilde{u}_x\psi_x)_x, \tag{4.14}$$

where it is understood that derivatives of $\tilde{u}$ appearing on the right-hand side are evaluated at $(x+\psi(x,t),t)$. Using (4.6) to replace $\tilde{u}_x$ and $\tilde{u}_t$, respectively, by $\bar{u}_x + v_x - (\bar{u}_x + v_x)\frac{\psi_x}{1+\psi_x}$ and $\bar{u}_t + v_t - (\bar{u}_x + v_x)\frac{\psi_t}{1+\psi_x}$, and moving the resulting $v_t\psi_x$ term to the left-hand side of (4.14), we obtain

$$\begin{aligned}(1+\psi_x)v_t - v_{xx} &= -(df(\bar{u})v)_x + Q(v)_x + \bar{u}_x\psi_t \\ &\quad - ((\bar{u}_x + v_x)\psi_x)_x + \left((\bar{u}_x + v_x)\frac{\psi_x^2}{1+\psi_x}\right)_x.\end{aligned} \tag{4.15}$$

Taking the L^2 inner product in x of $\sum_{j=0}^K \frac{\partial_x^{2j} v}{1+\psi_x}$ against (4.15), integrating by parts, and rearranging the resulting terms, we arrive at the inequality

$$\partial_t \|v(\cdot,t)\|_{H^K}^2 \le -\theta\|\partial_x^{K+1}v\|_{L^2}^2 + C(\|v\|_{H^K}^2 + \|(\psi_t(\cdot,t),\psi_x(\cdot,t))\|_{H^K}^2)$$

for some $\theta > 0$, $C > 0$, so long as $\|\tilde{u}\|_{H^K}$ remains bounded, and $\|v\|_{H^K}$ and $\|(\psi_t(\cdot,t),\psi_x(\cdot,t))\|_{H^{k+1}}$ remain sufficiently small. Using the Sobolev interpolation $\|v\|_{H^K}^2 \le \|\partial_x^{K+1}v\|_{L^2}^2 + \tilde{C}\|v\|_{L^2}^2$ for $\tilde{C} > 0$ sufficiently large, we obtain $\partial_t\|v(\cdot,t)\|_{H^K}^2 \le -\tilde{\theta}\|v\|_{H^K}^2 + C(\|v\|_{L^2}^2 + \|(\psi_t(\cdot,t),\psi_x(\cdot,t))\|_{H^K}^2)$, from which (4.13) follows by Gronwall's inequality. ∎

4.4. Integral representation/ψ-evolution scheme. By Proposition 4.3, we have, applying Duhamel's principle to (4.7),

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} G(x,t;y)v_0(y)\,dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} G(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds + \psi(x,t)\bar{u}'(x). \end{aligned} \tag{4.16}$$

Defining ψ implicitly as

$$\begin{aligned} \psi(x,t) &= -\int_{-\infty}^{\infty} e(x,t;y)v_0(y)\,dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} e(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds, \end{aligned} \tag{4.17}$$

following [26, 24, 11, 12], where e is defined as in (3.1), and substituting in (4.16) the decomposition $G = \bar{u}'(x)e + \tilde{G}$ of Corollary 3.3, we obtain the *integral representation*

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} \tilde{G}(x,t;y)v_0(y)\,dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \tilde{G}(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds, \end{aligned} \tag{4.18}$$

and, differentiating (4.17) with respect to t, and recalling that $e(x,s;y) \equiv 0$ for $s \le 1$,

$$\begin{aligned} \partial_t^j \partial_x^k \psi(x,t) &= -\int_{-\infty}^{\infty} \partial_t^j \partial_x^k e(x,t;y)u_0(y)\,dy \\ &\quad - \int_0^t \int_{-\infty}^{+\infty} \partial_t^j \partial_x^k e(x,t-s;y)(-Q_y + R_y + S_t + S_{yy})(y,s)\,dy\,ds. \end{aligned} \tag{4.19}$$

Equations (4.18) and (4.19) together form a complete system in the variables $(v, \partial_t^j\psi, \partial_x^k\psi)$, $0 \le j \le 1$, $0 \le k \le K+1$, from the solution of which we may afterward recover the shift ψ via (4.17). From the original differential equation (4.7) together with (4.19), we readily obtain short-time existence and continuity with respect to t of solutions $(v, \psi_t, \psi_x) \in H^K$ by a standard contraction-mapping argument based on (4.13), (4.17), and (3.3).

