ON THE STABILITY OF PERIODIC SOLUTIONS OF NONLINEAR DISPERSIVE EQUATIONS

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ABSTRACT

In this work, we consider varying aspects of the stability of periodic traveling wave solutions to nonlinear dispersive equations. In particular, we are interested in deriving universal geometric criterion for the stability of particular third order nonlinear dispersive PDE's. We begin by studying the spectral stability of such solutions to the generalized Korteweg-de Vries (gKdV) equation. Using the integrable structure of the ODE governing the traveling wave solutions of the gKdV, we are able to determine the role of the null-directions of the linearized operator in the stability of the traveling wave to perturbations of long-wavelength by conducting what amounts to a rigorous Whitham theory calculation. By then considering the characteristic polynomial of the monodromy map (the periodic Evans function) in a neighborhood of the origin in the spectral plane, we derive two separate instability indices. The first is a modulational stability index which, assuming a particular non-degeneracy condition holds, determines a rigorous normal form of the spectrum in the neighborhood of the origin, and yields necessary and sufficient criterion for the traveling wave to be modulationally stable. The second is an orientation index which counts modulo 2 the total number of periodic eigenvalues of the linearized operator with the positive real axis. This is essentially a generalization of the solitary wave stability index. Both of these indices are expressible in terms of a map between a parameter space which parameterizes the periodic traveling waves of the gKdV to the conserved quantities of the governing PDE. Moreover, we show how our general methods can be used to derive transverse-modulational instability indices, i.e. in analyzing the stability of such solutions to long-wavelength transverse perturbations in higher dimensional equations.

We also study the nonlinear stability of periodic traveling wave solutions of the

gKdV within the class of solutions having the same period. In particular, by conducting a detailed analysis of the Hamiltonian system satisfied by the traveling wave profile, we prove that in many cases the periodic spectral instability index mentioned above determines the orbital stability of the underlying traveling wave. However, the signs of two other indices play a role in our analysis, neither of which are present when one considers exponentially decaying solutions: this stands in stark contrast to the solitary wave theory, where such a solution is nonlinearly stable if and only if it is spectrally stable (assuming a particular non-degeneracy condition holds).

Finally, we show how our results extend to other classes of dispersive equations. In particular, we derive modulational and finite-wavelength instability indices for the generalized Benjamin-Bona-Mahony (gBBM) equation, as well as the generalized Camassa-Holm equation. Moreover, we prove a transverse instability result for the gBBM equation analogous to that for the gKdV.

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LIST OF ABBREVIATIONS

gBBM	Generalized Benjamin-Bona-Mahony
gCH	Generalized Camassa-Holm
gKdV	Generalized Korteweg-de Vries
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
ZK-gBBM	Zakharov-Kuznetsov Generalized Benjamin-Bona-Mahony
ZK-gKdV	Zakharov-Kuznetsov Generalized Korteweg-de Vries

LIST OF SYMBOLS

$L^p(X)$	The Hilbert space of all measurable functions $f : X \subset \mathbb{R} \to \mathbb{R}$ such that $\int_X \|f(x)\|^p dx < \infty$, equipped with the norm $\|f\|_{L^p} := \left(\int_X \|f(x)\|^p dx\right)^{1/p}$.
$L^2_{\text{per}}(a, b)$	The Hilbert space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that f is $b - a$ periodic and $f \in L^2([a, b])$.
$\langle f, g\rangle$	The standard inner-product between elements f and g of the space $L^2(X)$, $X \subset \mathbb{R}$, defined by $\langle f, g\rangle = \int_X f(x)g(x)dx$.
$H^s(X; \mathbb{R})$	The set of all real valued tempered distributions such that the distributional derivatives $\frac{d^n}{dx^n} f$ belong to $L^2(X)$ for $0 \leq n \leq x$.
$\text{spec}(A)$	Spectrum of the operator A as considered on $L^2(\mathbb{R})$.
$\sigma(A)$	The eigenvalues of an $n \times n$ matrix A.
$\|z\|$	The standard modulus of an element of $z \in \mathbb{C}$ defined by $\|z\|^2 = z\bar{z}$.
S^1	The set of all $\lambda \in \mathbb{C}$ such that $\|\lambda\| = 1$.
$A^\dagger$	The adjoint operator of an operator A.
$\langle f\rangle$	The functional $f \mapsto \int_a^b f(x)dx$ defined on $L^1([a, b])$. The set $[a, b]$ on which this integral is taken will be clear from context.
$\mathbb{R}i$	The set $\{x \in \mathbb{C} : x - \bar{x} = 0\}$.
$\mathbb{R}^+$	The set $\{x \in \mathbb{R} : x \geq 0\}$.
$B(a, R)$	The set $\{x \in \mathbb{C} : \|x - a\| < R\}$.

CHAPTER 1

Introduction

Among the most fundamental problems in the field of differential equations is that of stability of particular classes of solutions. Essentially, this question addresses the robustness of such solutions, i.e. their ability to persist under perturbation. For differential equations governing physical processes such information is of practical importance since solutions which are unstable do not manifest in physical situations, except possibly as transient phenomena. Thus, if ones goal is to restrict a particular physical system to a stable configuration, this question is clearly fundamental to such analysis. In particular, the study of this question aims at giving researchers practical and efficient "rules of thumb," derived from rigorous mathematical proof, to ascertain the stability of mathematical solutions arising in various physical models.

Within the context of nonlinear dispersive equations, probably the most famous model equation is the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x, \tag{1.1}$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. This equation was first formulated by Boussinesq [13], but is named after Diederik Korteweg and Gustav de Vries who studied it in [42]. Korteweg and de-Vries used the KdV as a model equation for the propagation of shallow water waves along a canal, but it has since been shown to arise as a model in many physical systems including long internal waves, ion acoustic waves in plasmas, and gravity waves. This equation can be viewed as a nonlinear perturbation of the Airy equation

$$u_t = u_{xxx}, \tag{1.2}$$

which is a linear dispersive equation with dispersion relation[1] $\omega = -\xi^3$. That is, (1.2) admits plane wave solutions of the form $u_\xi(x,t) = e^{i(\xi x - \xi^3 t)}$ for each $\xi \in \mathbb{R}$. It follows that plane wave solutions of (1.2) of different spatial frequencies propagate in time at different velocities: in particular, the higher the frequency the faster the temporal propagation. This encompasses the dispersing effects of the Airy equation and suggests that solutions of (1.1) which decay as $x \to \pm\infty$ should disperse as solutions of (1.2). While this is indeed the case[2] in many situations, notice that the nonlinearity present in (1.1) is of Burger's type, and hence one expects the nonlinearity to "focus" solutions and cause wave breaking. This mix of linear dispersion and Burger's nonlinearity leads to a quite curious phenomenon of the KdV: there exists solitary wave solutions termed "solitons" which are traveling wave solutions of the form $u(x,t) = u_c(x - ct)$ for $c > 0$ which propagate to the right in time[3]. Clearly, such solutions neither disperse nor develop singularities and exists as a result of the delicate balance of dispersion and nonlinearity in (1.1). Moreover, these solutions are easily seen to satisfy the scaling relationship $u_c(x) = cu_1(\sqrt{c}x)$, and hence the velocity of such a soliton solution is directly related to the amplitude, with frequency scaling as $c^{-1/2}$.

These soliton solutions are best described by using the complete integrability of (1.1) using the inverse scattering transform. The scattering transform for the KdV can be considered as a nonlinear analogue of Fourier transform which can be used to solve the Cauchy problem for the Airy equation. In particular, it follows from this theory that, in general, given any sufficiently smooth initial data solutions of (1.1) exist globally in time and as $t \to \infty$ the solution decouples into a "radiation" component[4], which propagates rightward and disperses like a solution of (1.2), and a nonlinear superposition of soliton

[1] Notice this is different than the group velocity of a plane wave solution of (1.2), which is $-\frac{d\omega}{d\xi} = 3\xi^2$. This suggests that such plane wave solutions propagate to the right for all $t > 0$.

[2] There are also radiative terms arising in connection with the complete integrability of the KdV. More will be mentioned on this subject later.

[3] Technically, actual solitons exist as a result of the complete integrability. They are characterized not only by their persistent shapes, but also by their interactions with one another.

[4] The radiation terms arise in connection with the Lax pair formulation mentioned above. In this theory, one associates to initial datum u_0 of (1.1) a Hill operator $\mathcal{L} = -\partial_x^2 + u_0$ acting on the space $L^2(\mathbb{R};\mathbb{R})$. The corresponding solution of (1.1) is characterized by the spectrum of $\mathcal{L}$ and various transmission and reflection coefficients. The radiative component of the solution is associated with the continuous spectrum of $\mathcal{L}$, while the soliton components are associated with the point spectrum.

components which propagate to the left. This topic of the complete integrability is a vast and interesting area in its own right, but its use is limited by the fact that many equations arising in physical applications are not completely integrable. For instance, the generalized KdV (gKdV) equation

$$u_t = u_{xxx} + f(u)_x, \tag{1.3}$$

is known to be completely integrable only when f is a cubic polynomial, which corresponds to the Gardner equation

$$u_t = u_{xxx} + \left(\omega u^2 + \varepsilon u^3\right)_x$$

where $(\omega, \varepsilon) \in \mathbb{R}^2$, which includes the KdV and modified KdV as special cases.

While the gKdV equation (1.3) does not in general possess exact "soliton solutions", which requires complete integrability, it does admit solitary wave solutions of the form $u(x,t) = u_c(x - ct)$ where the wave profile u_c decays exponentially at $\pm\infty$. This is easily verified by phase plane analysis, where the solitary wave corresponds to an orbit homoclinic to zero[5]. The stability of such solutions can no longer be ascertained by the inverse scattering transform, as in the case of the KdV, but nonetheless the stability theory is well understood. In this case, one can use variational methods in order to characterize the solitary wave as a critical point of a nonlinear functional acting on an appropriate space. The nonlinear stability of such a solution is then determined by the classification of this critical point as a local minima or not. For more details, see section 1.1 of this chapter. When considering the spectral stability of such solutions, one encounters a spectral problem on $L^2(\mathbb{R})$ of the form

$$\partial_x \mathcal{L}[u_c]v = \mu v,$$

where the operator $\mathcal{L}[u_c] = -\partial_x^2 - f'(u) + c$ is a self adjoint second order differential

[5]More generally, a solitary wave of a nonlinear PDE is required to be asymptotically constant. Hence, they correspond to either orbits homoclinic to an equilibrium solution or to orbits heteroclinic between equilibrium solutions.

operator with asymptotically constant coefficients, and $\partial_x \mathcal{L}[u_c]$ is the Frechét derivative (linearization) of (1.3) about the solitary wave u_c. The choice of the space $L^2(\mathbb{R})$ corresponds to considering spectral stability of the solution to spatially localized perturbations. Spectral stability of the solitary wave u_c is then equivalent to the requirement that the spectrum of the operator $\partial_x \mathcal{L}[u_c]$ acting on $L^2(\mathbb{R})$, denoted spec$(\partial_x \mathcal{L}[u_c])$, is a subset of the imaginary axis[6]. Since the linearized operator $\partial_x \mathcal{L}[u_c]$ has asymptotically constant coefficients, its essential spectrum can be characterized by Weyl's theorem and is seen to coincide with that of the operator $-\partial_x^3$ acting on $L^2(\mathbb{R})$. Thus the essential spectrum of the linearized operator is confined to the imaginary axis and the spectral stability of such a solitary wave solution is characterized by the point spectrum of $\partial_x \mathcal{L}[u_c]$, i.e. by the $L^2(\mathbb{R})$ eigenvalues of the linearized operator.

This ability to reduce spectral stability of a solution to analyzing *point* spectrum of an operator is very nice feature of the solitary wave theory which is typically not the case in other contexts. In particular, if one considers traveling wave solutions $u(x,t) = u_c(x - ct)$ of (1.3) where the wave profile u_c is a T-periodic function of its argument, then the concept of spectral stability become a much more delicate issue. This stems from the fact that since the coefficients of the linearized operator $\partial_x \mathcal{L}[u_c]$ are periodic, the L^2 spectrum consists entirely of continuous spectrum. In particular, there are *no* L^2 eigenvalues of the operator $\partial_x \mathcal{L}[u_c]$ in this case: for more information, see section 1.2 of this introduction. In order to circumvent this difficulty, it is often easier to restrict the class of admissible perturbations of the underlying periodic wave u_c. Indeed, if one considers perturbations which belong to the real Hilbert space of T-periodic functions of $\mathbb{R}$ which are square integrable over a period, which we denote $L^2_{\text{per}}([0,T])$, then the spectrum of the linearized operator $\partial_x \mathcal{L}[u_c]$ becomes discrete and coincides with the point spectrum. Moreover, by considering this class of admissible perturbations one can utilize the variational methods familiar from the solitary wave theory to study the nonlinear stability of the T-periodic traveling wave solution to

[6]In general, spectral stability occurs if the spectrum does not intersect the open right half plane in $\mathbb{C}$. However, the Hamiltonian structure of the linearized operator implies the set spec$(\partial_x \mathcal{L}[u_c])$ is symmetric about the real and imaginary axis, from which our claim follows.

T-periodic perturbations. While this may make the stability theory of such solutions easier on a mathematical level, it has the clear disadvantage of artificially restricting the class of admissible perturbations by requiring them to have the same periodic structure as the underlying wave. Moreover, these variational methods fundamentally rely on integration by parts, and the periodicity of the perturbations is needed in order to handle the boundary terms in a clean way. As such, there has been no progress in developing a nonlinear stability theory of such solutions to *non-periodic* perturbations. From the prospective of applications, however, it is desirable to consider either localized perturbations, corresponding to the $L^2(\mathbb{R})$ case, or uniformly bounded continuous perturbations[7].

The purpose of this thesis is to consider the stability properties of the periodic traveling wave solutions of (1.3) and related dispersive equations. We study both the spectral stability to localized perturbations, as well as nonlinear stability to periodic perturbations. A nice relationship between these two stability theories arises which more or less parallels that of the solitary wave stability theory. In particular, we show that (assuming a nondegeneracy condition is satisfied) such solutions of the KdV are nonlinearly stable to periodic perturbations if and only if they are spectrally stable to such perturbations. Moreover, we are able to hint at a *possible* way to extend orbital stability calculations to considering classes of perturbations which are periodic with period *close* to the underlying wave: this remark is very formal and we make no attempt at developing such a theory in this thesis, although we hope to study this problem extensively in the future.

The outline of this thesis is as follows. In the next three sections of this chapter, we review elements of stability theory as it applies to solitary and periodic solutions. The next three chapters are devoted to the stability analysis of periodic traveling wave solutions of the gKdV equation (1.3). In chapter 2, we study the spectral stability of such solutions to both periodic and long-wavelength perturbations. We develop stability indices in each of these cases which we express as Jacobians of particular maps between

[7]By standard results in Floquet theory, the spectral stability to such perturbations is equivalent to spectral stability to localized perturbations.

spaces parameterizing the periodic traveling wave solutions. In chapter 3, we extend our results on the spectral stability to periodic perturbations to study the nonlinear stability of such solutions to periodic perturbations. The methods are variational in nature, and thus there is a close parallel to the solitary wave theory. In chapter 4, we study the stability of periodic traveling wave solutions of (1.3) as solutions to higher-dimensional nonlinear partial differential equations: in particular, we study the stability of such solutions to perturbations which have long-wavelength in the transverse direction of the propagation of our solution. Finally, in chapter 5 we show how the techniques from chapter 2 and 4 can be extended to other nonlinear dispersive equations. In particular, we conduct a spectral stability analysis of the generalized Benjamin-Bona-Mahony (gBBM) equation

$$u_t - u_{xxt} + u_x + f(u)_x = 0$$

and the generalized Camassa-Holm equation

$$u_t - u_{xxt} = 2u_x u_{xx} + u u_{xxx} + (f(u)/2)_x - k u_x,$$

as well as conduct a transverse stability analysis for such solutions of the gBBM equation.

1.1 Solitary Wave Stability Theory

As the work in this thesis concerns the stability of periodic solutions of certain classes of dispersive PDE, we find it appropriate to begin with a review of the parallel results in the solitary wave context. In this theory, one considers asymptotically constant solutions of nonlinear equations of the form

$$\mathcal{D}u_t = N(u), \tag{1.4}$$

where $\mathcal{D}$ is an invertible operator and N is a nonlinear differential operator, both of which are considered as acting on $L^2(\mathbb{R})$. When considering traveling wave solutions of the form $u(x,t) = u_c(x - ct)$, one encounters the ordinary differential equation

$$N(u_c) - c\mathcal{D}u_{c,x} = 0, \tag{1.5}$$

from which simple phase plane analysis can be used to prove the existence of asymptotically constant solutions.

In the study of the spectral stability of such a solution, one must determine the $L^2(\mathbb{R})$ spectrum of the operator

$$\mathcal{L} := N'(u_c) - c\mathcal{D}\partial_x,$$

and spectral stability is equivalent to the condition that $\text{spec}(\mathcal{L})$ does not intersect the open right half plane. As previously mentioned, the essential spectrum of $\mathcal{L}$ does not usually play a role in the spectral stability analysis, as it is characterized by Weyl's theorem and, in most cases, seen to lie in the stable half space[8]. For example, if one considers solutions of the gKdV equation (1.3) which are asymptotically zero, then the essential spectrum of the associated linearized operator $\partial_x\left(-\partial_x^2 - f'(u_c) + c\right)$ corresponds to the L^2 spectrum of the constant coefficient operator $-\partial_x^3 + (c - f'(0))\partial_x$, which is easily seen to be the imaginary axis by the Fourier transform. Thus, the spectral stability of asymptotically constant solutions is (in general) characterized completely by its point spectrum in $L^2(\mathbb{R})$.

As an elementary example, consider the nonlinear reaction diffusion equation

$$u_t = u_{xx} - f(u) \tag{1.6}$$

where f is sufficiently smooth, and suppose that $u(x,t) = u(x)$ is a solitary wave solution. Then it is easily seen that such solutions are spectrally stable if and only if

[8] For Hamiltonian systems, as the ones studied in this thesis, the essential spectrum is usually seen to lie on $\mathbb{R}i$.