4.5. Nonlinear iteration. Associated with the solution (u, ψ_t, ψ_x) of integral system (4.18)–(4.19), define

$$\zeta(t) := \sup_{0 \le s \le t} \|(v, \psi_t, \psi_x)\|_{H^K}(s)(1+s)^{1/4}. \tag{4.20}$$

Lemma 4.6. *Let $E_0 := \|v_0\|_{L^1 \cap H^K}$ be sufficiently small. Then for all $t \ge 0$ for which $\zeta(t)$ is finite and sufficiently small, we have, for some $C > 0$ and $E_0 := \|v_0\|_{L^1 \cap H^K}$,*

$$\zeta(t) \le C(E_0 + \zeta(t)^2). \tag{4.21}$$

Proof. By (4.9)–(4.10) and definition (4.20),

$$\|(Q,R,S)\|_{L^1\cap L^\infty} \le \|(v,v_x,\psi_t,\psi_x)\|_{L^1}^2 + \|(v,v_x,\psi_t,\psi_x)\|_{L^\infty}^2 \le C\zeta(t)^2(1+t)^{-\frac{1}{2}}, \tag{4.22}$$

so long as $|\psi_x| \le \|\psi_x\|_{H^K} \le \zeta(t)$ remains small, and likewise (using the equation to bound t-derivatives in terms of x-derivatives of up to two orders)

$$\|(\partial_t + \partial_x^2)S\|_{L^1\cap L^\infty} \le \|(v,\psi_x)\|_{W^{2,1}}^2 + \|(v,\psi_x)\|_{W^{2,\infty}}^2 \le C\zeta(t)^2(1+t)^{-\frac{1}{2}}. \tag{4.23}$$

By standard semigroup theory [18, 2] the full solution operator $S(t) = e^{Lt}$ satisfies

$$\|S(t)g\|_{L^p(\mathbb{R})} \le C\|g\|_{L^p(\mathbb{R})}$$

for all $t \ge 0$, and hence, by applying this short-time bound in conjunction with Corollary 3.3 with $q = 1$ and $d = 1$, noting that the map $g \mapsto \int_{-\infty}^{\infty} e(x,t;y)g(y)dy$ is a bounded linear functional from $L^p \to L^p$, we obtain the estimate

$$\left\|\int_{-\infty}^{\infty} \tilde{G}(\cdot,t;y)v(y,0)dy\right\|_{L^p(\mathbb{R})} \le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0$$

for all $2 \le p \le \infty$. Similarly, applying Corollary 3.3 with $q = 1$ and $d = 1$ to representations (4.18)–(4.19), we obtain for any $2 \le p < \infty$

$$\begin{aligned}\|v(\cdot,t)\|_{L^p(x)} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 \\ &\quad + C\zeta(t)^2\int_0^t (t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}}(1+t-s)^{-\frac{1}{4}}(1+s)^{-\frac{1}{2}}ds \\ &\le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)}\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}\|(\psi_t,\psi_x)(\cdot,t)\|_{W^{K+1,p}} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 \\ &\quad + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)-1/2}(1+s)^{-\frac{1}{2}}ds \\ &\le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{2}(1-1/p)}.\end{aligned} \tag{4.25}$$

Notice that the above bounds do not hold in the case $p = \infty$ due to terms of size $\log(1+t)$ arising from integrating over $\left[\frac{t}{2}, t\right]$. Using (4.13) and (4.24)–(4.25), we obtain $\|v(\cdot,t)\|_{H^K(x)} \le C(E_0+\zeta(t)^2)(1+t)^{-\frac{1}{4}}$. Combining this with (4.25), $p = 2$, rearranging, and recalling definition (4.20), we obtain (4.6). ■