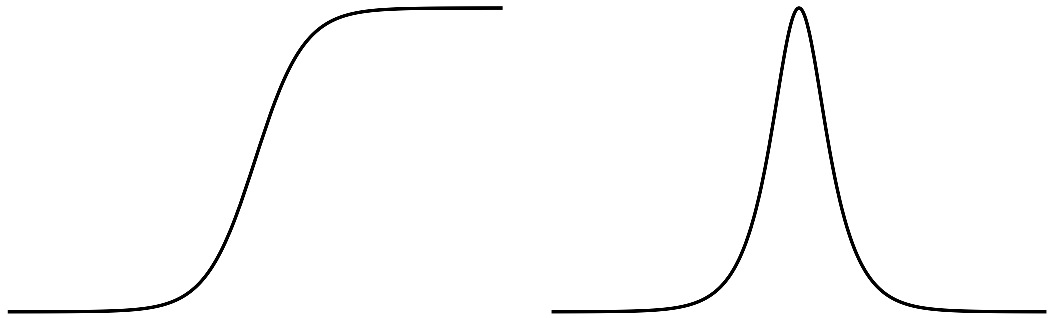

Figure 1.1: If u is a front, as in the left figure, then $u_x > 0$ for all x and hence such solutions of the reaction-diffusion equation (1.6) spectrally stable. If u is a pulse, as in the right figure, then u_x is clearly not the ground state and hence such solutions are unstable.

they are monotone [25]: as a result, fronts are stable but pulses are not (see Figure .1). To see this, notice that if we linearize about u we obtain the equation

$$v_t = v_{xx} - f'(u_c)v$$

considered on $L^2(\mathbb{R})$. Since this equation is autonomous in time, taking the Laplace transform in time leads to the spectral problem

$$Lv = \mu v$$

on $L^2(\mathbb{R})$, where $L := \partial_x^2 - f'(u_c)$. Notice that L is a self-adjoint operator on $L^2(\mathbb{R})$ and is a relatively compact perturbation of the negative operator ∂_x^2. By translation invariance, it is clear that u_x belongs to the kernel of L and hence $\mu = 0$ is an L^2 eigenvalue. If the profile u is monotone, it follows that u_x does not vanish on $\mathbb{R}$ and hence $\mu = 0$ is the ground-state eigenvalue of L (see Theorem 11.8 in [47]). It follows that the set spec(L) does not intersect the open right half plane and hence one has spectral stability in this case. Similarly, if u is not monotone, then u_x will have at least one zero on $\mathbb{R}$ and hence $\mu = 0$ is not the ground state eigenvalue in this case. It follows that the ground state eigenvalue of L must be positive, which implies spectral instability.

The nonlinear stability of asymptotically constant solutions has also been the subject of great study over the last century. To discuss the relevant results, we now restrict ourselves to the generalized KdV equation (1.3). The solitary wave solutions of the form $u(x,t) = u_{c_0}(x - c_0 t)$ of this equation are known to be nonlinearly stable if the functional

$$d(c) = \mathcal{E}(u_c) + c_0\mathcal{P}(u_c)$$

is convex at c_0, where $\mathcal{E}$ is the corresponding Hamiltonian and $\mathcal{P}$ denotes the square of the L^2 norm. The proof is quite technical and will not be reproduced here. The main point to note are that there is a one parameter family of solitary waves parameterized by the wave speed c since the translation invariance can be modded out[9]. Thus, the functional d can be considered as acting on the set of (equivalence classes of) solitary wave solutions of (1.3). By construction, c_0 is a critical point of d and hence convexity at c_0 is equivalent to the condition that d has a local minimum at c_0. Moreover, the functional $\mathcal{P}$ is conserved under the gKdV flow, and hence one could interpret this stability condition as saying that such a solution is nonlinearly stable if the Hamiltonian $\mathcal{E}$ restricted to the manifold $X_0 := \{u \in L^2(\mathbb{R}) : \mathcal{P}(u) = \mathcal{P}(u_{c_0})\}$ has a local minimum at u_{c_0}: the wave speed c_0 present in the definition of $d(c)$ acts as a Lagrange multiplier for this minimization problem.

The corresponding nonlinear instability theory follows the same lines: the solitary wave solution u_{c_0} of (1.3) is nonlinearly unstable if the functional d is concave at c_0, i.e. if $d''(c_0) < 0$. It follows that the solution u_{c_0} is a local maximum of the Hamiltonian $\mathcal{E}$ restricted to the manifold X_0. Stability along the transition curve $d''(c_0) = 0$ is a more delicate issue and will not be discussed here. Thus, assuming that the wavespeed c_0 is such that $d''(c_0) \neq 0$, it follows that the corresponding solitary wave solution of (1.3) will be nonlinearly stable if and only if $d''(c_0) > 0$. This point is important for understanding the work presented in this thesis: in order to determine certain notions of stability, one must often require that particular non-degeneracy conditions are met. In our case of periodic traveling wave solutions of (2.1), we will have a corresponding

[9] This is usually done by requiring $u_x(0) = 0$.

non-degeneracy condition: such a condition is to be expected by the above comments.

In the case of the gKdV, it is possible to use the explicit structure of the equation to show that

$$d''(c_0) = \frac{\partial}{\partial c}\mathcal{P}(u_c)\big|_{c=c_0}.$$

Thus, the nonlinear stability of such a solution is determined by whether the momentum of the wave is an increasing function of the wave speed at c_0. What is remarkable here is that the nonlinear stability criterion can be formulated in terms of the derivative of a conserved quantity of the gKdV flow in terms of the single parameter (modulo translations) which parameterizes the family of solitary waves. In particular, this condition is geometric in nature: the non-degeneracy condition that $d''(c_0) \neq 0$ is equivalent to requiring that c_0 is not a critical point of the function $c \mapsto \mathcal{P}(u_c)$. As a result, the mapping $c \mapsto \mathcal{P}(u_c)$ is a diffeomorphism of a neighborhood of c_0 onto a neighborhood of $\mathcal{P}(u_{c_0})$, and hence solitary waves u_c with $|c - c_0| \ll 1$ can be locally parameterized by the momentum instead of the wave speed and the stability index $d''(c_0)$ is precisely the Jacobian of this map at c_0. This geometric formulation of the above nonlinear stability index will serve as a model for the analysis present throughout this thesis: all stability indices will be shown to be expressible in terms of the Jacobians between parameters which define the periodic traveling wave solutions and the corresponding conserved quantities of the gKdV flow.

In the case of nonlinear instability of a solitary wave solution of (1.3), it was shown by Pego and Weinstein [56, 57] that the mechanism behind this instability is that as one crosses the transition curve $\frac{\partial}{\partial c}\mathcal{P}(u_c) = 0$ from a region of stability to a region of instability, two real eigenvalues of the linearized operator $\partial_x \mathcal{L}$ bifurcate from the origin in a pair symmetric to the imaginary axis. Unlike the case of finite-dimensional Hamiltonian systems, however, this bifurcation does not result from a pair of imaginary eigenvalue colliding at the origin and bifurcating. Their proof is based on the Evans function framework put forward by Evans [21, 22, 23, 24], and the relevant points will be outlined in chapter 3. As we will see, the corresponding theory in the periodic case developed in chapters 2, 3, and 4 yields precisely such a dichotomy, assuming the non-degeneracy condition is met, in the case of the KdV. In particular, in chapter 3 we show such solutions of the KdV one either has orbital stability or spectral instability assuming, of course, the non-degeneracy condition is met.

1.2 Periodic Stability Theory

We now move on to discuss elements of periodic stability theory and the spectral stability techniques used throughout this thesis. We are concerned with the stability theory of periodic traveling wave solutions to various classes of non-linear PDE's of the form

$$\mathcal{D}u_t = N(u),\ (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.7}$$

where $\mathcal{D}$ is an invertible operator and N is a non-linear differential operator, both considered on the real Hilbert space $L^2(\mathbb{R})$. In particular, we will be most interested in the case when $N(u)$ is a third order operator and (1.7) is dispersive and the flow induced by (1.7) admits (at least) three conserved quantities: such equations include the gKdV, gBBM, and the gCH equations. Upon linearizing about such a T-periodic solution, i.e. calculating the Frechet derivative at the T-periodic traveling wave solution, and taking the Laplace transform in time one encounters a spectral problem on $L^2(\mathbb{R})$ of the form

$$N'(u)v = \mu \mathcal{D} v, \tag{1.8}$$

where $N'(u)$ is a linear operator with periodic coefficients. It follows that this thesis deals primarily with the spectral theory for linear operators with periodic coefficients. In order to understand the nature of such spectra, this section is devoted to reviewing the basic results of Floquet theory and the periodic Evans function.

Since the operator $\mathcal{A} := \mathcal{D}^{-1}N'(u)$ has T-periodic coefficients the spectrum is best described using Floquet's theorem, which we now state.

Theorem 1 (Floquet's Theorem)**.** *Let $\Phi(x)$ be a fundamental solution matrix for the*

T-periodic first order system on $\mathbb{R}^n$

$$\dot{\mathbf{y}} = \mathbf{A}(x)\mathbf{y}, \quad x \in \mathbb{R}, \tag{1.9}$$

i.e. the columns of the matrix $\Phi(x)$ are linearly independent for each $x \in \mathbb{R}$ and each column is a (vector) solution to the system (1.9)*. Define the associated matrix operator $\mathbf{M} = \Phi(T)\Phi(0)^{-1}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{M}$, and $\mu \in \mathbb{C}$ is such that $e^{\mu T} = \lambda$, then there is a non-trivial solution of* (1.9) *of the form*

$$y(x) = e^{\mu x}p(x)$$

where p is a T-periodic function. Moreover, every solution of (1.9) *can be written in this way.*

The matrix $\mathbf{M}$ in mentioned in Floquet's theorem is known as the monodromy operator, or the period map, of the T-periodic system (1.9). The proof of Floquet's theorem is well known and can be found in [17] for example. In contrast to the solitary wave case, the spectrum of the linearization about a periodic wave is completely essential and coincides with the continuous spectrum. Indeed, notice that it follows from Floquet's theorem that the point spectrum of an operator with periodic coefficients acting on $L^2(\mathbb{R})$ is empty: if ψ is a solution of the (1.9), it follows from Floquet's theorem that there exists a $\mu \in \mathbb{C}$ and a T-periodic function p such that $\psi(x) = e^{\mu x}p(x)$. In particular, for any $N \in \mathbb{Z}$ we have

$$\psi(NT) = e^{N\mu T}p(0) = \lambda^N p(0)$$

where $\lambda = e^{\mu T}$ is an eigenvalue of $\mathbf{M}$. If the solution $\psi(x)$ decays as $x \to \infty$ it immediately follows that it must be unbounded as $x \to -\infty$, from which our claim follows. Moreover, since $\mathcal{A}^\dagger$ also has periodic coefficients and is defined on $L^2(\mathbb{R})$, it follows that the point spectrum of $\mathcal{A}^\dagger$ is empty and hence the residual spectrum of $\mathcal{A}$ is empty. Thus, the spectrum of the operator $\mathcal{A}$ is completely essential and coincides with the continuous spectrum. Moreover, one could use a variant of Weyl's theorem to conclude that $\mu \in \operatorname{spec}(\mathcal{A})$ if and only if there exists a non-trivial uniformly bounded solution of the equation $\mathcal{A}v = \mu v$ (see page 140 of [36]). Thus, the set $\operatorname{spec}(\mathcal{A})$ consists entirely of $L^\infty(\mathbb{R})$ eigenvalues.

From the above remarks, it follows that solutions of a T-periodic first order system of the form (1.9) are stable (in the sense of Lyapunov) if and only if the monodromy matrix $\mathbf{M}$ has an eigenvalue on the unit circle, which we denote S^1. This immediately leads us to the following definition.

Definition 1. *Suppose the eigenvalue problem* (1.8) *is written as a first order system as*

$$Y_x = \mathbf{H}(x, \mu)Y$$

where $x \in \mathbb{R}$ *and* $Y(x) \in \mathbb{R}^3$. *The monodromy operator* $\mathbf{M}(\mu)$ *is defined to be the map*

$$\mathbf{M}(\mu) = \Phi(T, \mu)$$

where $\Phi(x, \mu)$ *solves the initial value problem*

$$\Phi_x = \mathbf{H}(x, \mu)\Phi, \quad \Phi(0, \mu) = \mathbf{I}$$

where $\mathbf{I}$ *is the* 3×3 *identity matrix. We say* $\mu \in \operatorname{spec}(\mathcal{A})$ *if there exists a non-trivial bounded function* ψ *such that* $\mathcal{A}\psi = \mu\psi$. *Moreover, we say the* T*-periodic solution of* (1.7) *is spectrally stable if and only if the stability spectrum of the corresponding linearized operator does not intersect the open right half plane in* $\mathbb{R}$.

As an immediately corollary of Floquet's theorem and the above definitions, we have the following result which will be used heavily throughout this work.

Theorem 2. *We have* $\mu \in \operatorname{spec}(\mathcal{A})$ *if and only if there exists a* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$ *and*

$$\det\left(\mathbf{M}(\mu) - \lambda\mathbf{I}\right) = 0.$$

This theorem allows one to encode the spectrum of the linear operator $\mathcal{A}$ as the solution set to a function of two complex variables. Following Gardner [29, 30] we define the periodic Evans function $D : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ to be the characteristic polynomial of the corresponding period map, i.e. by the formula

$$D(\mu, \lambda) = \det\left(\mathbf{M}(\mu) - \lambda \mathbf{I} \right).$$

It follows that $\mu \in \operatorname{spec}(\mathcal{A})$ if and only if $D(\mu, \lambda) = 0$ for some $\lambda \in S^1$. As a result, we will frequently work with the function $D(\mu, e^{i\kappa})$ [10] for $(\mu, \kappa) \in \mathbb{C} \times (\mathbb{R}/2\pi\mathbb{Z})$, but the more general definition defined on all of $\mathbb{C}^2$ is useful for certain analyticity arguments. Notice, however, that the notation at this point becomes a bit-cumbersome. In particular, since the L^2 spectrum of the linearized operator is completely essential, one must take care in defining the multiplicity of an element of $\operatorname{spec}(\mathcal{A})$. Clearly, one can speak of the geometric and algebraic multiplicities of the elements of $\operatorname{spec}(\mathcal{A})$ as L^∞ eigenvalues. However, these notions are not particularly helpful in our analysis. In order to define a useful notion of the multiplicity of an element of $\operatorname{spec}(\mathcal{A})$, we make the following definitions.

Definition 2. *A point $\mu \in \operatorname{spec}(\mathcal{A})$ is called a λ-eigenvalue of $\mathcal{A}$ if $|\lambda| = 1$ and λ is an eigenvalue of the corresponding monodromy matrix $\mathbf{M}(\mu)$. The geometric λ-multiplicity of μ is given by the dimension of the kernel of the matrix $\mathbf{M}(\mu) - \lambda\mathbf{I}$. Moreover, the algebraic λ-multiplicity of μ is the dimension of the generalized null space of the operator $\mathcal{A}$ acting on the space $\{v \in L^2(\mathbb{R}) : v(x+T) = \lambda v(x)\}$.*

Throughout this thesis, we will spend much of our time studying the 1-eigenvalues of the operator $\mathcal{A}$. Since such elements of $\operatorname{spec}(\mathcal{A})$ correspond to eigenvalues in the space $L^2_{\text{per}}([0,T])$, we slightly abuse notation and refer to the 1-eigenvalues as "periodic eigenvalues" of the operator $\mathcal{A}$. Similarly, we refer to the algebraic 1-multiplicity of a periodic eigenvalue of $\mathcal{A}$ as just the periodic multiplicity. In order to make a connection of these notions of multiplicity to the periodic Evans function, we point out the following

[10] This is actually the definition of the periodic Evans function introduced by Gardner.

Theorem of Gardner [29].

Theorem 3. *The periodic Evans function $D(\mu, \lambda)$ is analytic in the complex variables μ and λ. Moreover, for a fixed $\lambda \in S^1$ and $\mu \in \mathrm{spec}(\mathcal{A})$ a λ-eigenvalue, the multiplicity of the root μ of the function $D(\,\cdot\,, \lambda)$ is precisely the algebraic λ-multiplicity of μ.*

The first claim of this theorem is clear. Since $\partial_{\bar{\mu}} H(x, \mu) = 0$, it follows that $\partial_{\bar{\mu}} D(\mu, \lambda) = 0$. Similarly, one finds that $\partial_{\bar{\lambda}} D(\mu, \lambda) = 0$. The second claim is considerably more complicated and the reader is referred to Proposition 2.5 of [29]: as we will see shortly, this result is clear for constant coefficient equations. Due to the joint analyticity of D on the complex variables μ and λ, a basic result from several complex variable theory immediately yields the following corollary.

Corollary 1. *The spectrum* $\mathrm{spec}(\mathcal{A})$ *contains no isolated points in* $\mathbb{C}$.

Proof. We recall the following theorem from which the corollary follows: if for fixed λ^* the function $D(\mu, \lambda^*)$ has a zero of order k at μ^* and is holomorphic in a polydisc about (μ^*, λ^*) then there is some smaller polydisc about (μ^*, λ^*) so that for every λ in a disc about λ^* the function $D(\mu, \lambda)$ (with λ fixed) has k roots in the disc $|\mu - \mu^*| < \delta$. For details see the text of Gunning[34]. □

As an elementary example, we consider the constant coefficient linear PDE

$$u_t = Au_{xxx} + Bu_{xx} + Cu_x \tag{1.10}$$

where $x, t \in \mathbb{R}$ and A, B, C are fixed real constants. This equation takes the form (1.7) with $\mathcal{D} = 1$ and $N(u) = L$, where $L := A\partial_x^3 + B\partial_x^2 + C\partial_x$. Since $N(\cdot)$ is linear, $N'(u) = L$ and hence the stability of a T-periodic stationary solution to (1.10) is governed by the L^2 spectrum of the linear operator L. Now, writing the equation $Lv = \mu v$ as the first order system of form (1.9), with $\mathbf{H}(x, \mu)$ identically equal to the constant (in the variable x) matrix

$$\mathbf{H}(\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ A^{-1}\mu & -A^{-1}C & -A^{-1}B \end{pmatrix},$$

it follows that for equation (1.10) the monodromy operator can be expressed as $\mathbf{M}(\mu) = \exp(\mathbf{H}(\mu)T)$. Thus, we have the factorization

$$\begin{aligned} D(\mu, e^{i\kappa}) &= \det\left(\exp(\mathbf{H}(\mu)T) - e^{i\kappa}\mathbf{I}\right) \\ &= \prod_{j=1}^{3}\left(e^{\omega_j(\mu)T} - e^{i\kappa}\right) \end{aligned}$$

where $\omega_j(\mu)$ denotes the eigenvalues of $\mathbf{H}(\mu)$, i.e. they are roots of the characteristic equation

$$A\omega^3 + B\omega^2 + C\omega - \mu = 0. \tag{1.11}$$

In this case, we see that if $D(\mu, \lambda) = 0$, then the algebraic λ-multiplicity of μ is equal to the algebraic multiplicity of λ as an eigenvalue of $\mathbf{M}(\mu)$. Indeed, it is clear that for a fixed $\kappa_0 \in \mathbb{R}$ and μ_0 such that $D(\mu_0, e^{i\kappa_0}) = 0$, the algebraic multiplicity of $e^{i\kappa_0}$ as an eigenvalue of the matrix $\mathbf{M}(\mu_0)$ is precisely the order of the root μ_0 of the equation $D(\mu, e^{i\kappa_0}) = 0$. Moreover, the geometric $e^{i\kappa_0}$-multiplicity of μ is precisely the geometric multiplicity of $e^{i\kappa_0}$ as an eigenvalue of $\mathbf{M}(\mu_0)$. Thus, in the constant coefficient case it is clear that the above definitions agree with our more classical definitions of multiplicities of eigenvalues.