Proof of Theorem 1.2. By short-time H^K existence theory, $\|(v,\psi_t,\psi_x)\|_{H^K}$ is continuous so long as it remains small; hence ζ remains continuous so long as it remains small. By (4.6), therefore, it follows by continuous induction that, assuming $C > 1$ without loss of generality, $\zeta(t) \le 2CE_0$ for $t \ge 0$, if $E_0 < \frac{1}{4C^2}$, yielding by (4.20) the result (1.15) for $p = 2$. Applying (4.24)–(4.25), we obtain (1.15) for $2 \le p \le p_*$ for any $p_* < \infty$, with uniform constant C. Taking $p_* > 4$ and estimating

$$\|Q\|_{L^2},\ \|R\|_{L^2},\ \|S\|_{L^2}(t) \le \|(v,\psi_t,\psi_x)\|_{L^4}^2 \le CE_0(1+t)^{-\frac{3}{4}}$$

in place of the weaker (4.22) (again using (4.24)–(4.25)), and then applying Corollary 3.3 with $q = 2$, $d = 1$, we finally obtain (1.15) for $2 \le p \le \infty$ by a computation similar to (4.24)–(4.25); we omit the details of this final bootstrap argument. Estimate (1.16) then follows using (3.3) with $q = d = 1$, by

$$\begin{aligned}\|\psi(\cdot,t)\|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}E_0 + C\zeta(t)^2\int_0^t (1+t-s)^{-\frac{1}{2}(1-1/p)}(1+s)^{-\frac{1}{2}}ds\\ &\le C(1+t)^{-\frac{1}{2p}}(E_0+\zeta(t)^2),\end{aligned}$$

together with the fact that $\tilde{u}(x,t) - \bar{u}(x) = v(x-\psi,t) + \bar{u}(x) - \bar{u}(x-\psi)$, so that $|\tilde{u}(\cdot,t) - \bar{u}|$ is controlled by the sum of $|v|$ and $|\bar{u}(x) - \bar{u}(x-\psi)| \sim |\psi|$. This yields stability for $\|u - \bar{u}\|_{L^1\cap H^K}|_{t=0}$ sufficiently small, as described in the final line of the theorem. ■

5. Nonlinear stability in dimension two. We now briefly sketch the extension to dimension $d = 2$. Given a solution $\tilde{u}(x,t)$ of (1.4), define the nonlinear perturbation variable

$$v = u - \bar{u} = \tilde{u}(x_1 + \psi(x,t), x_2, t) - \bar{u}(x_1), \tag{5.1}$$

where

$$u(x,t) := \tilde{u}(x_1 + \psi(x,t), t) \tag{5.2}$$

and $\psi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is to be chosen later.

Lemma 5.1. *For v, u as in (5.2),*

$$u_t + \sum_{j=1}^d f^j(u)_{x_j} - \sum_{j=1}^d u_{x_jx_j} = (\partial_t - L)\,\bar{u}'(x_1)\psi(x,t) + \partial_x R + \partial_t S + T, \tag{5.3}$$

where

$$\begin{aligned}R &= O(|(v,\psi_t,\psi_x)||(v,v_x,\psi_t,\psi_x)|),\\ S &:= -v\psi_{x_1} = (|v||\psi_x|),\\ T &:= O(|\psi_x|^3 + |(v,\psi_x)||\psi_{xx}|).\end{aligned}$$

Proof. Similarly to the proof of Lemma 4.1, this proof follows by a straightforward computation. Using the fact that $\tilde{u}_t + \sum_j df^j(\tilde{u})\tilde{u}_{x_j} - \sum_j \tilde{u}_{x_jx_j} = 0$, it follows that

$$\begin{aligned}u_t + \sum_j df^j(u)u_{x_j} - \sum_j u_{x_jx_j} &= \tilde{u}_{x_1}\psi_t - \tilde{u}_t\psi_{x_1} + \sum_{j\ne1} df^j(\tilde{u})\tilde{u}_{x_1}\psi_{x_j}\\ &\quad - \sum_{j\ne1}\tilde{u}_{x_jx_1}\psi_{x_j} - \sum_j(\tilde{u}_{x_1}\psi_{x_j})_{x_j},\end{aligned} \tag{5.4}$$

where it is understood that derivatives of $\tilde{u}$ appearing on the right-hand side are evaluated at $(x + \psi(x,t), t)$. Moreover, by another direct calculation, using the fact that $L(\bar{u}'(x_1)) = 0$ by translation invariance, we have

$$(\partial_t - L)\,\bar{u}'(x_1)\psi = \bar{u}_{x_1}\psi_t - \bar{u}_t\psi_{x_1} + \sum_{j\ne1} df^j(\bar{u})\bar{u}_{x_1}\psi_{x_j} - \sum_{j\ne1}\bar{u}_{x_jx_1}\psi_{x_j} - \sum_j(\bar{u}_{x_1}\psi_{x_j})_{x_j}.$$

Subtracting, and using (4.5) and

$$
\begin{aligned}
\bar{u}_{x_j} + v_{x_j} &= \tilde{u}_{x_j} + \tilde{u}_{x_1}\psi_{x_j},\\
\bar{u}_t + v_t &= \tilde{u}_t + \tilde{u}_{x_1}\psi_t,
\end{aligned}
\tag{5.5}
$$

so that

$$
\begin{aligned}
\tilde{u}_{x_j} - \bar{u}_{x_j} - v_{x_j} &= -(\bar{u}_{x_1} + v_{x_1})\frac{\psi_{x_j}}{1+\psi_{x_1}},\\
\tilde{u}_t - \bar{u}_t - v_t &= -(\bar{u}_{x_1} + v_{x_1})\frac{\psi_t}{1+\psi_{x_1}},
\end{aligned}
\tag{5.6}
$$

we obtain

$$
\begin{aligned}
u_t + \sum_j df^j(u)u_{x_j} - \sum_j u_{x_jx_j} = (\partial_t - L)\bar{u}'(x_1)\psi + v_{x_1}\psi_t - v_t\psi_{x_1}&\\
+ \sum_{j\neq 1}(df^j(\tilde{u})\tilde{u}_{x_1} - df^j(\bar{u})\bar{u}_{x_1})\psi_{x_j}&\\
- \sum_{j\neq 1}(\tilde{u}_{x_jx_1} - \bar{u}_{x_jx_1})\psi_{x_j} - \sum_j((\tilde{u}_{x_1} - \bar{u}_{x_1})\psi_{x_j})_{x_j}&.
\end{aligned}
$$

Using $v_{x_1}\psi_t - v_t\psi_{x_1} = (v\psi_t)_{x_1} - (v\psi_{x_1})_t$,

$$
df^j(\tilde{u})\tilde{u}_{x_1} = f(u)_{x_1} - df^j(\tilde{u})\tilde{u}_{x_1}\psi_{x_1} = f(u)_{x_1}(1-\psi_x) - df^j(\tilde{u})\tilde{u}_{x_1}\psi_{x_1}^2,
$$

and $\tilde{u}_{x_jx_1} = (\tilde{u}_{x_j})_{x_1} - \tilde{u}_{x_jx_1}\psi_{x_1} = (\tilde{u}_{x_j})_{x_1}(1-\psi_{x_1}) + \tilde{u}_{x_jx_1}\psi_{x_1}^2$ and rearranging, we obtain