Continuing this example, it follows that the zero set of D consists of all $\mu \in \mathbb{C}$ and $k \in \mathbb{R}$ such that (1.11) has a root of the form

$$\omega(\mu) = i\kappa/T \pmod{2\pi i/T}.$$

Hence, $\mu \in \operatorname{spec}(L)$ if and only if the matrix $\mathbf{H}(\mu)$ has an eigenvalue on the imaginary axis. With this motivation in mind, setting $\lambda = i\kappa/T$ in the characteristic equation (1.11) leads to the dispersion relation

$$iA\kappa^3 + BT\kappa^2 - iCT^2\kappa + T^3\mu = 0,$$

which recovers the standard characterization of the spectrum of L via the Fourier Transform (modulo the superfluous factor of the period T). Since the constants A, B, and C were assumed to be real, it follows that the underlying periodic wave is spectrally stable if and only if $B \leq 0$. Notice in this example the $L^2(\mathbb{R})$ spectrum of the operator $N'(u) = L$ is purely continuous, which agrees with Corollary 1.

We are now able to briefly describe the main methods used in the spectral stability analysis of the operator $\mathcal{A}$ via the periodic Evans function. Suppose we have a T-periodic solution of (1.7). In the examples we will consider, the operator N will be invariant under translations in the x-variable. By Noether's theorem, it follows that $\mu = 0$ belongs to the set spec($\mathcal{A}$). Moreover, in each of the these cases we will be able to construct two other non-trivial bounded solutions of $\mathcal{A}v = 0$, and a linear combination of these can be taken to be T-periodic. Thus, the function $D(\mu, 1)$ will have a root of multiplicity at least two at $\mu = 0$, i.e. $\mu = 0$ is a periodic eigenvalue of $\mathcal{A}$ of 1-multiplicity at least two. Due to the Hamiltonian nature of equation (1.7), we are also able to explicitly construct a T-periodic generalized eigenfunction corresponding to $\mu = 0$. Thus, $\mu = 0$ is a periodic eigenvalue of $\mathcal{A}$ of 1-multiplicity at least three. Moreover, this multiplicity is exactly three assuming a particular non-degeneracy condition is met. From the theory of branching solutions to nonlinear equations, it follows there will be three branches of solutions $\mu_i(\kappa)$, $i = 1, 2, 3$ which solve the equation $D(\mu(\kappa), e^{i\kappa}) = 0$ for $\kappa \in \mathbb{R}$ sufficiently small. That is, the three periodic eigenvalues of $\mathcal{A}$ at $\mu = 0$ will bifurcate from the $\mu = 0$ state when one considers perturbations which are $\widetilde{T}$-periodic with $|T - \widetilde{T}| \ll 1$. In general, the three roots can be expressed in terms of a power series in fractional powers of κ: a Puiseux series. Hence, the spectrum in a neighborhood of the origin will not in general be analytic in the Floquet parameter κ. To calculate this expansion to first order, we use develop a perturbation theory appropriate to a Jordan block to determine the dominant balance of the equation $D(\mu, e^{i\kappa})$ in a neighborhood of $(\mu, \kappa) = (0, 0)$. From this, we are able to use a Newton diagram (see Appendix) to determine the leading order expansion of the roots $\mu_i(\kappa)$. In the case that the first order term of the μ_i has positive real part, it follows that spec($\mathcal{A}$) has a non-trivial intersection with the open right half plane in $\mathbb{C}$ thus implying exponential instability of the underlying periodic wave due to the existence of an exponentially (temporally) growing eigen-mode of the linearization (Frechet derivative) at the periodic traveling wave.

This spectral instability in a neighborhood of the origin is what is commonly referred to as a modulational instability. Since the L^∞ eigenfunctions of the linearized operator represent the admissible perturbations of the underlying wave, it follows that modulational instability analysis is equivalent to studying the stability of the periodic wave profile to perturbations of long-wavelength, i.e. slow modulations in the underlying wave[11]. There is well-developed physical theory along these lines known as Whitham modulation theory [64, 65]. This theory essentially begins by working with a corresponding Lagrangian and by noticing that there are two scales to consider: a slow scale of the perturbation and a fast scale of the oscillation of the underlying wave. Since the variation in the perturbation is minimal over a single period of the underlying solution, we can formally average the dynamics of (1.7) over a period. Considering variations of this average Lagrangian yields a system of equations for the variations of the free-parameters as functions of the slow variables which is in general hyperbolic. Modulational stability or instability is then determined by whether the characteristics for this hyperbolic system are real or complex. While the resulting modulation theory of Whitham provides formal modulational instability results, most rigorous calculations along these lines occur only in the completely integrable settings. A future goal of the modulational instability analysis presented in this thesis is to provide rigorous justifications of the formal averaging of Whitham theory in the non-integrable setting: while this thesis presents the rigorous results for the gKdV and related equations, we have not yet carried out a formal Whitham modulation calculation in this context.

This thesis also concerns the development of a stability theory to perturbations of the same period as the underlying wave[12]. Note that since there are no isolated points of the

[11]The L^∞ eigenfunctions of $\mathcal{A}v = \mu v$ with $|\mu| \ll 1$ correspond to perturbations of the underlying wave with period *close* to that of the underlying wave. Thus, on compact intervals of space and time the perturbation *looks* co-periodic with the solution, but as one moves farther out in space and time one sees noticeable variation.

[12]We will often refer to such perturbations as simply periodic perturbations, with the implication that we require the period to be that of the underlying wave.

spectrum, such an instability immediately implies the existence of a curve of spectrum supported away from zero, and hence a periodic spectral instability immediately implies a finite-wavelength instability (in analogy with the long-wavelength theory mentioned above). This theory is developed by first noticing that periodic eigenvalues of $\partial_x\mathcal{L}[u]$ are precisely the roots of the function $D(\mu, 1)$. As pointed out in chapter 2, such periodic eigenvalues must either be purely imaginary or real: thus if one is interested in developing a stability theory to such perturbations, it is enough to consider roots of the function

$$D(\,\cdot\,;1):\mathbb{R}\to\mathbb{R}.$$

By analyzing the function $D(\mu, 1)$ for $\mu \in \mathbb{R}^+$, it is clear that if it the values for μ near the origin and near $+\infty$ have opposite signs, then there are an odd number positive periodic eigenvalues (counting multiplicity[13]). It should be clear that this index is a generalization of the orientation index familiar from the solitary wave theory. Indeed, we will prove in each of our cases that this index recovers the solitary wave stability theory in a large period limit: see chapter 3. Although this result is known in many cases by the results of Gardner, we provide new proofs using simple asymptotic estimates to determine the sign of the above orientation index in this limit. We also prove this index is intimately related to the nonlinear stability of the underlying wave to periodic perturbations[14].

1.3 Basic Examples

In this section, we present two elementary examples employing our methods of modulational instability analysis. To begin, consider the modulational stability of a T-periodic solution of the third order constant coefficient linear system on $\mathbb{R}^n$

$$u_t = \mathbf{A}u_{xxx} + \mathbf{C}u_x$$

[13]More precisely, counting the periodic multiplicity, i.e. the algebraic 1 multiplicity.
[14]Again, by periodic perturbations we mean perturbations which are T-periodic.

where $u(x,t) \in \mathbb{R}^n$ for each fixed $(x,t) \in \mathbb{R}^2$, and $\mathbf{A}$ and $\mathbf{C}$ are now constant $n \times n$ real matrices. The spectral stability of such a solution is determined by the spectrum of the linearized operator $L = \mathbf{A}\partial_x^3 + \mathbf{C}\partial_x$ acting on $L^2(\mathbb{R})$. Since this spectral problem is invariant under the transformation $(x,\mu) \mapsto (-x,-\mu)$, it follows that the set spec(L) is symmetric and hence spectral stability is equivalent to the condition spec $(L) \subset \mathbb{R}i$. Straight forward computations show that the characteristic equation of the corresponding $\mathbf{H}(x,\mu) = \mathbf{H}(\mu)$ takes the form

$$\det\left(\mathbf{A}\lambda^3 + \mathbf{C}\lambda - \mu\mathbf{I}\right) = 0. \tag{1.12}$$

Thus, $D(\mu, e^{i\kappa})$ admits the factorization

$$D(\mu, e^{i\kappa}) = \prod_{j=1}^{3n}\left(e^{\lambda_j(\mu)T} - e^{i\kappa}\right)$$

where the set $\{\lambda_j(\mu)\}_{j=1}^{3n}$ denotes the $3n$ roots of (1.12). It follows that $\mu \in$ spec(L) if and only if $\mathbf{H}(\mu)$ has an eigenvalue on the imaginary axis. Setting $\lambda = ik$ for some $k \in \mathbb{R}$ and substituting into the characteristic equation (1.12) yields the dispersion relation

$$\det\left(i\mathbf{A}k^3 - i\mathbf{C}k + \mu\mathbf{I}\right) = 0.$$

In particular, notice that since (1.12) has a n-fold root of $\mu = 0$ at $k = 0$, the function $D(\mu, 1)$ has an n-fold root at $\mu = 0$. Hence, $\mu = 0$ is a periodic eigenvalue of L of periodic-multiplicity n. A simple calculation for the n roots of (1.12) bifurcating from $\mu(0) = 0$ yields the following asymptotic expansion:

$$\mu_j(k) = 0 - ic_jk + \mathcal{O}(k^2).$$

where the c_j represent the eigenvalues of the matrix $\mathbf{C}$. In particular, notice that in this case the n roots μ_j are analytic in the variable λ. Recalling that $k \in \mathbb{R}$ and that the matrix $\mathbf{C}$ is real, it follows that a necessary condition for stability in this setting is the hyperbolicity of the matrix $\mathbf{C}$, i.e. $\sigma(\mathbf{C})$ must be real.

As a slightly more sophisticated example, which is more in line with the modulational stability analysis in this thesis, consider a smooth T-periodic stationary solution $u(x)$ of the second order reaction-diffusion equation

$$u_t = u_{xx} - f(u), \tag{1.13}$$

where f is sufficiently smooth ($f \in C^2(\mathbb{R})$ for example)[15]. This equation takes the general form (1.7) with $\mathcal{D} = -1$ and $N(u) = -u_{xx} + f(u)$: although this equation is not dispersive, it provides a nice model example on which to base the spectral stability theory of the present work. Linearizing around u and taking the Laplace transform in time leads to a spectral problem of the form $Lv = \mu v$ considered on $L^2(\mathbb{R})$, where L is the operator [16]

$$L := N'(u) = -\partial_x^2 + f'(u)$$

with densely defined domain $C^\infty(\mathbb{R})$. It is clear by integration by parts that L is symmetric on $L^2(\mathbb{R})$. Moreover, since u is clearly bounded on $\mathbb{R}$, it follows that L is a self-adjoint operator on $L^2(\mathbb{R})$, and hence its spectrum is purely real. It follows that u will be a spectrally stable solution of (1.13) if and only if

$$\text{spec}(L) \subset \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda \geq 0\}.$$

However, this is easily seen not to be the case unless $f'(u) \equiv 0$: by differentiating (1.13) with respect to x implies that $Lu_x = 0$, and hence $\mu = 0$ is a periodic eigenvalue of L. Since u_x vanishes at some point on $[0, T]$, it can not correspond to the ground state eigenvalue and hence one has spectral instability in this case. Even with this in mind, we can still ask meaningful stability questions. In particular, we can seek conditions for such a solution to be *modulationally* stable, i.e. conditions to ensure that

[15] In order for periodic solutions of (1.13) to exist, slightly more must be assumed. By simple phase plane analysis, such solutions exist if and only if f has a local maximum at some point in $\mathbb{R}$. We will return to this point at the end of this section, but for now we assume that such a solution exists, and hence f satisfies any hypothesis needed to make this assumption valid.

[16] Notice that we have changed notations from the analogous example in section 1.1. This does not make a large difference in the resulting stability theories, and makes L a relatively compact perturbation of the *positive* operator $-\partial_x^2$.

spec$(L) \cap B(0,R) \subset \mathbb{R}^+$ for some sufficiently small $R > 0$. By defining $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$, where $\mathbf{M}(\mu)$ is the monodromy map corresponding to the spectral problem $Lv = \mu v$, it is then clear from Theorem 5 from chapter 2 that this solution is modulationally unstable if and only if the sign of $a'(0)$ is such that spec(L) intersects the negative real axis near $\mu = 0$. Although this trivially follows from the Theorem 5, we now prove the following theorem using the periodic Evans function techniques outlined in the previous section in order to give the reader a feeling for the analysis present throughout this thesis.

Theorem 4. *A T-periodic smooth solution of* (1.13) *is modulationally stable if and only if $a'(0) < 0$.*

To prove this theorem, notice that if we write (2.20) as a first order system of form (1.9) with

$$\mathbf{H}(x,\mu) = \begin{pmatrix} 0 & 1 \\ f'(u) - \mu & 0 \end{pmatrix}.$$

Since $\operatorname{tr}(\mathbf{H}(x,\mu)) = 0$ for all $x \in \mathbb{R}$ and $\mu \in \mathbb{C}$, it follows that $\det(\mathbf{M}(\mu)) = 1$ for all $\mu \in \mathbb{C}$ where $\mathbf{M}(\mu)$ is the corresponding monodromy operator. Since u_x is periodic with the same period as u, it follows that there is an invertible matrix V and a real number σ such that

$$V\,\mathbf{M}(0)V^{-1} = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}.$$

Thus, generically, 1 is an eigenvalue of $\mathbf{M}(0)$ of algebraic multiplicity two and geometric multiplicity one. Using the analyticity of $\mathbf{M}(\mu)$ near $\mu = 0$, we can expand the periodic Evans function case near $(\mu, \lambda) = (0,1)$ as

$$\begin{aligned} D(\mu,\lambda) &= \det(\mathbf{M}(\mu) - \mathbf{I} - \eta \mathbf{I}) \\ &= \eta^2 - (a(\mu) - 2)\eta + b(\mu) \end{aligned} \tag{1.14}$$

where $\eta = \lambda - 1$, $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$, and $b(\mu) = \det(\mathbf{M}(\mu) - \mathbf{I})$. Our first goal is to determine the dominant balance of the equation $D(\mu,\lambda)$ in a neighborhood of $(\mu,\lambda) =$

$(0,1)$. To this end, we use the specific structure of the linearized operator L to prove that $a(\mu) = 2 - b(\mu)$ for all $\mu \in \mathbb{C}$. This follows from the fact that for any fundamental matrix solution $\Phi(x)$ for the system $Y'(x) = \mathbf{H}(x,\mu)Y(x)$, with $\mathbf{H}(x,\mu)$ defined as above, which satisfies $\Phi(0) = \mathbf{I}$ one has

$$\Phi(x)^{\dagger}\,\mathbf{J}\Phi(x) = \mathbf{J}$$

for all $x \in \mathbb{R}$, where $\mathbf{J}$ is the standard skew symmetric matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To see this, notice that the matrix $\mathbf{H}_0(x,\mu) := -\mathbf{J}\,\mathbf{H}(x,\mu)$ is self adjoint since $\mu \in \mathbb{R}$. Therefore, differentiating the expression $\Phi(x)^{\dagger}\,\mathbf{J}\Phi(x)$ with respect to x yields

$$\begin{aligned}
\partial_x\left(\Phi(x)^{\dagger}\,\mathbf{J}\Phi(x)\right) &= \Phi_x(x)^{\dagger}\,\mathbf{J}\Phi(x) + \Phi(x)\,\mathbf{J}\Phi_x(x) \\
&= \left(\mathbf{J}\,\mathbf{H}_0(x,\mu)\Phi(x)\right)^{\dagger}\,\mathbf{J}\Phi(x) + \Phi(x)\,\mathbf{J}\left(\mathbf{J}\,\mathbf{H}_0(x,\mu)\Phi(x)\right) \\
&= -\Phi(x)^{\dagger}\,\mathbf{H}_0(x,\mu)\,\mathbf{J}^2\Phi(x) + \Phi(x)\,\mathbf{J}^2\,\mathbf{H}_0(x,\mu)\Phi(x) \\
&= \Phi(x)^{\dagger}\,\mathbf{H}_0(x,\mu)\Phi(x) - \Phi(x)\,\mathbf{H}_0(x,\mu)\Phi(x) \\
&= 0.
\end{aligned}$$

Since $\Phi(0) = \mathbf{I}$ by assumption, the claim follows. Therefore, since $\det(\mathbf{M}(\mu)) = 1$, it follows that the periodic Evans function satisfies

$$\begin{aligned}
D(\mu,\lambda) &= \det\left(\mathbf{M}(\mu)^{-1} - \lambda\mathbf{I}\right) \\
&= \lambda^2 \det\left(\mathbf{M}(\mu) - \lambda^{-1}\mathbf{I}\right) \\
&= (\lambda-1)^2 + (a(\mu)-2)\,\lambda(\lambda-1) + b(\mu)\lambda^2. \qquad (1.15)
\end{aligned}$$

Comparing the $\mathcal{O}(1)$ terms between (1.14) and (1.15) implies that

$$a(\mu) + b(\mu) = 2$$

as claimed. In fact, this is essentially the only useful information the symmetry $\mathbf{M}(\mu) \sim \mathbf{M}(\mu)^{-\dagger}$ yields, since comparing the $\mathcal{O}(\lambda)$ terms in (1.14) and (1.15) yields a trivial identity and comparing the $\mathcal{O}(\lambda^2)$ yields the same information as the $\mathcal{O}(1)$ terms.

By the above identity, and by the analyticity of the functions a and b, it follows that in a sufficiently small neighborhood of $(\mu, \lambda) = (0, 1)$, the equation $D(\mu, \lambda) = 0$ has the asymptotic expansion

$$\eta^2 - a'(0)\eta\mu - a'(0)\mu - \frac{a''(0)}{2}\mu^2 + \mathcal{O}(3) = 0, \tag{1.16}$$

where $\mathcal{O}(3)$ represents terms which are degree three and higher in the variables μ and η. In order to prove Theorem 4, we distinguish between two cases. First, consider the case where $a'(0) \neq 0$. It follows from (1.16) that $D_\mu(0, 1) \neq 0$ and hence $\mu = 0$ is a simple root of $D(\mu, 1) = 0$, i.e. $\mu = 0$ is a simple periodic eigenvalue of the operator L. Moreover, using a Newton diagram, it follows that there is a single branch of spectrum bifurcating from the $\mu = 0$ state which admits the asymptotic expansion

$$\mu(\kappa) = \frac{-\kappa}{a'(0)} + \mathcal{O}(\kappa^2),$$

where again we used the fact that $\eta = i\kappa + \mathcal{O}(\kappa^2)$ for $|\kappa| \ll 1$. Thus, it immediately follows that the periodic solution u of (1.13) exhibits a modulational instability in this case if and only if $a'(0) > 0$.

Next, if $a'(0) = 0$ it follows that $\mu = 0$ is a periodic eigenvalue of L of (periodic) multiplicity two. Indeed, by (1.16) we see that in this case the function $D(\mu, 1)$ has a root at $\mu = 0$ of multiplicity two. Taking $\eta = i\kappa + \mathcal{O}(\kappa^2)$ for $\kappa \in \mathbb{R}$, it follows by (1.16) that there are two branches of spectrum bifurcating from the $\mu = 0$ state which admit the asymptotic expansion

$$\mu_{\pm}(\kappa) = \pm \left(\frac{-2\,\mathrm{sign}(a''(0))}{|a''(0)|} \right)^{1/2} \kappa + \mathcal{O}(\kappa)^2.$$

Since $\mathrm{spec}(L) \subset \mathbb{R}$, it follows that the quantity $a''(0)$ must be negative (as predicted by Theorem 5), and hence one always has modulational instability in this case. This completes our proof of Theorem 4

By the above analysis, it follows that the quantity $a'(0)$ serves as a "modulational instability index" in the sense that its sign completely determines the stability of periodic stationary solutions of (1.13) to perturbations of long-wavelength. This index can also be described in terms of properties of the solution u: in particular, its period. Indeed, notice that (1.13) restricted to stationary solutions can be integrated to yield

$$-\frac{1}{2}u_x^2 + F(u) = E \tag{1.17}$$

for some constant E, where F is a function such that $F' = f$ and $F(0) = 0$. The constant E is usually interpreted as the "energy" of the corresponding solution of the ODE. We assume that E is such that the function $F(x)$ has a local maximum greater than the energy level E, thus guaranteeing the existence of periodic solutions for initial data sufficiently close to this maximum. We have already seen that u_x is a periodic solution of the equation $Lv = 0$. We now set $u(0) = u_-$ and note that, due to translation invariance, we may assume that $u_x(0) = 0$ thus factoring out the (continuous) symmetry group of equation (1.13) restricted to stationary solutions. By differentiating (1.17) with respect to E we see that $\frac{\partial}{\partial E}u(x;E) = u_E$ also solves $Lv = 0$, and hence we may use u_x and u_E to explicitly calculate $\mathbf{M}(0)$ in a basis. Indeed, defining $y_1(x) = \left(\frac{du_-}{dE}\right)^{-1} u_E$ and $y_2(x) = -\left(f'(u_-)\right)^{-1} u_x(x)$, it follows from direct calculation that

$$y_1(0) = 1, \quad y_2(0) = 0$$
$$y_1'(0) = 0, \quad y_2'(0) = 1$$

Thus, it follows by calculating $u_E(T)$ by the chain rule that we have

$$\mathbf{M}(0) = \begin{pmatrix} 1 & f(u_-)\frac{du_-}{dE}T_E \\ 0 & 1 \end{pmatrix}$$

where $T_E = \frac{d}{dE}T$. Notice that differentiating (1.17) with respect to E and evaluating at $x = 0$ shows that $f'(u_-)\frac{du_-}{dE} = 1$. Next, using variation of parameters, one can express $\frac{d}{d\mu}y_j$ and $\frac{d}{d\mu}y_j'$ in terms of the y_j and y_j' (see [49]). Finally, using the facts that $\det(\mathbf{M}(0)) = 1$ and $\operatorname{tr}(\mathbf{M}(0)) = a(0) = 2$, a bit of algebra eventually yields the expression

$$a'(0) = \operatorname{sign}(y_1'(T)) \int_0^T \left(\sqrt{|y_1'(T)|}y_2 + \operatorname{sign}(y_1'(t)) \frac{y_1(T) - y_2'(T)}{2\sqrt{|y_1'(T)|}} y_1 \right)^2 dx.$$

It now follows directly that

$$\operatorname{sign}\big(a'(0)\big) = \operatorname{sign}(T_E).$$

Thus, by our previous work it follows that T_E serves as a modulational instability index for the stationary periodic solutions of equation (1.13), in the sense that such a solution is modulationally stable if and only if $T_E < 0$. In particular, this shows the modulational stability of such a solution can be determined by properties of the solution itself[17].

Using the above example as a model, we can state our general method for deriving modulational stability indices for equations of form (1.7)

1. First, determine a basis for the kernel of the operator $N'(u)$. Use this to understand the Jordan structure of $\mathbf{M}(0)$.

2. Using symmetries inherent in the equation (1.7), as well as a perturbation theory appropriate to a Jordan block, determine the dominant balance of the equation $D(\mu, \lambda) = 0$ in a neighborhood of $(\mu, \lambda) = (0, 1)$. This will in general encode

[17] In chapter 3, we will prove that $T_E > 0$ for a large class of nonlinearities if the solution does not bound a homoclinic orbit. It follows that such solutions of the reaction-diffusion equation (1.13) are always modulationally unstable.

several branches of spectrum bifurcating from the $\mu = 0$ state as a polynomial in the spectral parameter μ and the Floquet exponent κ.

3. Analyze the above polynomial to determine conditions which guarantee the spectral branches bifurcating from $\mu = 0$ all remain in the stable closed left half plane of $\mathbb{C}$.

In the cases we will encounter in this thesis, we will be able to reduce the n^{th} order stationary traveling wave ODE $N(u) - c\mathcal{D}u_x = 0$ to quadrature which, by Noether's theorem, will generate an $n+1$-dimensional manifold of periodic traveling wave solutions. As in the previous example, under this integrability assumption we will be able to describe the modulational stability index in terms of properties of the underlying T-periodic solution itself: in particular, we can express it in terms of gradients of the conserved quantities of the PDE (1.7) and the period. The result is a geometric theory for modulational stability theory of T-periodic traveling wave solutions for the equations considered. As mentioned before, most of our analysis can be viewed as rigorous Whitham modulation theory calculations and so it is not surprising that the resulting stability indices have a geometric point of view.

It should be noted that there is no analogue of an orientation index for (1.13). Indeed, by standard asymptotics the monodromy $\mathbf{M}(\mu)$ satisfies

$$\mathbf{M}(\mu) \approx \exp\left(\mathbf{A}_\infty(\mu)T\right)$$

where the matrix $\mathbf{A}_\infty(\mu)$ is defined by

$$\mathbf{A}_\infty = \left(\begin{array}{cc} 0 & 1 \\ -\mu & 0 \end{array}\right).$$

The eigenvalues of $\mathbf{A}_\infty(\mu)$ are $\lambda_\pm(\mu) = \pm i\sqrt{\mu}$ and hence $a(\mu) = 2\cos\left(T\sqrt{\mu}\right)$. It follows that

$$D(\mu, 1) \approx 2 - 2\cos\left(T\sqrt{\mu}\right)$$

for $\mu \gg 1$. In particular, it is clear that the sign of $D(\mu, 1)$ does not have a well defined limit as $\mu \to \infty$, which is to be expected by Theorem 5. In the dispersive equations we will consider in this thesis, this limit is guaranteed to exist since it will be shown that if $\mu \in \mathbb{R}$ is sufficiently large, then μ is not a periodic eigenvalue of the linearization.

CHAPTER 2

Spectral Stability Analysis of the Generalized Korteweg-de Vries Equation

This chapter is devoted to analyzing the spectral stability of a family of periodic traveling wave solutions to the generalized Korteweg-de Vries (gKdV) equation. The methods used throughout this chapter are quite general and apply to many other equations: we choose the gKdV mostly because of its simplicity in applying these methods and partly due to its historical significance in the solitary wave context. In later chapters, we will examine how these same techniques can be applied to the situations of periodic traveling wave solutions of the generalized Benjamin-Bona-Mahony and generalized Camassa-Holm equations. Throughout this chapter, however, we will only be concerned with developing a spectral stability theory for such solutions of the gKdV.

The goal of this chapter is twofold. First, we develop a spectral instability theory for stationary periodic traveling wave solutions of the gKdV to perturbations of long wavelength: so called modulational, or sideband, instability. In particular we are interested in understanding the role played by the null directions of the linearized operator in the stability of the traveling wave to perturbations of long wavelength. We develop an index, which we call the modulational instability index, whose sign gives necessary and sufficient conditions for spectral instability in a sufficiently small neighborhood of the origin in the spectral plane (assuming a particular non-degeneracy condition holds). Secondly, we develop a spectral instability theory for such solutions to perturbations of the same period. In particular, we develop an index which counts (modulo 2) the number of non-zero real periodic eigenvalues of the linearized operator. Since the spectrum of the linearized operator is continuous, this immediately implies instability with respect to finite-wavelength perturbations. This index is shown to be a generalization of the one which governs the stability of the solitary wave, in the sense that through it we regain the solitary wave spectral stability theory as the period tends to infinity. Both of these indices are shown to be expressible in terms of maps between a space of constants parameterizing the traveling waves and the conserved quantities of the governing PDE. These two indices together provide a good deal of information about the spectrum of the linearized operator.

Finally, we sketch the connection of this calculation to a study of the linearized operator - in particular we perform a perturbation calculation in terms of the Floquet parameter. This suggests a geometric interpretation attached to the vanishing of the finite-wavelength instability index previously mentioned.

2.1 Introduction and Preliminaries

In this chapter, we consider the spectral stability of periodic traveling wave solutions to the generalized KdV (gKdV) equation

$$u_t = u_{xxx} + (f(u))_x \tag{2.1}$$

where $f(\cdot) \in C^2(\mathbb{R})$ is a suitably smooth nonlinearity satisfying certain convexity assumptions to be discussed later. This equation admits periodic traveling wave solutions of the form

$$u(x,t) = u(x+ct), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+$$

where the wave-speed c is assumed to be positive and $u(\cdot)$ is a periodic function of its argument. Of particular interest is the case of power law nonlinearity $f(u) = u^{p+1}$, which in the cases $p = 1,\ 2$ represents the equations for traveling wave solutions to the KdV and MKdV, respectively. Obviously such traveling wave solutions are reducible to quadrature: they satisfy

$$u_{xx} + f(u) - cu = a \tag{2.2}$$

$$\frac{u_x^2}{2} + F(u) - c\frac{u^2}{2} = au + E. \tag{2.3}$$

In particular, we are interested in the spectrum of the linearized operator (in the moving coordinate frame)

$$\mu v = v_{xxx} + (vf'(u))_x - cv_x$$

in two related settings. First, we study the spectrum in a neighborhood of $\lambda = 0$. Physically this amounts to long-wavelength perturbations of the underlying wave profile: in essence slow modulations of the traveling wave. There is a well developed physical theory, commonly known as Whitham modulation theory[64, 65], for dealing with such problems. On a mathematical level the origin in the spectral plane is distinguished by the fact that the ordinary differential equation giving the traveling wave profile is completely integrable. Thus the tangent space to the manifold of traveling wave profiles can be explicitly computed, and the null-space to the linearized operator can be built up from elements of this tangent space. We show that these considerations give a rigorous normal form for the spectrum of the linearized operator in the vicinity of the origin providing that certain genericity conditions are met. Assuming that these genericity conditions are met we are able to show the following: there is a discriminant Δ which can be calculated explicitly. If this discriminant is positive then the spectrum in a neighborhood of the origin consists of the imaginary axis[1] with multiplicity three. If this discriminant is negative the spectrum of the linearization in the neighborhood of the origin consists of the imaginary axis (with multiplicity one) together with two curves which leave the origin along lines in the complex plane, implying instability. Long wavelength theories are invariably geometric in nature, and the one presented here is no exception: both the instability index and the genericity conditions admit a natural geometric interpretation.

Secondly, we are interested in determining sufficient conditions for the existence of unstable spectrum supported away from $\lambda = 0$. Here, this is accomplished by calculating an orientation index using Evans function techniques: essentially comparing the behavior of the Evans function near the origin with the asymptotic behavior near in-

[1]Note that this does *not* imply spectral stability since there is the possibility of bands of spectrum off of the imaginary axis away from the origin.

finity. Physically, such an instability amounts to an instability with respect to finite wavelength perturbations. The derived stability index is a generalization of the one which governs stability of solitary waves. In fact, in the case of power-law nonlinearity and wave speed $c > 0$, we show that in a long wavelength limit the sign of this index (which is actually what determines stability) agrees with the sign of the solitary wave stability index derived by, for example, Pego and Weinstein[57, 56].

The work presented in this chapter uses ideas from both stability theory and modulation theory, and as such there is an extensive background literature. Most obviously is the stability theory of solitary wave solutions to KdV type equations which was pioneered by Benjamin[7] and further developed by Bona[10], Grillakis[31], Grillakis,Shatah and Strauss[32, 33], Bona, Souganides and Strauss[11], Pego and Weinstein[57, 56], Weinstein[62, 63] and others. In this theory the role of the discriminant is played by the derivative of the momentum with respect to wave-speed. Our discriminant is somewhat more complicated, which is to be expected: as shown in the next section, the solitary waves homoclinic to the origin are a codimension two subset of the family of periodic solutions, so one expects that the general stability condition will more complicated. There are also a number of calculations of the stability of periodic solutions to perturbations of the same period, or to perturbations of twice the period, due to Angulo Pava[3], Angulo Pava, Bona and Scialom[4] and others. In this setting the linearized operator has a compact resolvent, so the spectrum is purely discrete, and the arguments are similar in spirit to those for the solitary wave stability. In contrast we consider the linearized operator as acting on $L^2(\mathbb{R})$, which corresponds to localized perturbations, where one must understand the continuous spectrum of the operator.

A stability calculation in the spirit of modulation theory was given by Rowlands[59] for the cubic nonlinear Schrödinger equation. Other stability calculations in the same spirit, but differing greatly in details and approach, were given by Gallay and Hărăguş [27, 28], Hărăguş and Kapitula [35], Deconinck and Kapitula [19], Bridges and Rowlands [15], and Bridges and Mielke [5]. The work of Gardner [30] is also related, though it should be noted that the long-wavelength limit in Gardner is very different from the one we consider here: in the former it is the traveling wave itself which has a long period, while in our calculation the period is fixed and we are considering *perturbations* of long period. The ideas represented in the current chapter owes a debt to the substantial literature on Whitham theory for integrable systems developed by Lax and Levermore [43, 44, 45], Flashka, Forest and McLaughlin [26], and many others. We note, however, that the calculation outlined in this paper is not an integrable calculation. The papers that are perhaps closest to that presented here are those by Oh and Zumbrun[52, 53, 54] and Serre [61] on the stability of periodic solutions to viscous conservation laws, where similar results relating the behavior of the linearized spectral problem in a neighborhood of the origin to a formal theory of slow modulations are proved.

The results of this chapter are most explicit in the case of power law nonlinearity. In this case, due to the scaling invariance we can always assume that $c \in \{-1, +1\}$. Indeed, it is easy to check that if $u(x,t;c)$ solves (2.1) with the nonlinearity $f(u) = u^{p+1}$, then

$$|c|^{1/p} u \left(|c|^{1/2} x, |c|^{3/2} t, \operatorname{sgn}(c) \right) \tag{2.4}$$

solves (2.1) with wave speed $\operatorname{sgn}(c)$. This scaling induces a natural scaling on the periodic traveling wave solutions of (2.1), which we are able to exploit in order to determine asymptotic estimates of the mentioned instability indices in the limit as the periodic waves approach the homoclinic orbit, i.e. as the period increases to infinity.

The rest of this chapter is organized as follows: in the next section we determine the basic properties of the periodic traveling wave solutions of (2.1) which will be used extensively throughout this thesis. In section 2.2, we lay out some basic general properties of the spectrum of the linearized operator. In the second section we explicitly compute the monodromy map and associated periodic Evans function at the origin. A perturbation analysis in the neighborhood of the origin gives a local normal form for the Evans function. In section 2.3 we develop similar results from the point of view of the linearized operator: we compute the generalized periodic null-space of the linearized operator in terms of the tangent space to the ordinary differential equation defining the traveling wave. The structure of this null-space (under some genericity conditions) reflects that of the monodromy map at the origin, and a similar perturbation analysis gives a local normal form for the spectrum. While the two approaches are in principle the same some calculations are more easily carried out in one framework than the other: in general the analysis of the linearized operator seems considerably more complicated than our approach via the periodic Evans function. Finally, we end with some concluding remarks.

It should be noted we restrict neither the size of the periodic solution nor the period. Moreover, all of our analysis applies to both localized and bounded perturbations of the underlying wave. Also in this chapter "stability" will always means spectral stability.