$$
\begin{aligned}
u_t + \sum_j df^j(u)u_{x_j} - \sum_j u_{x_jx_j} = (\partial_t - L)\bar{u}'(x_1)\psi + (v\psi_t)_{x_1} - (v\psi_{x_1})_t&\\
+ \sum_{j\neq 1}(f^j(u) - f^j(\bar{u}))_{x_1}\psi_{x_j}&\\
- \sum_{j\neq 1} f(u)_{x_1}\psi_{x_1}\psi_{x_j} - \sum_{j\neq 1} df^j(\tilde{u})\tilde{u}_{x_1}\psi_{x_1}^2\psi_{x_j}&\\
- \sum_{j\neq 1}(\tilde{u}_{x_j} - \bar{u}_{x_j})_{x_1}\psi_{x_j} + \sum_{j\neq 1}(\tilde{u}_{x_j})_{x_1}\psi_{x_1}\psi_{x_j}&\\
+ \sum_{j\neq 1}\tilde{u}_{x_jx_1}\psi_{x_1}^2\psi_{x_j}&\\
- \sum_j (v_{x_1}\psi_{x_1})_{x_j} - \sum_j\left((\bar{u}_{x_1} + v_{x_1})\frac{\psi_{x_j}\psi_{x_1}}{1+\psi_{x_1}}\right)_{x_j}&.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
(f^j(u) - f^j(\bar{u}))_{x_1}\psi_{x_j} &= (f^j(u) - f^j(\bar{u})\psi_{x_j})_{x_1} - (f^j(u) - f^j(\bar{u}))\psi_{x_jx_1},\\
f(u)_{x_1}\psi_{x_1}\psi_{x_j} &= (f(u)\psi_{x_1}\psi_{x_j})_{x_1} - f(u)(\psi_{x_1}\psi_{x_j})_{x_1}
\end{aligned}
$$

and

$$(\tilde u_{x_j}-\bar u_{x_j})_{x_1}\psi_{x_j}=((\tilde u_{x_j}-\bar u_{x_j})\psi_{x_j})_{x_1}-(\tilde u_{x_j}-\bar u_{x_j})\psi_{x_jx_1},$$

with $|f^j(u)-f^j(\bar u)|=O(|v|)$ and $|\tilde u_{x_j}-\bar u_{x_j}|=O(|v|)$, we obtain the result. ■

Proof of Theorem 1.3. The result of Lemma 5.1 is the only part of the analysis that differs essentially from that of the one-dimensional case. The cancellation and nonlinear damping arguments go through exactly as before to yield the analogues of Propositions 4.3 and 4.5. Likewise, we obtain a Duhamel representation

$$\begin{aligned} v(x,t)&=\int_{-\infty}^{\infty}\tilde G(x,t;y)v_0(y)\,dy\\ &\quad+\int_0^t\int_{-\infty}^{\infty}\tilde G(x,t-s;y)(R_y+S_t+T)(y,s)\,dy\,ds \end{aligned}\tag{5.7}$$

and

$$\begin{aligned} \partial_t^j\partial_x^k\psi(x,t)&=-\int_{-\infty}^{\infty}\partial_t^j\partial_x^k e(x,t;y)u_0(y)\,dy\\ &\quad-\int_0^t\int_{-\infty}^{+\infty}\partial_t^j\partial_x^k e(x,t-s;y)(R_y+S_t+T)(y,s)\,dy\,ds \end{aligned}\tag{5.8}$$

analogous to that of (4.18)–(4.19), forming a closed system in variables (v,ψ_x,ψ_t).

To obtain the analogue of Lemma 4.6, completing the proof of nonlinear stability, we now define

$$\begin{aligned} \eta(t):=&\sup_{0\le s\le t}\|v\|_{H^K}(s)(1+s)^{1/2}+\sup_{0\le s\le t,\,2\le p\le\infty}\|v\|_{W^{1,p}}(s)(1+s)^{1-\frac1p}\\ &+\sup_{0\le s\le t}\|(\psi_t,\psi_x)\|_{H^K}(s)(1+s)^{1-\varepsilon}\\ &+\sup_{0\le s\le t,\,2\le p\le\infty}\|(\psi_t,\psi_x)\|_{W^{1,p}}(s)(1+s)^{\frac32-\frac1p-\varepsilon} \end{aligned}\tag{5.9}$$

and demonstrate that for all $t\ge0$ for which $\eta(t)$ is finite, there exists a constant $C>0$ such that

$$\eta(t)\le C\left(E_0+\eta(t)^2\right),$$

where, as before, $E_0:=\|v_0\|_{L^1\cap H^K}$.