2.2 Properties of the Stationary Periodic Traveling Waves

In this section, we recall the basic properties of the periodic traveling wave solutions of (2.1). For each non-zero $c \in \mathbb{R}$, a traveling wave solution of (2.1) with wave speed c is a solution of the traveling wave ordinary differential equation

$$u_{xxx} + f(u)_x - cu_x = 0, \tag{2.5}$$

i.e. they are solutions of (2.1) which are stationary in the moving coordinate frame defined by $x + ct$. Clearly, such solutions are reducible to quadrature and satisfy

$$u_{xx} + f(u) - cu = a, \tag{2.6}$$

$$\frac{1}{2}u_x^2 + F(u) - \frac{c}{2}u^2 - au = E, \tag{2.7}$$

where a and E are real constants of integration, and F satisfies $F' = f$, $F(0) = 0$. In order to ensure the existence of periodic orbits of (2.5), we require that the effective potential

$$V(u; a, c) = F(u) - \frac{c}{2}u^2 - au$$

has a non-degenerate local minimum. Notice this places a restriction on the allowable parameter regime for our problem. This motivates the following definition.

Definition 3. *We define the set* $\Omega \subset \mathbb{R}^3$ *to be the open set consisting of all triples* (a, E, c) *such that the solution* $u(x) = u(x; a, E, c)$ *of* (2.7) *is periodic and its orbit in phase space does not bound a homoclinic orbit.*

Remark 1. *Taking into account the translation invariance of* (2.1)*, it follows that for each* $(a, E, c) \in \Omega$ *we can construct a one-parameter family of periodic traveling wave solutions of* (2.1)*: namely*

$$u_\xi(x, t) = u(x + ct + \xi; a, E, c)$$

where $\xi \in \mathbb{R}$*. Thus, the periodic traveling waves of* (2.1) *constitute a four dimensional manifold of solutions. However, this added constant of integration does not play an important role in our theory. In particular, we can mod out the continuous symmetry of* (2.1) *by requiring all periodic traveling wave solutions to satisfy the conditions* $u_x(0) = 0$ *and* $V'(u(0)) < 0$*. As a result, each periodic solution of* (2.5) *is an even function of the variable* x*.*

In order to understand the set Ω, we now characterize this set in the special case of the KdV equation. In this case, the effective potential takes the form

$$V(u; a, c) = \frac{1}{6}u^3 - \frac{c}{2}u^2 - au.$$

If $2a < -c^2$, then $V'(u; a, c)$ is an increasing function of u and hence no bounded solutions exist. Thus, we must have $2a \geq -c^2$. Fix now a positive wave speed c. For each $a > -\frac{c^2}{2}$ the potential has a (unique) local minima at $u_m = c + \sqrt{c^2 + 2a}$ and a (unique) local maximum at $u_M = c - \sqrt{c^2 + 2a}$, and we define the critical energies

$$E_-(a, c) = V(u_m; a, c) = -\frac{c^2}{3}\left(c + \sqrt{c^2 + 2a}\right) - \frac{a}{3}\left(3c + 2\sqrt{c^2 + 2a}\right)$$
$$E_+(a, c) = V(u_M; a, c) = \frac{c^2}{3}\left(-c + \sqrt{c^2 + 2a}\right) + \frac{a}{3}\left(-3c + 2\sqrt{c^2 + 2a}\right)$$

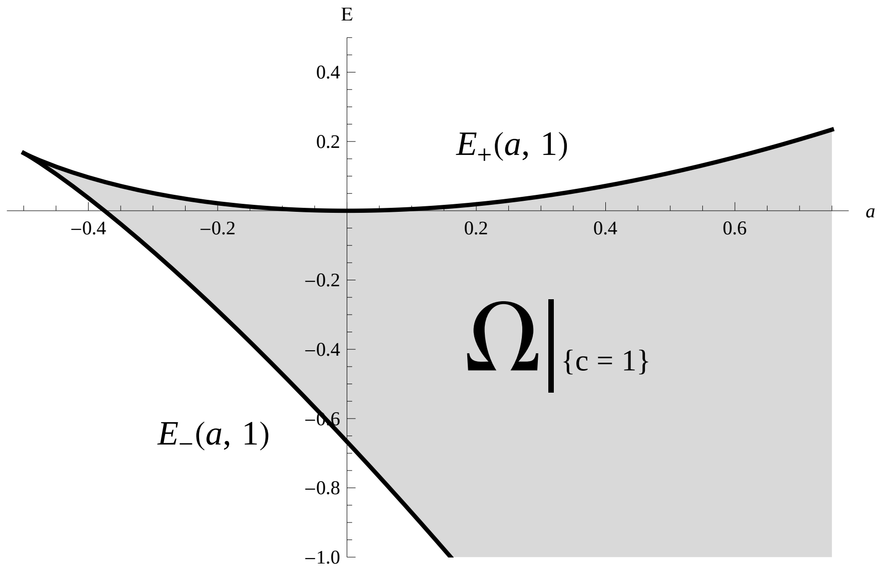

Figure 2.1: The shaded region corresponds to the open set $\Omega \subset \mathbb{R}^3$ restricted to the hyperplane $c = 1$ for the KdV equation.

In order to ensure bounded solutions of (2.5), we must then require

$$E_-(a, c) \leq E \leq E_+(a, c).$$

We thus have three possibilities for the bounded solutions of (2.5) for fixed a and c:

1. If $E = E_-(a, c)$, the corresponding solution is the equilibrium solution

$$u(x; a, E_-(a, c), c) = c.$$

2. If $E = E_+(a, c)$, then either the solution is constant (corresponding to the case when $2a + c^2 = 0$) or else is homoclinic to u_M as $x \to \pm\infty$. The classical solitary wave corresponds to the solution $u(x; 0, E_+(0, c), c)$, and hence the solitary waves form a co-dimension two subset of the set of bounded solutions of (2.5).

3. If $E_-(a, c) < E < E_+(a, c)$, then the solution $u(x; a, E, c)$ is periodic with minimal period $T(a, E, c)$.

Thus, in the case of the KdV equation, the set Ω is given explicitly by

$$\{(a,E,c)\in\mathbb{R}^3 : c>0,\ 2a>-c^2,\ E_-(a,c)<E<E_+(a,c)\}$$

For convenience, the graph of the restriction of this set to the hyperplane $\{c=1\}$ is provided in Figure 2.1. Similar constructions can be used to describe the set Ω, and hence the set of bounded solutions of (2.5), for general non-linearities.

Throughout this paper, we will always assume that our periodic traveling waves correspond to an (a,E,c) within the open region Ω, and that the roots $u_\pm$ of $E = V(u;a,c)$ with $V(u;a,c)<E$ for $u\in(u_-,u_+)$ are simple. It follows that $u_\pm$ are C^1 functions of a, E, c on Ω, and that $u(0)=u_-$. Moreover, given $(a,E,c)\in\Omega$, we define the period of the corresponding solution to be

$$T = T(a,E,c) := 2\int_{u_-}^{u_+}\frac{du}{\sqrt{2\left(E-V(u;a,c)\right)}}. \tag{2.8}$$

The above interval can be regularized at the square root branch points u_-, u_+ by the following procedure: Write $E-V(u;a,c)=(r-u_-)(u_+-r)Q(u)$ and consider the change of variables $u=\frac{u_++u_-}{2}+\frac{u_+-u_-}{2}\sin(\theta)$. Notice that $Q(u)\neq 0$ on $[u_-,u_+]$. It follows that $du=\sqrt{(u-u_-)(u_+-u)}d\theta$ and hence (2.8) can be written in a regularized form as

$$T(a,E,c) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{Q\left(\frac{u_++u_-}{2}+\frac{u_+-u_-}{2}\sin(\theta)\right)}}.$$

In particular, we can differentiate the above relation with respect to the parameters a, E, and c within the parameter regime Ω. Similarly the mass, momentum, and Hamiltonian of the traveling wave are given by the first and second moments of this density, i.e.

$$M(a,E,c) = \langle u\rangle = \int_0^T u(x)\,dx = 2\int_{u_-}^{u_+}\frac{u\,du}{\sqrt{2\left(E-V(u;a,c)\right)}} \tag{2.9}$$

$$P(a,E,c) = \langle u^2\rangle = \int_0^T u^2(x)\,dx = 2\int_{u_-}^{u_+}\frac{u^2\,du}{\sqrt{2\left(E-V(u;a,c)\right)}} \tag{2.10}$$

$$H(a,E,c) = \left\langle\frac{u_x^2}{2}-F(u)\right\rangle = 2\int_{u_-}^{u_+}\frac{E-V(u;a,c)-F(u)}{\sqrt{2\left(E-V(u;a,c)\right)}}\,du. \tag{2.11}$$

Notice that these integrals can be regularized as above, and represent conserved quantities of the gKdV flow restricted to the manifold of periodic traveling wave solutions. In particular one can differentiate the above expressions with respect to the parameters (a, E, c). Notice that in the derivation of the gKdV [5], the solution u can represent either the horizontal velocity of a wave profile, or the density of the wave. Thus, the functional M can be interpreted as a "mass" since it is the integral of the density over space. Similarly, the functional P can be interpreted as a "momentum" since it is the integral of the density times velocity over space.

Throughout this paper, a large role will be played by the gradients of the above conserved quantities. However, by the Hamiltonian structure of (2.5), the corresponding derivatives of the period, mass, and momentum restricted to a periodic traveling wave $u(\cdot; a, E, c)$ with $(a, E, c) \in \Omega$ are related as they arise as the elements of the Hessian of a scalar function. In particular, if we define the classical action

$$K(a, E, c) = \oint u_x \, du = \int_0^T u_x^2 \, dx = 2 \int_{u_-}^{u_+} \sqrt{2(E - V(u; a, c))} \, du \tag{2.12}$$

(which is not itself conserved under the ODE flow induced by (2.5)) the derivative of this map as a function $K : \Omega \to \mathbb{R}$ is given explicitly by

$$D_{a,E,c}K(a, E, c) = \left(M(a, E, c), T(a, E, c), \frac{1}{2}P(a, E, c) \right),$$

where $D_{a,E,c} = \langle \frac{\partial}{\partial a}, \frac{\partial}{\partial E}, \frac{\partial}{\partial c} \rangle$. This is easily verified by differentiating (2.12) and recalling that $E - V(u_\pm; a, c) = 0$. Moreover, using the fact that T, M, P, and H are C^1 functions of parameters (a, E, c) on the domain Ω, we have the following relationship between the gradients of the conserved quantities of the gKdV flow restricted to the periodic traveling waves:

$$E \; D_{a,E,c}(T) + a \; D_{a,E,c}(M) + \frac{c}{2} \; D_{a,E,c}(P) + D_{a,E,c}(H) = 0. \tag{2.13}$$

Indeed, notice that by integrating (2.7) over the interval $[0, T(a, E, c)]$ yields

$$\frac{1}{2}K + \int_0^T F(u(x))dx = \frac{c}{2}P + aM + ET. \tag{2.14}$$

Moreover, from the definition of the Hamiltonian $H(a, E, c)$, it is clear that

$$\frac{1}{2}K - \int_0^T F(u(x))dx = H. \tag{2.15}$$

Thus, adding equations (2.14) and (2.15) and taking the partial derivatives with respect to a, E, and c yields

$$\begin{pmatrix} T_a & M_a & P_a & H_a \\ T_E & M_E & P_E & H_E \\ T_c & M_c & P_c & H_c \end{pmatrix} \begin{pmatrix} E \\ a \\ \frac{c}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

from which (2.13) follows. Thus, although the subsequent theory is developed most naturally in terms of the quantities T, M, and P, it is possible to restate our results in terms of M, P and H so long as $E \neq 0$: this is desirable since these have a natural interpretation as conserved quantities of the partial differential equation (2.1).

We now discuss the parametrization of the periodic solutions of (2.5) in more detail. A major technical necessity throughout this paper is that the constants of motion for the PDE flow defined by (2.1) provide (at least locally) a good parametrization for the periodic traveling wave solutions. In particular, we assume for a given $(a, E, c) \in \Omega$ that the conserved quantities (H, M, P) are good local coordinates for the periodic traveling waves near (a, E, c). More precisely, we assume the map $(a, E, c) \to (H(a, E, c), M(a, E, c), P(a, E, c))$ has a unique C^1 inverse in a neighborhood of (a, E, c). If we adopt the notation

$$\{f, g\}_{x,y} = \begin{vmatrix} f_x & g_x \\ f_y & g_y \end{vmatrix}$$

for 2×2 Jacobians, and $\{f, g, h\}_{x,y,z}$ for the analogous 3×3 Jacobian, it follows this is possible exactly when $\{H, M, P\}_{a,E,c}$ is non-zero, which is equivalent to $\{T, M, P\}_{a,E,c} \neq 0$ if $E \neq 0$ by (2.13). This non-degeneracy condition will be shown to hold for power-law nonlinearities $f(u) = u^{p+1}$ in a neighborhood of the solitary wave $(a, E, c) = (0, 0, c)$

if and only if $p \neq 4$: at $p = 4$, $\{T, M, P\}_{a,E,c}$ switches signs near the solitary wave. We will see that this sign change signals the switch from stability to instability in the solitary wave setting, and we will relate the vanishing of $\{T, M, P\}_{a,E,c}$ to the structure of the generalized periodic null-space of the linearized operator.

2.3 Linearization and Floquet Theory

We now begin our study of spectral stability of the periodic waves $u(x) = u(x; a, E, c)$ under small perturbation. To this end, suppose $(a, E, c) \in \Omega$ and consider a small perturbation of the periodic wave $u(x; a, E, c)$ of the form

$$\psi(x, t; a, E, c) = u(x; a, E, c) + \varepsilon v(x, t) + \mathcal{O}(\varepsilon^2),$$

where $0 < |\varepsilon| \ll 1$ is a small parameter. Substituting this into (2.1) and collecting the $\mathcal{O}(\varepsilon)$ terms yields the linearized equation $\partial_x \mathcal{L}[u]v = -v_t$, where $\mathcal{L}[u] := -\partial_x^2 - f'(u) + c$ is a linear differential operator with periodic coefficients. As the linearized equation is autonomous in time, we seek separated solutions of the form $v(x, t) = e^{-\mu t} v(x)$, which yields the spectral problem

$$\partial_x \mathcal{L}[u]v = \mu v. \tag{2.16}$$

Note that we consider the linearized operator $\partial_x \mathcal{L}[u]$ as a closed linear operator acting on a Banach space X with domain $D(\partial_x \mathcal{L}[u])$. In literature, several choices for X have been studied, each of which corresponding to different classes of admissible perturbations v. In our case, we consider $X = L^2(\mathbb{R}; \mathbb{R})$ and $D(\partial_x \mathcal{L}[u]) = H^3(\mathbb{R})$, corresponding to spatially localized perturbations. As mentioned in the introduction the resulting spectrum is purely continuous, which is a serious impediment to implementing many of the stability techniques familiar from solitary wave theory. Indeed, if we consider the linearized operator $\partial_x \mathcal{L}[u]$ as acting on $L^2_{per}([0, T]; \mathbb{R})$, then the resolvent is a compact operator and hence the spectrum consists of isolated eigenvalues of finite-multiplicity. However, by considering $L^2(\mathbb{R}; \mathbb{R})$, the spectrum is purely continuous and consists of no isolated eigenvalues of finite multiplicity. This fact is usually avoided all together by implementing a Floquet-Bloch decomposition of the linearized operator: we will outline this approach in section 2.6. However, the main approach taken in this work is to utilize the integrable structure of the traveling wave ODE (2.5) in order to explicitly construct the tangent space to the manifold $\mathcal{M}$ of periodic traveling wave solutions of (2.1), i.e.

$$\mathcal{M} = \{g(\,\cdot\,; a, E, c) : g \text{ solves (2.5) and } (a, E, c) \in \Omega\},$$

at $u(\,\cdot\,; a, E, c)$, and to utilize perturbation theory to determine the structure of the spectrum in a sufficiently small neighborhood of the origin.

In order to understand the structure of the spectrum of $\partial_x \mathcal{L}[u]$, we the following definitions familiar from Floquet theory.

Definition 4. *The monodromy matrix* $\mathbf{M}(\mu)$ *associated to the spectral problem* (2.16) *is defined to be the period map*

$$\mathbf{M}(\mu) = \Phi(T, \mu),$$

where $\Phi(x, \mu)$ *satisfies the initial value problem*

$$\Phi_x = \mathbf{H}(x; \mu)\Phi, \quad \Phi(0, \mu) = \mathbf{I} \tag{2.17}$$

with $\mathbf{I}$ *the* 3×3 *identity matrix and*

$$\mathbf{H}(x; \mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu - u_x f''(u) & -f'(u) + c & 0 \end{pmatrix}.$$

As mentioned in section 1.2, it follows that the spectrum of the linearized operator $\partial_x \mathcal{L}[u]$ acting on $L^2(\mathbb{R})$ is entirely essential and coincides with the continuous spectrum. In particular, it follows by a variant of Weyl's theorem that $\mu \in \text{spec}(\partial_x \mathcal{L}[u])$ if and only if there exists a non-trivial uniformly bounded solution of the equation $\partial_x \mathcal{L}[u]v = \mu v$.

This leads us to the following definition.

Definition 5. *We say* $\mu \in \mathrm{spec}(\partial_x \mathcal{L}[u])$ *if there exists a non-trivial bounded function* ψ *such that* $\partial_x \mathcal{L}[u]\psi = \mu\psi$ *or, equivalently if there exists a* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$ *and*

$$\det[\,\mathbf{M}(\mu) - \lambda \mathbf{I}] = 0.$$

Moreover, we say the periodic solution $u(x; a, E, c)$ *is spectrally stable if* $\mathrm{spec}(\partial_x \mathcal{L}[u])$ *does not intersect the open right half plane.*

Notice that due to the Hamiltonian nature of the problem, $\mathrm{spec}(\partial_x \mathcal{L}[u])$ is symmetric with respect to reflections across the real and imaginary axes. Thus, spectral stability occurs if and only if $\mathrm{spec}(\partial_x \mathcal{L}[u]) \subset \mathbb{R}i$. Following Gardner [29, 29], we define the periodic Evans function for this problem to be

$$D(\mu, \lambda) = \det\left(\,\mathbf{M}(\mu) - \lambda \mathbf{I}\right). \tag{2.18}$$

It follows that $\mu \in \mathrm{spec}(\partial_x \mathcal{L}[u])$ if and only if there exists a $\lambda \in S^1$ such that $D(\mu, \lambda) = 0$. A brief outline of the basic properties of this function was given in section 1.2 of the introduction. In particular, recall that $D(\mu, \lambda)$ is analytic in the complex variables μ and λ, and that for a fixed $\lambda_0 \in S^1$ the multiplicity of a root of the equation $D(\mu, \lambda_0) = 0$ is exactly the algebraic λ_0-multiplicity of the λ_0-eigenvalue μ. Throughout this thesis, we will be primarily concerned with the periodic eigenvalues of $\partial_x \mathcal{L}[u]$, which corresponds to the solutions of the equation $D(\mu, 1) = 0$. In this case, the multiplicity of a root of $D(\mu, 1) = 0$ is precisely the dimension of the generalized periodic null-space of the linearized operator $\partial_x \mathcal{L}[u]$.

Since we are primarily concerned with roots of $D(\mu, \lambda)$ with λ on the unit circle we will frequently work with the function $D(\mu, e^{i\kappa})$, which is actually the function considered by Gardner. It follows that the set $\mathrm{spec}(\partial_x \mathcal{L}[u])$ consists of precisely the L^∞ eigenvalues of the linearized operator $\partial_x \mathcal{L}[u]$. Moreover, if we define a projection operator $\pi_1 : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ by $\pi_1(z_1, z_2) = z_1$ for $(z_1, z_2) \in \mathbb{C} \times \mathbb{R}$, then the projection of the zero set of $D(\mu, e^{i\kappa})$ in $\mathbb{C} \times \mathbb{R}$ via π_1 is precisely the set $\mathrm{spec}(\partial_x \mathcal{L}[u])$.

As we will see below, it follows from the integrable structure of (2.16) that the function $D(\mu, 1)$ has a zero of multiplicity (generically) three at $\mu = 0$. As $\lambda = e^{i\kappa}$ varies on S^1 there will be in general three branches $\mu_j(\kappa)$ of roots of $D(\mu_j(\kappa), e^{i\kappa})$ for κ small which bifurcate from the origin. Assuming these branches are analytic[2] in κ, it follows that a necessary condition for spectral stability is thus

$$\frac{\partial}{\partial\kappa}\mu_j(\kappa)\Big|_{\kappa=0} \in \mathbb{R}i. \tag{2.19}$$

This observation leads us to the use of perturbation methods in the study of the spectrum of $\partial_x\mathcal{L}[u]$ near the origin, i.e. modulational instability analysis of the underlying periodic traveling wave. As we will see, the first order terms of a Taylor series expansion of the three branches $\mu_j(\kappa)$ can be encoded as roots of a homogeneous cubic polynomial, and hence spectral stability is determined by the sign of the associated discriminant. Moreover, it follows by the Hamiltonian structure of (2.16) that in fact $\text{spec}(\partial_x\mathcal{L}[u]) \subset \mathbb{R}i$ if (2.19) holds and the branches are distinct.

The main technical result we need to ensure the analyticity of the spectrum bifurcating from the $\mu = 0$ state is that the Hamiltonian structure of the linearized operator implies the operators $\mathbf{M}(\mu)$ and $\mathbf{M}(-\mu)^{-1}$ are related in a very specific way. This is the content of the following lemma.

Lemma 1. *The matrices* $\mathbf{M}(\mu)$ *and* $\mathbf{M}(-\mu)^{-1}$ *are similar.*

Proof. The idea of this proof is clear: the spectral problem $\partial_x\mathcal{L}[u]v = \mu v$ is invariant under the transformation $(x, \mu) \to (-x, -\mu)$: this is a reflection of the Hamiltonian structure of the linearized operator $\partial_x\mathcal{L}[u]$. To prove this from a rigorous framework, notice from the form of the operator $\mathcal{L}[u]$, we know that if $\mu \in \text{spec}(\partial_x\mathcal{L}[u])$ with $\partial_x\mathcal{L}[u]g(x) = \mu g(x)$, it follows that $\partial_x\mathcal{L}[u]g(-x) = -\mu g(-x)$. Thus, using the notation of (2.17), it follows that $\Phi(x, -\mu) \sim \Phi(-x, \mu)$. The lemma follows by evaluating at $x = T$. ∎

[2] In general, for each j, the theory of branching solutions of non-linear equations guarantees the existence of a natural number m_j such that $\mu_j(\cdot)$ is an analytic function of κ^{1/m_j}. As we will see in our case, the Hamiltonian nature of the linearized operator $\partial_x\mathcal{L}[u]$ assures that $m_j = 1$, and hence the roots are in fact analytic functions of the Floquet parameter.

While it is possible to prove analyticity of the spectrum near $\mu = 0$ with out the use of Lemma 1 by using variation of parameters, we find that the Lemma 1 is useful in understanding the mechanism behind this highly non-generic bifurcation.

The goal of this chapter is to determine various asymptotic expansions of the periodic Evans function. In the next section, we will study the asymptotics of (2.18) as $\mu \to \infty$. This will provide information about the global structure of the spectrum of the linearized operator $\partial_x \mathcal{L}[u]$, as well as providing us with a finite wavelength instability index which counts modulo 2 the number of intersections of the spectrum with the positive real axis. We then study the asymptotics of (2.18) in the limit $(\mu, \kappa) \to (0, 0)$, which yields a "modulational stability index" in terms of the derivatives of the monodromy operator at the origin.

2.4 Global Structure of $\text{spec}(\partial_x \mathcal{L}[u])$ and $\text{spec}\,(\mathcal{L}[u])$

In this section, we review some basic global features of the spectrum of the linear operators $\mathcal{L}[u]$ and $\partial_x \mathcal{L}[u]$ which are useful in a local analysis near $\mu = 0$, as well as the global analysis. We also state some important properties of the Evans function $D(\mu, \lambda)$ which are vital to the foregoing analysis.

We begin with analyzing the spectrum of the operator $\mathcal{L}[u]$. As this is a self adjoint Hill-type operator with periodic coefficients, it follows the spectrum on $L^2(\mathbb{R})$ is real and purely continuous. Moreover, we have the following theorem concerning the global structure of the set $\text{spec}(\mathcal{L}[u])$.

Theorem 5 (Oscillation Theorem)**.** *To every differential equation of the form*

$$-\partial_x^2 v + Q(x)v = \mu v, \tag{2.20}$$

where $Q(x)$ is a T-periodic function of x, there belong two monotonically increasing infinite sequences of real numbers $\{\lambda_j\}_{j=0}^{\infty}$ and $\{\lambda_j'\}_{j=1}^{\infty}$ such that (2.20) *has a solution of period T if and only if $\mu = \lambda_j$ and an anti-periodic solution of period $2T$ (i.e. a solution ϕ such that $\phi(x+T) = -\phi(x)$) if and only if $\mu = \lambda_j'$. These sequences satisfy* *the inequalities*

$$\lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \dots$$

as well as the relations $\lim_{j\to\infty} \lambda_j^{-1} = \lim_{j\to\infty} (\lambda_j)'^{-1} = 0$. *Moreover, if* $\mathbf{m}(\mu)$ *is the corresponding monodromy operator, define the function* $a(\mu) = \operatorname{tr}(\mathbf{m}(\mu))$. *Then the function* $a(\mu)$ *satisfies the following properties:*

1. $\{\mu \in \mathbb{R} : a(\mu) = 2\} = \{\lambda_j\}_{j=1}^{\infty}$ *and* $\{\mu \in \mathbb{R} : a(\mu) = -2\} = \{\lambda_j\}_{j=1}^{\infty}$
2. $a(\mu) > 2$ *for all* $\mu < \lambda_0$.
3. *One has* $a'(\mu) < 0$ *on* $(\lambda_j, \lambda_{j+1}')$ *for* $j \geq 0$ *and* $a'(0) > 0$ *on* $(\lambda_j', \lambda_{j-1})$ *for* $j \geq 1$.

In particular, the solutions of (2.20) *are stable if* $a(\mu) \in (-2, 2)$ *and are stable at* $\mu = \lambda_j$ *if and only if* $a'(\lambda_j) = 0$, *in which case one has* $a''(0) < 0$. *Similarly, the solutions* (2.20) *are stable at* $\mu = \lambda_j'$ *if and only if* $a'(\lambda_j') = 0$, *in which case one has* $a''(0) > 0$.

The proof of this result is well known and can be found in many texts on the subject: see for example [49]. As a result, we have the following non-degeneracy condition which states conditions on the nonlinearity for the map $E \to T(a_0, E, c_0)$ to be invertible for a fixed (a_0, c_0) such that $(a_0, E, c_0) \in \Omega$.

Lemma 2. *Let* $(a_0, E_0, c_0) \in \Omega$ *and* $u = u(\,\cdot\,; a_0, E_0, c_0)$ *denote the corresponding periodic solution of* (2.5) *with wave speed* c_0 *and period* $T = T(a_0, E_0, c_0)$. *If the non-linearity* f *in* (2.1) *is such that* $f'(u)$ *is co-periodic with* u, *then* $T_E > 0$ *at* (a_0, E_0, c_0).

The proof of this lemma will be postponed until the next chapter where it is more natural: it requires a more detailed analysis of the periodic spectrum of the linear operator $\mathcal{L}[u]$ (see Lemma 10). In particular, this lemma guarantees that $T_E > 0$ for all $(a, E, c) \in \Omega$ if $f(u) = u^{p+1}$ for some $p \geq 1$. In Lemma 10 of Chapter 3, we will see this implies that $\mu = 0$ is a simple T-periodic eigenvalue of $\mathcal{L}[u]$, and that this operator has exactly one negative eigenvalue. By Theorem 3.1 of [57], it follows that for such nonlinearities the number of unstable T-periodic eigenvalues of $\partial_x \mathcal{L}[u]$ with positive real part can be at most one. Since the spectrum of the linearized operator $\partial_x \mathcal{L}[u]$ is symmetric about the real and imaginary axis, it follows that all unstable eigenvalues of the operator $\partial_x \mathcal{L}[u]$ acting on the space $L^2_{\text{per}}([0,T])$ are real. This important observation will play a large role in understanding the orientation index derived in section 2.5, and will also play a large role in understanding the orbital stability of such solutions to T-periodic perturbations conducted in Chapter 3.

Next, we prove some basic properties of the spectrum of the linearized operator $\partial_x \mathcal{L}[u]$ considered on $L^2(\mathbb{R})$.

Proposition 1. *The spectrum* $\text{spec}\,(\partial_x \mathcal{L}[u])$ *has the following properties:*

- *There are no isolated points of the spectrum. In particular, the spectrum consists of piecewise smooth arcs.*

- *The entire imaginary axis is contained in the spectrum, i.e.* $\mathbb{R}i \subset \text{spec}(\partial_x \mathcal{L}[u])$. *Further for* $|\mu|$ *sufficiently large along the imaginary axis the multiplicity is one, i.e. the multiplicity of the root* $D(\cdot, \lambda(\cdot)) = 0$ *is one.*

- $\mathbb{R} \cap \text{spec}(\partial_x \mathcal{L}[u])$ *consists of a finite number of points. In particular there are no bands on the real axis.*

Proof. The first claim, that the spectrum is never discrete, was given in the introduction. Notice that it is clear from the implicit function theorem that μ is a smooth function of λ as long as $\frac{\partial D}{\partial \mu} = \text{tr}(\text{cof}^t(\mathbf{M}(\mu) - \lambda \mathbf{I})\,\mathbf{M}'_\mu) \neq 0$, where cof represents the standard cofactor matrix. Moreover, since $D(\mu, \lambda_0)$ is analytic in μ for a fixed λ_0, it follows that, in general, the spectrum will consist of piecewise smooth curves. We will show directly later that the spectrum is in fact an analytic function of λ on S^1, at least in a neighborhood of the origin.

The second claim is an easy symmetry calculation. First, notice from the definition of $\mathbf{H}(x,\mu)$ in (2.17) that $\text{tr}(\mathbf{H}(x,\mu)) = 0$ for all $x \in \mathbb{R}$, $\mu \in \mathbb{C}$. Thus, Abel's formula implies that

$$\det(\Phi(x,\mu)) = \exp\left(\int_0^x \text{tr}(\mathbf{H}(x,\mu))dx\right)\det(\Phi(0,\mu))$$

and hence $\det(\mathbf{M}(\mu)) = 1$ for all $\mu \in \mathbb{C}$. Thus, it is possible to find functions $a, d : \mathbb{C} \to \mathbb{C}$ such that

$$D(\mu, \lambda) = -\lambda^3 + a(\mu)\lambda^2 + d(\mu)\lambda + 1$$

for all $\mu, \lambda \in \mathbb{C}$. In particular, notice that $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$. By Lemma 1 it follows that

$$\begin{aligned}
\det[\mathbf{M}(-\mu) - \lambda\mathbf{I}] &= \det[\mathbf{M}^{-1}(\mu) - \lambda\mathbf{I}] \\
&= -\lambda^3 \det[\mathbf{M}^{-1}(\mu)] \det[\mathbf{M}(\mu) - \lambda^{-1}] \\
&= -\lambda^3 \left(-\lambda^{-3} + a(\mu)\lambda^{-2} + d(\mu)\lambda^{-1} + 1\right) \\
&= -\lambda^3 - d(\mu)\lambda^2 - a(\mu)\lambda + 1
\end{aligned}$$

and hence we have the identity $d(\mu) = -a(-\mu)$ for all $\mu \in \mathbb{C}$.

Now, since the entries of the matrix $\mathbf{H}(x, \mu)$ are real for $\mu \in \mathbb{R}$, it follows that $\mathbf{M}(\mu)$ is a real matrix on the real axis. Thus, its eigenvalues are either real or occur in complex conjugate pairs, and hence $a(\mu)$ is real on the real axis. It follows from Schwarz reflection that for $\mu \in \mathbb{R}i$, we have $a(-\mu) = a(\overline{\mu}) = \overline{a(\mu)}$ and the Evans function takes the form

$$D(\mu, \lambda) = -\lambda^3 + a(\mu)\lambda^2 - \overline{a(\mu)}\lambda + 1,$$

from which it follows that

$$D(\mu, \lambda) = -\lambda^3 \overline{D\left(\mu; \left(\overline{\lambda}\right)^{-1}\right)}$$

for all $\mu, \lambda \in \mathbb{C}$. Hence for imaginary μ the eigenvalues of the monodromy are symmetric with respect to the unit circle with the same multiplicities. Since the monodromy has three eigenvalues, it follows that at least one must lie on the unit circle and hence $\mathbb{R}i \subset \operatorname{spec}(\partial_x \mathcal{L}[u])$ as claimed.

To see that the multiplicity of the spectrum on the imaginary axis is eventually one, we note that by standard asymptotics the monodromy $\mathbf{M}(\mu)$ satisfies

$$\mathbf{M}(\mu) \approx e^{\mathbf{A}(\mu)T} \text{ for } |\mu| \gg 1,$$

where $\mathbf{A}(\mu)$ is defined by

$$\mathbf{A}(\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & 0 & 0 \end{pmatrix}.$$

The three eigenvalues of $e^{\mathbf{A}(\mu)T}$ are given by

$$\lambda_1 = e^{-\mu^{1/3}T}, \quad \lambda_2 = e^{-\mu^{1/3}\omega T}, \quad \text{and} \quad \lambda_3 = \mathrm{e}^{-\mu^{1/3}\overline{\omega}\mathrm{T}} \tag{2.21}$$

where $\omega = e^{2\pi i/3}$ is the principle third root of unity. If $\mu \in \mathbb{R}^+ i$ it follows that $\lambda_1 = \exp\left(-|\mu|^{1/3}e^{i\pi/6}T\right)$ and since $\cos(\pi/6) > 0$ we have $|\lambda_1| \to 0$ as $\mathbb{R}^+ i \ni \mu \to \infty$. Similarly, $\lambda_2 = \exp\left(-|\mu|^{1/3}e^{5\pi/6}T\right)$ and $\lambda_3 = \exp\left(|\mu|^{1/3}i\right)$ so that $|\lambda_2| \to \infty$ as $\mathbb{R}^+ i \ni \mu \to \infty$ and $|\lambda_3| = 1$. Thus, for $\mu \in \mathbb{R}^+ i$ large, we have that μ is an eigenvalue of multiplicity one. Similarly, we can show $|\lambda_1| \to \infty$, $|\lambda_3| \to 0$ as $\mathbb{R}^+ i \ni \mu \to -\infty$ and $|\lambda_2| = 1$ for $\mu \in \mathbb{R}^- i$, $|\mu| \gg 1$. Therefore, it follows that $\mu \in \operatorname{spec}(\partial_x \mathcal{L}[u])$ with multiplicity one for $\mu \in \mathbb{R}i$, $|\mu| \gg 1$.