First, observe that, by (5.9), the differentiated source terms R and S satisfy

$$\|(R,S)\|_{L^1\cap L^\infty}\le\|(v,v_x,\psi_t,\psi_x)\|_{H^2}^2\le C\eta(t)^2(1+t)^{-1},$$

and

$$\begin{aligned} \|(R_x,S_t)\|_{L^2\cap L^\infty}&\le\|(v,v_x,v_{xx},\psi_x,\psi_{xx},\psi_{xxx},\psi_t,\psi_{tx})\|_{H^1}\|(v,v_x,\psi_x,\psi_t)\|_{L^\infty}\\ &\le C\eta(t)^2(1+t)^{-\frac32}, \end{aligned}$$

while the undifferentiated source term T satisfies a faster decay rate

$$\|T\|_{L^1\cap L^\infty}\le\|(v,\psi_x)\|_{H^2}\|\psi_{xx}\|_{H^2}+\|\psi_x\|_{H^2}^2\le C\eta(t)^2(1+t)^{\varepsilon-\frac32}.$$

Applying Corollary 3.3 with $d = 2$, $q = 1$ for undifferentiated source term T and for differentiated source terms R, S on $[0, \frac{t}{2}]$ and with $d = 2$, $q = 2$ for R_x and S_t on $[\frac{t}{2}, t]$ thus yields in place of (4.24)–(4.25) the estimates

$$\begin{aligned}
&\|v(\cdot,t)\|_{L^p(x)} \\
&\quad\le C(1+t)^{-(1-1/p)}E_0 + C\eta(t)^2\int_0^{\frac{t}{2}}(1+t-s)^{-\frac12}(t-s)^{-(1-\frac1p)}(1+s)^{-1}ds \\
&\qquad + C\eta(t)^2\int_{\frac{t}{2}}^t (t-s)^{-(\frac12-\frac1p)}(1+s)^{-\frac32}ds \\
&\qquad + C\eta(t)^2\int_0^t (1+t-s)^{-\frac12}(t-s)^{-(\frac12-\frac1p)}(1+s)^{\varepsilon-\frac32}ds \\
&\quad\le C(E_0+\eta(t)^2)(1+t)^{-(1-1/p)}
\end{aligned} \tag{5.10}$$

and, for $0 \le j \le 2$,

$$\begin{aligned}
\|\partial_x^j(\psi_x,\psi_t)(\cdot,t)\|_{L^p(x)} &\le C(1+t)^{-(1-\frac1p)-\frac12}E_0 \\
&\quad + C\eta(t)^2\int_0^t (1+t-s)^{-(1-\frac1p)-\frac12}(1+s)^{\varepsilon-\frac32}ds \\
&\quad + C\eta(t)^2\int_{\frac{t}{2}}^t (1+t-s)^{-(1-\frac1p)}(1+s)^{-\frac32}ds \\
&\le C(E_0+\eta(t)^2)(1+t)^{\varepsilon-(1-\frac1p)-\frac12},
\end{aligned} \tag{5.11}$$

valid for all $2 \le p \le \infty$. Likewise, differentiating (5.7), we may estimate $\|v_x(\cdot,t)\|_{L^p(x)}$ by exactly the same estimate as in (5.10) since a further x-derivative does not harm the argument.

Together with the nonlinear damping estimate, these establish the analogue of Lemma 4.6 as in the one-dimensional case, from which we obtain nonlinear stability and sharp estimates as claimed. We omit the details, which are entirely similar to, but substantially simpler than, those of the one-dimensional case. ■

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