The final claim follows from a similar asymptotic calculation together with an analyticity argument. As noted above, for μ real the eigenvalues of the monodromy are either all real or one real and one complex conjugate pair. It follows that if $\mu \in \operatorname{spec}(\partial_x \mathcal{L}[u]) \cap \mathbb{R}$, then 1 or -1 must be an eigenvalue of the monodromy. Indeed, it is clear that if $\mu \in \operatorname{spec}(\partial_x \mathcal{L}[u])$ is real and $\mathbf{M}(\mu)$ has all real eigenvalues, then either 1 or -1 must be an eigenvalue of $\mathbf{M}(\mu)$. If, for such μ, $\mathbf{M}(\mu)$ has a complex conjugate pair of eigenvalues, it must be that 1 is an eigenvalue of $\mathbf{M}(\mu)$ since $\det(\mathbf{M}(\mu)) = 1$. Thus if a point on the real axis is in the spectrum then either $D(\mu, 1)$ or $D(\mu, -1)$ must vanish. Since the functions $D(\mu, \pm 1)$ are entire functions, it follows that are either identically zero or the zero set has no finite accumulation points. However, the large μ asymptotics for $\mu \in \mathbb{R}$ implies that neither of the functions $D(\mu, \pm 1)$ can identically vanish, and hence their zero sets must be discrete. Furthermore, the large μ asymptotics implies that for sufficiently large μ along the real axis $\mu \notin \mathrm{spec}\,(\partial_x \mathcal{L}[u])$, so the spectrum is confined to a compact subset of the real line. Since we have already argued the sets $\{\mu \in \mathbb{C} : D(\mu, \pm 1) = 0\}$ have no finite accumulation points, it follows that there are only a finite number of real eigenvalues. □

Remark 2. *Note that, in the calculation of Hărăguş and Kapitula [35] the real eigenvalues play a slightly different role than other eigenvalues off of the imaginary axis. The fact that there are only a finite number of these indicates that there are only a finite number of values of the Floquet parameter for which there are real eigenvalues:* $\kappa_r(\gamma) = 0$ *for all but a finite number of values of the Floquet parameter* γ *in their notation.*

2.5 The Orientation Index: Finite-Wavelength Instabilities

We now move on to study the structure of $\mathrm{spec}(\partial_x \mathcal{L}[u]) \cap \mathbb{R}$ more carefully. We will suppose throughout this section that $\mu \in \mathbb{R}$. Clearly, the condition $D(\mu, 1) = 0$ is sufficient to guarantee $\mu \in \mathrm{spec}(\partial_x \mathcal{L}[u])$. In particular, such roots correspond to the periodic eigenvalues of the operator $\partial_x \mathcal{L}[u]$. Since $D(\mu, 1) = a(\mu) - a(-\mu)$ by Proposition 1, it is clear that $D(0, 1) = 0$ and hence $\mu = 0$ is a T-periodic eigenvalue of $\partial_x \mathcal{L}[u]$. The question is whether $D(\mu, 1)$ has any other real roots. If it does, then the underlying periodic solution $u(x; a, E, c)$ is spectrally unstable due to the presence of a real nonzero element of $\mathrm{spec}(\partial_x \mathcal{L}[u])$. In order to detect this periodic instability, we calculate the orientation index

$$\mathrm{sign}\,(D(\infty, 1))\,\frac{\partial^m}{\partial \mu^m} D(\mu, 1)\big|_{\mu=0}$$

in terms of the function $a(\mu) = \mathrm{tr}(\,\mathbf{M}(\mu))$, where m is the first positive integer such that the m^{th} derivative of $D(\cdot, 1)$ at $\mu = 0$ is non-zero. Note that it is clear that the negativity of this index implies the set $\{\mu \in \mathbb{R}^* : D(\mu, 1) = 0\}$ is non-empty. Indeed, negativity would imply the sign of $D(\mu, 1)$ for small positive μ is opposite that of the sign of $D(\mu, 1)$ for large positive μ. Since $D(\mu, 1)$ is continuous on $\mathbb{R}$, our claim follows.

As we will see in the next section, Lemma 1 implies that $D_\mu(0,1) = 0$. Since $D_{\mu\mu}(0,1)$ clearly vanishes, we see that $m \geq 3$. Moreover, in the next section we will use the integrable structure of the traveling wave ODE (2.5) in order to calculate this index in terms of the conserved quantities of the traveling wave. However, we merely mention these results now as to not distract from the modulational instability analysis of the next section. In order to calculate the above orientation index, we start by determining the asymptotic behavior of $D(\mu,1)$ as $\mu \to \infty$.

Lemma 3. *The function $D(\,\cdot\,,1) : \mathbb{R} \to \mathbb{R}$ is an odd function which satisfies the asymptotic relation*

$$\lim_{\mathbb{R}\ni\mu\to\pm\infty} D(\mu,1) = \mp\infty.$$

Proof. Clearly, $D(\,\cdot\,,1)$ is an odd function of its argument, and hence it is sufficient to consider the limit as $\mu \to \infty$. To begin, define a new variable $\rho = \mu^{1/3}T$. Then from the asymptotic relations (2.21) we have

$$\begin{aligned} a(\rho) &= e^{-\rho} + e^{-(-1+\sqrt{3}i)\rho/2} + e^{-(-1-\sqrt{3}i)\rho/2} \\ \widetilde{a}(\rho) &= e^{\rho} + e^{-(1+\sqrt{3}i)\rho/2} + e^{-(1-\sqrt{3}i)\rho/2} \end{aligned}$$

where $\widetilde{a}(\rho)$ is the trace when you take $\mu \to -\mu$. It follows that $D(\mu,1) = a(\rho) - \widetilde{a}(\rho)$ behaves like $-e^{\rho}$ for large positive ρ, i.e. $\mu \gg 0$. This completes the proof. □

From these results, we have the following theorem relating the sign of $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ to the stability of the underlying periodic wave.

Theorem 6. *If $a'''(0) = \operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0)) > 0$, then the number of roots of $D(\mu,1)$ (i.e. the number of periodic eigenvalues) on the positive real axis is odd. In particular* $\operatorname{spec}(\partial_x\mathcal{L}[u]) \cap \mathbb{R}^* \neq \emptyset$ *and the eigenvalue problem* (2.16) *is spectrally unstable.*

Proof. We show in lemma 2 that $D(0,1) = D_\mu(0,1) = D_{\mu\mu}(0,1) = 0$ and $D_{\mu\mu\mu}(0,1) = 2a'''(0)$. Thus, if $a'''(0) > 0$, then $D(\mu;1)$ is positive for small positive values of μ. Since $D(\mu,1)$ is negative for sufficiently large μ we know that $D(\pm\mu^*,0) = 0$ for some $\mu^* \in \mathbb{R}\setminus\{0\}$, which completes the proof. In the next section we establish the following formula for $D_{\mu\mu\mu}(0,1)$, the first non-vanishing derivative:

$$D_{\mu\mu\mu}(0,1) = -3 \begin{vmatrix} T_a & M_a & P_a \\ T_E & M_E & P_E \\ T_c & M_c & P_c \end{vmatrix} = 6 \begin{vmatrix} K_{aa} & K_{aE} & K_{ac} \\ K_{aE} & K_{EE} & K_{cE} \\ K_{ac} & K_{cE} & K_{cc} \end{vmatrix} = \frac{3}{E} \begin{vmatrix} M_a & P_a & H_a \\ M_E & P_E & H_E \\ M_c & P_c & H_c \end{vmatrix},$$

where again K is the classical action of the traveling wave ODE. Hence this "orientation index" can be expressed in terms of the Jacobian of the map between the constants of integration of the traveling wave ordinary differential equation $(a, E; c)$ and the conserved quantities M, P, and H of the gKdV, assuming $E \neq 0$. This orientation index is analogous to the quantity which is calculated in the stability theory of the solitary waves, and we will see this connection explicitly in later sections. □

Recall from our remarks in the previous section that the number of negative eigenvalues of $\mathcal{L}[u]$ on $L^2_{\text{per}}(\mathbb{R})$ bounds above the number of periodic eigenvalues of $\partial_x \mathcal{L}[u]$ with positive real part. Since $\mathcal{L}[u]$ has precisely one negative eigenvalue if $T_E \geq 0$ (see Lemma 10), and since the set $\text{spec}(\partial_x \mathcal{L}[u])$ is symmetric about the real and imaginary axis, it follows that all unstable periodic eigenvalues of the linearized operator must be real. Thus, the set $\{\mu \in \mathbb{R}^* : D(\mu, 1) = 0\}$ is empty if $a'''(0) < 0$, and contains precisely one element if $a'''(0) > 0$. Indeed, since $D(\mu, 1)$ is continuous on $\mathbb{R}$, the condition $a'''(0) < 0$ implies the cardinality of non-zero real roots is even, and hence must in fact be zero. This immediately yields the following corollary to Theorem 6.

Corollary 2. *If $(a_0, E_0, c_0) \in \Omega$, then the corresponding periodic traveling wave solution of* (2.1) *is exponentially unstable to periodic perturbations if $a'''(0) > 0$, and is spectrally stable to such perturbations if $a'''(0) < 0$.*

It is important to notice the instability detected by Theorem 6 is an instability with respect to *finite (bounded) wavelength* perturbations. In the next section we will derive a *modulational* stability index which detects instability with respect to arbitrarily long wavelength perturbations. See the comments at the end of the section 2.6.2. Notice the solitary wave solutions go unstable in the manner detected by Theorem 6, through the creation of an eigenvalue on the real axis. In general the periodic waves seem to first go unstable through the creation of a band of spectrum which does not intersect the real axis, and later there is a secondary bifurcation resulting in a real eigenvalue. This phenomenon appears to have first been observed by Kapitula and Hărăguş. While we don't have a general proof of this we do show that, in the case of power law nonlinearity, there is a real periodic eigenvalue as well as a band of unstable spectrum connected to the origin. It is also worth noting that the analogous calculation for $D(\mu,-1)$ shows that the number of anti-periodic eigenvalues is always even. While this is not particularly useful in our analysis, it does eliminate some possible modes of instability.

2.6 Local Analysis of the Periodic Evans Function

In this section, we turn our attention to studying the asymptotic behavior of $D(\mu,e^{i\kappa})$ as $\mu\to 0$. We begin by proving that $D(0,e^{i\kappa})$ has a zero of multiplicity three at $\kappa=0$ by directly computing the Jordan normal form of the matrix $\mathbf{M}(0)$. In particular, we will see that $\lambda=1$ is an eigenvalue of $\mathbf{M}(0)$ of algebraic multiplicity three and geometric multiplicity (generically) two. This calculation gives us a starting point for our analysis of the spectrum of $\partial_x\mathcal{L}[u]$ in a neighborhood of the origin.

Using perturbation theory appropriate to a Jordan block, as well as the Hamiltonian symmetry inherent in (2.16), we prove the three roots of $D(\mu,e^{i\kappa})$ bifurcate from $\mu=0$ analytically in κ in a neighborhood of $\kappa=0$, and derive a necessary and sufficient condition for modulational instability of the underlying periodic wave $u(x;a,E,c)$, $(a,E,c)\in\Omega$, in terms of the constants of motion of (2.1) restricted to the manifold of periodic traveling wave solutions with $(a,E,c)\in\Omega$. Note that this conclusion is somewhat unexpected: normally the eigenvalues of a non-trivial Jordan block do not bifurcate analytically but instead admit a Puiseux series in fractional powers. However because of the symmetries of the problem the admissible perturbations are severely restricted, resulting in a non-generic bifurcation.

2.6.1 Calculation of the Period Map at the Origin

The first major calculation we present is an explicit calculation of the monodromy matrix at the origin in terms of the derivatives of the underlying periodic solution u with respect to the parameters (a, E, c). We do this by first computing a matrix valued solution to the ordinary differential equation satisfying the wrong initial condition: $\mathbf{U}(0,0)$ is non-singular but not the identity. One can then multiply on the right by $\mathbf{U}^{-1}(0,0)$ to find the monodromy matrix. We find that (as expected) the monodromy operator $\mathbf{M}(\mu)$ has a non-trivial Jordan form when $\mu = 0$. Our goal is then to utilize perturbation theory of Jordan blocks to calculate the normal form of the characteristic polynomial in a neighborhood of $\mu = 0,\ \lambda = 1$, where λ is the eigenvalue parameter of the monodromy operator. We find that, due to the symmetry inherent in Lemma 1, the Jordan blocks bifurcate in a very non-generic way.

To begin we write the above third order eigenvalue problem as a first order system as in (2.17). Recall that $\det(\mathbf{M}(\mu)) = 1$ for all $\mu \in \mathbb{C}$ since $\mathrm{tr}(\mathbf{H}(x;\mu)) = 0$ for all $(x,\mu) \in \mathbb{R} \times \mathbb{C}$. In order to calculate a matrix solution $\Phi(x;\mu)$, we must first find three linearly independent solutions of the corresponding first order system. In general, this is a daunting task, but since the above system with $\mu = 0$ arises as the Frechet derivative (linearization) of an integrable ordinary differential equation this can be done by considering infinitesimal variations of the four-parameters defining the family of periodic traveling wave solutions of (2.1), and thus generating the tangent space. As noted earlier the solutions $u(x-x_0; a, E, c)$ constitute a 4-dimensional solution manifold of (2.1) parameterized by x_0, a, E, c. Heuristically then, the tangent space is spanned by the generators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial a}$, and $\frac{\partial}{\partial E}$. The action of the generator $\frac{\partial}{\partial c}$ is somewhat different and is connected with the generalized null-space: this is because of its dynamical nature within the family of periodic traveling wave solutions. Although this is trivially verified, as in the proof of the following proposition, we outline here a proof based more on the dynamics of the linearized equation.

Recall that $u = u(\,\cdot\,; a_0, E_0, c_0)$ is a member of a four parameter family of periodic traveling waves. Differentiating the representation of this family with respect to the wave speed and evaluating at the identity (c=0) yields

$$\frac{\partial}{\partial c}u(x+ct+\xi;a_0,E_0,c_0+c)|_{c=0} = tu_x(x+\xi;a_0,E_0,c_0)+u_c(x+\xi;a_0,E_0,c_0). \quad (2.22)$$

It now follows from the dynamical version of the spectral problem $\partial_x\mathcal{L}[u]v = \mu v$, i.e.

$$\partial_x\mathcal{L}[u]v = -v_t,$$

that $\partial_x\mathcal{L}[u]u_c = -u_x$. Indeed, notice that if $\partial_x\mathcal{L}[u]\psi_1 = u_x$, then if we let $v(x,t) = a_0(t)u_x + a_1(t)\psi_1(t)$, for some coefficient functions a_0 and a_1 and require that v solves $\partial_x\mathcal{L}[u]v = -v_t$, we see that the coefficient functions satisfy the first order system

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

It follows that a_1 is a constant and $v = a_1\left(u_x - t\psi_1\right)$. Comparing this with (2.22) it follows that $\partial_x\mathcal{L}[u]u_c = -u_x$ as claimed. This action will become important in the next section as it will allow us to explicitly compute the $\mathcal{O}(|\mu|)$ variation of the eigenfunction bifurcating from the u_x state in terms of u_c, and hence the calculation of the $\mathcal{O}(|\mu|^2)$ variation reduces to a first order calculation. We summarize these observations within the following proposition. Again, the above claim can be proved trivially from the structure of the traveling wave ODE, as in the following Proposition, however we find the above proof to be useful in understanding why the generator $\frac{\partial}{\partial c}$ acts differently from the others: any time an equation has a symmetry which is dynamical in nature (like scaling in the wave-speed or Galilean invariance, for example) the resulting action yields an element of the *generalized* null space of the linearized operator at the origin.

Proposition 2. *A basis of solutions to the third order system*

$$Y_x = \mathbf{H}(x;0)Y$$

is given by

$$Y_1^t = (u_x, u_{xx}, u_{xxx})$$

$$Y_2^t = (u_a, u_{ax}, u_{axx})$$

$$Y_3^t = (u_E, u_{Ex}, u_{Exx}).$$

A particular solution to the inhomogeneous problem

$$Y_x = \mathbf{H}(x;0)Y + W$$

where $W^t = (0, 0, u_x)$ *is given by*

$$Y_3^t = (u_c, u_{cx}, u_{cxx}).$$

Proof. A straightforward calculation: simply differentiate (2.6) to see that $\mathcal{L}[u]u_x = 0$, $\mathcal{L}[u]u_a = -1$, $\mathcal{L}[u]u_E = 0$, and $\mathcal{L}[u]u_c = -u$. Notice that it follows that $\partial_x\mathcal{L}[u]\,(-u_c) = u_x$. □

The fact that u_a, u_E are not periodic - they exhibit secular growth due to the variation of the period with respect to the parameters - gives an indication that the eigenspaces of the monodromy at $\mu = 0$ are not semi-simple, and hence we expect the existence of a non-trivial Jordan block of the monodromy map $\mathbf{M}(0)$. This is the main result of this section whose proof we now present.

By Proposition 2, three linearly independent solutions of (2.17) corresponding to $\mu = 0$ are given by

$$Y_1(x) = \begin{pmatrix} u' \\ u'' \\ u''' \end{pmatrix}, \quad Y_2(x) = \begin{pmatrix} u_a \\ u_a' \\ u_a'' \end{pmatrix}, \quad \text{and} \quad Y_3(x) = \begin{pmatrix} u_E \\ u_E' \\ u_E'' \end{pmatrix}, \tag{2.23}$$

where we have suppressed the dependence on the variables x, a, E, and c. By hypothesis, for any $(a, E, c) \in \Omega$ the solution u satisfies

$$u(0; a, E, c) = u_- = u(T; a, E, c) \tag{2.24}$$

$$\partial_x u(0; a, E, c) = 0 = \partial_x u(T; a, E, c) \tag{2.25}$$

$$\partial_{xx} u(0; a, E, c) = -V'(u_-; a, c) = \partial_{xx} u(T; a, E, c). \tag{2.26}$$

Moreover, from equation (2.1) it follows that

$$u_{xxx}(0; a, E, c) = c u_x(0; a, E, c) - \frac{d}{dx} \left(f(u(x; a, E, c)) \right) \big|_{x=0} = 0.$$

Defining $\mathbf{U}(x, 0) = [Y_1(x), Y_2(x), Y_3(x)]$ to be the corresponding solution matrix in this basis, a direct calculation yields

$$\mathbf{U}(0,0) = \begin{pmatrix} 0 & \partial_a u_- & \partial_E u_- \\ -V'(u_-) & 0 & 0 \\ 0 & 1 - V''(u_-)\partial_a u_- & -V''(u_-)\partial_E u_- \end{pmatrix}. \tag{2.27}$$

Note that differentiating the relation $E - V(u_-) = 0$ gives the relation $-V'(u_-)\partial_E u_- = -1$, and hence $\det(\mathbf{U}(0,0)) = -1$. Thus, these solutions are linearly independent at $x = 0$, and hence for all x. By using similar methods then, we can compute $\mathbf{U}(T, 0)$ and right-multiply by $\mathbf{U}^{-1}(0, 0)$ to determine the monodromy $\mathbf{M}(0)$.

The matrix $\mathbf{U}(T, 0)$ can be calculated by differentiating (2.24)-(2.26) with respect to the parameters a and E by use of the chain rule. Notice this time the dependence of the period on a and E enters in an important way. For example, differentiating the relation (2.24) with respect to the parameter E gives

$$\partial_E u(T; a, E, c) = u_E(T; a, E, c) + \frac{\partial u}{\partial x}(T; a, E, c)\, T_E(a, c, E) = \partial_E u_-.$$

Since the derivative vanishes at the period points this implies $\partial_E u(T) = \partial_E u_-$. However, differentiating (2.25) with respect to E gives

$$\partial_E u_x(T; a, E, c) = u_{xx}(T; a, E, c)T_E + u_{Ex}(T; a, E, c) = 0$$

which implies $u_{Ex}(T; a, E, c) = V'(u_-)T_E$. Continuing in this manner gives the following expression for the change in this matrix solution across the period:

$$\mathbf{U}(T, 0) = \mathbf{U}(0, 0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & V'(u_-; a, c)T_a & V'(u_-; a, c)T_E \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.28}$$

In particular, we find that $\mathbf{U}(T, 0) - \mathbf{U}(0, 0)$ is a rank one matrix, which naturally leads to the following proposition.

Proposition 3. *There exists a basis in $\mathbb{R}^3$ such that the monodromy matrix $\mathbf{M}(\mu)$ evaluated at $\mu = 0$ takes the following Jordan normal form:*

$$\mathbf{M}(0) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix} \tag{2.29}$$

where $\sigma \neq 0$ as long as T_a and T_E do not simultaneously vanish. In particular, the monodromy operator at $\mu = 0$ has a single eigenvalue of $\lambda = 1$ with algebraic multiplicity three and geometric multiplicity two as long as the period is not at a critical point with respect to the parameters a and E for fixed waves speed c.

Proof. Recall $\det(\mathbf{U}(0, 0)) = -V'(u_-)\partial_E u_- = -1$, so $\mathbf{U}(0, 0)$ is invertible. Multiplying the above expression for $\mathbf{U}(T, 0)$ on the right by the matrix $\mathbf{U}^{-1}(0, 0)$ yields

$$\mathbf{M}(\mu = 0) = I + \vec{w} \otimes \vec{v}\ \mathbf{U}^{-1}(0)$$

where $\vec{w} = (0,1,0)^T$ and $\vec{v} = (0, V'(u_-)T_a, V'(u_-)T_E)^T$. Next, notice that

$$\mathbf{U}(0,0) \begin{pmatrix} 0 & -T_a & -T_E \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \vec{w} \otimes \vec{v}$$

and hence defining $\mathbf{N} := \mathbf{U}^{-1}(0)\,\mathbf{M}(0)\,\mathbf{U}(0)$ gives the equation

$$\mathbf{N} = \begin{pmatrix} 1 & -T_a & -T_E \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$Ker(\mathbf{N} - \mathbf{I}) = span\left\{(1,0,0)^T,\ (0, T_E, -T_a)^T\right\}$$

Now, take $\vec{v_3} := (0, -T_a, -T_E)$ and notice that $\vec{v_3} \notin Ker(\mathbf{N} - \mathbf{I})$. The Jordan structure then follows by noticing then that $(\mathbf{N} - \mathbf{I})\vec{v_3} = (T_a^2 + T_E^2)(1,0,0)^T \in \ker(\mathbf{N} - \mathbf{I})$.

Finally, recall that if u and $f(u)$ are co-periodic, we have already stated that $T_E > 0$. In other situations we will assume that this condition is met unless otherwise stated. □

Knowing an orthogonally invariant structure for $\mathbf{M}(0)$ gives us a base point to begin our perturbation calculation. In particular, notice that since $\mathbf{M}(0)$ has 1 as an eigenvalue with geometric multiplicity two, it follows that $D(0,1) = D_\mu(0,1) = 0$, and hence the linearized operator $\partial_x \mathcal{L}[u]$ has a periodic eigenvalue of multiplicity at least three at the origin: recall that $D(\cdot, 1)$ is an odd function. This will be established rigorously in the next section as a result of Lemma 1. The important point is that the implicit function theorem fails for the equation $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0,0)$. However, we are able to correct this by finding the dominant balance of the equation $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0,0)$ and using a suitable change of coordinates. This is one of our main results in the next section.

2.6.2 Asymptotic Analysis of $D(\mu,\kappa)$ near $(\mu,\kappa)=(0,0)$

We now analyze the characteristic polynomial of $\mathbf{M}(\mu)$ in a neighborhood of $\mu=0$ by considering $\mathbf{M}(\mu)$ as a small perturbation of the matrix $\mathbf{M}(0)$ constructed above. It is well understood how the eigenvalues of a Jordan block bifurcate under perturbation: see Kato[41] or Moro, Burke and Overton[51]. It is worth noting, however, that in this case the bifurcation is highly non-generic due to the constraints imposed by the symmetry of the problem.

Recall from Proposition 1 that the spectrum near $\mu=0$ is continuous. By the analyticity of $\mathbf{M}(\mu)$ in a neighborhood of $\mu=0$, we can expand $\mathbf{M}(\mu)$ for small μ as

$$\mathbf{M}(\mu)=\mathbf{M}(0)+\mu\,\mathbf{M}_\mu(0)+\frac{\mu^2}{2}\mathbf{M}_{\mu\mu}(0)+\mathcal{O}\left(|\mu|^3\right)$$

where $\mathbf{M}_\mu(0)=[M_{i,j}^{(1)}]$ and $\mathbf{M}_{\mu\mu}(0)=[M_{i,j}^{(2)}]$. If one makes a similarity transform $\widetilde{\mathbf{M}}(\mu)=\mathbf{V}^{-1}\mathbf{M}(\mu)\mathbf{V}$ so that $\widetilde{\mathbf{M}}(0)$ is in the Jordan normal form (2.29) then a direct calculation using the above second order expansion of $\widetilde{\mathbf{M}}(\mu)$ implies that in a neighborhood of $\mu=0$, the Evans function can be expressed as

$$\begin{aligned}
D(\mu,e^{i\kappa}) &= \det\left((\widetilde{\mathbf{M}}(\mu)-I)-(e^{i\kappa}-1)\mathbf{I}\right)\\
&= -\eta^3+\eta^2\left(\mu\,\mathrm{tr}\left(\widetilde{\mathbf{M}}_\mu(0)\right)+\frac{\mu^2}{2}\mathrm{tr}(\widetilde{\mathbf{M}}_{\mu\mu}(0))\right)\\
&- \eta\left(\mu\widetilde{M}_{3,2}^{(1)}\sigma+\mu^2\left(\frac{1}{2}\left(\mathrm{tr}(\widetilde{\mathbf{M}}_\mu)\right)^2-\frac{1}{2}\mathrm{tr}(\widetilde{\mathbf{M}}_\mu^2)-\frac{\sigma}{2}\widetilde{M}_{3,2}^{(2)}\right)\right)\\
&- \sigma(\widetilde{M}_{1,1}^{(1)}\widetilde{M}_{3,2}^{(1)}-\widetilde{M}_{3,1}^{(1)}\widetilde{M}_{1,2}^{(1)})\mu^2\\
&+ \mu^3\left(\det\left(\widetilde{\mathbf{M}}_\mu(0)\right)+\sigma S\right)+\mathcal{O}\left(4\right),
\end{aligned}\qquad (2.30)$$

where $\eta=e^{i\kappa}-1$, S represents mixed terms from $\widetilde{\mathbf{M}}_\mu(0)$ and $\widetilde{\mathbf{M}}_{\mu\mu}(0)$, σ is as in Proposition 3, and the notation $\mathcal{O}(4)$ represents terms whose degree is four or higher. Notice there are no other μ^3 terms since $\mathbf{M}(0)-I$ has rank one. Notice that from the previous subsection, we know the equation $\partial_x\mathcal{L}[u]v=0$ has at least two T-periodic solutions. Thus, it must be that $D(\mu,1)=\mathcal{O}(|\mu|^3)$. However, we need more detailed analysis to understand the way in which these three periodic-eigenvalue bifurcate from the $\mu = 0$ state. To this end, we use the symmetry from Lemma 1 in order to determine the dominant balance of the equation $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0, 0)$.

A useful construction for implicit function calculations of this type is that of the Newton diagram, which is a subset of the non-negative integer lattice $\mathbb{Z}^+ \times \mathbb{Z}^+$: see the appendix for more details. A vertex (i, j) is included if the coefficient of $\eta^{3-i}\mu^j$ in (2.30) is *non-zero*, otherwise the vertex is not included. The lower convex hull of the Newton diagram is made up of a collection of line segments. For each line segment of the lower convex hull the implicitly defined function has j distinct solution branches, where j is the horizontal length of the segment, of the form

$$\eta_k = \sum_i \alpha_i^{(k)} \mu^{si},$$

where $\alpha_1^{(k)} \neq 0$, s is the slope of the line segment, and k ranges from 1 to j. This is equivalent to the method of "dominant balance" presented in textbooks on asymptotic methods but is somewhat more systematic. For more information, see the appendix.

As an application to our problem, notice that the Newton diagram corresponding to the perturbed matrix $\widetilde{\mathbf{M}}(0) + \mu\,\widetilde{\mathbf{M}}_\mu(0) + \mathcal{O}(|\mu|^2)$ implies that, generically, the three eigenvalues bifurcating from the $\mu = 0$ state are determined by the number $\sigma\widetilde{M}_{3,2}^{(1)}$, where σ is defined as in Proposition 3. If this number is non-zero, the Newton diagram implies one eigenvalue bifurcates as $\mu = \mu_1\kappa^2 + \mathcal{O}(\kappa^4)$ and two bifurcate as $\pm\mu_{2,3}\sqrt{\kappa} + \mathcal{O}(\kappa)$, where the μ_i are non-zero. In particular, the pair of eigenvalues bifurcating non-analytically from the $\mu = 0$ state are given by

$$\mu_0 = -\frac{1}{\sigma\widetilde{M}_{3,2}^{(1)}}\kappa + \mathcal{O}(\kappa^2), \quad \mu_\pm = \sqrt{\frac{6i\sigma\widetilde{M}_{3,2}^{(1)}}{D_{\mu\mu\mu}(0,1)}}\,\kappa^{1/2} + \mathcal{O}(\kappa),$$

where we take the standard square root branch. This immediately implies modulational instability of the underlying periodic wave $u(x; a, E, c)$ for any non-linearity f, which is a highly unusual assertion. Thankfully, the symmetry inherent in the linearized operator, namely that $\mathbf{M}(\mu) \sim \mathbf{M}(-\mu)^{-1}$, forces the quantity $\sigma\widetilde{M}_{3,2}^{(1)}$ to vanish, which results in

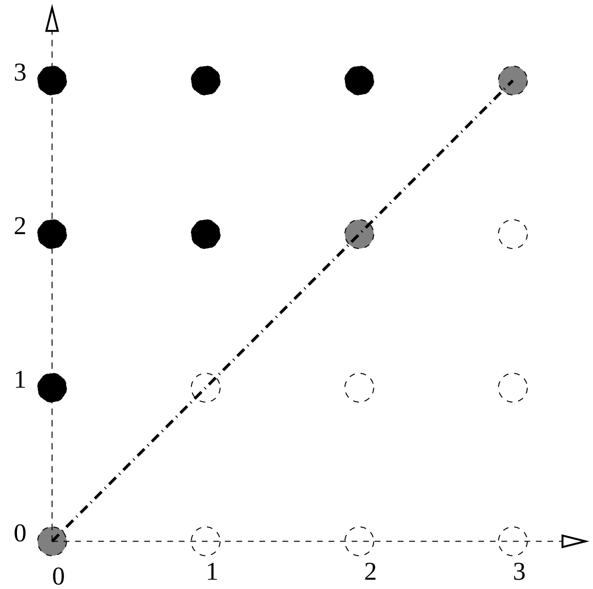

Figure 2.2: The Newton diagram corresponding to the asymptotic expansion of $D(\mu,\kappa) = 0$ in a neighborhood of $(\mu,\kappa) = (0,0)$ is shown to $O(|\mu|^3)$. Terms associated to open circles are shown to vanish due to the natural symmetries inherent in (2.1). The grey circles are non-vanishing terms which are a part of the lower convex hull. The black circles lie above the lower convex hull and thus do not contribute to the leading order asymptotics.

a non-generic bifurcation from the $\mu = 0$ state. Moreover, the vanishing of this product forces the use of a second order expansion of $\mathbf{M}(\mu)$ near $\mu = 0$ since the first order information has been shown to be insufficient. These results are proved in the following lemma.

Lemma 4. *The equation $D(\mu, e^{i\kappa}) = 0$ has the following normal form in a neighborhood of $(\mu,\kappa) = (0,0)$:*

$$-(i\kappa)^3 + \frac{i\kappa\mu^2}{2}\,\mathrm{tr}\,(\mathbf{M}_{\mu\mu}(0)) + \frac{\mu^3}{3}\,\mathrm{tr}\,(\mathbf{M}_{\mu\mu\mu}(0)) + \mathcal{O}(4) = 0,$$

whose Newton diagram is depicted in Figure 2.1, where $\mathcal{O}(4)$ represents terms of order four or higher in the variables κ and μ. Notice that if $\mathrm{tr}\,(\mathbf{M}_{\mu\mu\mu}(0)) = 0$, this only applies to two of the three branches bifurcating from the $\mu = 0$ state.

Proof. Let $a(\mu) = \mathrm{tr}(\mathbf{M}(\mu))$ as before and define the function b in a neighborhood of

$\mu = 0$ by

$$\det[(\mathbf{M}(\mu) - I) - (e^{i\kappa} - 1)I] = -\eta^3 + (a(\mu) - 3)\eta^2 + b(\mu)\eta + D(\mu, 1), \tag{2.31}$$

where $\eta = e^{i\kappa} - 1$. Notice in particular that $\eta = i\kappa + \mathcal{O}\left(\kappa^2\right)$ in a neighborhood of $\kappa = 0$. By (2.30), we have the expressions

$$\begin{aligned} a(\mu) &= \operatorname{tr}(\mathbf{M}(\mu)) = 3 + \mu \operatorname{tr}(\mathbf{M}_\mu(0)) + \frac{\mu^2}{2}\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) + \frac{\mu^3}{6}\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0)) + \mathcal{O}(|\mu|^4) \\ b(\mu) &= \frac{1}{2}\left(\operatorname{tr}((\mathbf{M}(\mu) - I)^2) - \operatorname{tr}(\mathbf{M}(\mu) - I)^2\right) \\ &= -\mu M^{(1)}_{3,2}\sigma - \mu^2\left(\frac{1}{2}\operatorname{tr}(\mathbf{M}_\mu)^2 - \frac{1}{2}\operatorname{tr}(\mathbf{M}_\mu^2) - \frac{\sigma}{2}\tilde{M}_{3,2}\right) + \mathcal{O}(|\mu|^3) \\ D(\mu, 1) &= -\sigma(\widetilde{M}^{(1)}_{1,1}\widetilde{M}^{(1)}_{3,2} - \widetilde{M}^{(1)}_{3,1}\widetilde{M}^{(1)}_{1,2})\mu^2 + (\det(\mathbf{M}_\mu(0)) + \sigma S)\,\mu^3 + \mathcal{O}(|\mu|^4) \end{aligned}$$

Recall from Proposition 1 that $D(\mu, 1)$ is an odd function of μ, and hence $D(\mu, 1) = \mathcal{O}(|\mu|^3)$, i.e. $D_{\mu\mu}(0, 1) = -2\sigma(\widetilde{M}^{(1)}_{1,1}\widetilde{M}^{(1)}_{3,2} - \widetilde{M}^{(1)}_{3,1}\widetilde{M}^{(1)}_{1,2}) = 0$.

Similarly, using (2.31) along with Lemma 1 we have

$$\begin{aligned} \det[\mathbf{M}(\mu) - \lambda\mathbf{I}] &= -\lambda^3 \det\left[\mathbf{M}(-\mu) - \frac{1}{\lambda}\right] \\ &= -(\lambda - 1)^3 - a(-\mu)\lambda(\lambda - 1)^2 + b(-\mu)\lambda^2(\lambda - 1) - D(-\mu, 1)\lambda^3. \end{aligned}$$

Comparing the λ^2 and λ^3 terms above with those in (2.31) we get the relations

$$\begin{cases} b(\mu) = 2a(\mu) - a(-\mu) - 3, \quad \text{and} \\ a(\mu) - b(\mu) + D(\mu, 1) = 3. \end{cases}$$

Since $D(\mu, 1) = \mathcal{O}(|\mu|^3)$, these relations imply $b'(0) = 3a'(0) = a'(0)$. Hence, $\operatorname{tr}(\mathbf{M}_\mu(0)) = a'(0) = 0$ and $-\sigma\widetilde{M}^{(1)}_{3,2} = b'(0) = 0$. By recalling $\sigma \neq 0$ from 3, this implies $\widetilde{M}^{(1)}_{3,2} = 0$. Similarly, we have $a''(0) = b''(0)$ and hence $b''(0) = \operatorname{tr}(\mathbf{M}_{\mu\mu}(0))$. Also, we have $b'''(0) = 3a'''(0)$ and $D_{\mu\mu\mu}(0, 1) = b'''(0) - a'''(0) = 2a'''(0)$, and hence $D_{\mu\mu\mu}(0, 1) = 2\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$. The corollary follows by analyzing equation (2.30) as well as the corresponding Newton diagram (see Figure 2.2). □

From this it follows that, in the neighborhood of the origin, the leading order piece of the periodic Evans function is a homogeneous cubic polynomial in the variables $\kappa,\ \mu$. Thus, to leading order, the spectrum in a neighborhood of the origin consists of three straight lines, one of which is guaranteed to lie along the imaginary axis by Proposition 1. Although the implicit function theorem still fails for the dominant balance obtained in Lemma 4, it is easily corrected by considering projective coordinates (as suggested by the homogeneity of the dominant balance), which leads to the following theorem.

Theorem 7. *With the above notation, define*

$$\Delta(f;u) = \frac{1}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))\right)^3 - 3\left(\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))\right)^2, \tag{2.32}$$

where f denotes the dependence on the non-linearity used in (2.1)*, and suppose that* $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ *is non-zero. If $\Delta > 0$, then the imaginary axis in the neighborhood in the origin is in the spectrum with multiplicity three. If $\Delta < 0$ then the imaginary axis in a neighborhood of the origin is in the spectrum with multiplicity one, together with two curves which leave the origin along lines in the complex plane: see Figure 2.3. In particular, in the latter case the periodic wave is modulationally unstable.*

Proof. Since the leading order piece of the Evans function is homogeneous it suggests working with a projective coordinate $y = \frac{i\mu}{\kappa}$. Making such a change of variables leads to the equation

$$1 - \frac{y^2}{2}\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) + \frac{y^3}{3}\operatorname{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) + \kappa E(\kappa, y) = 0,$$

where $E(\kappa, y)$ is continuous in a neighborhood of the origin. The implicit function theorem applies in a neighborhood of the points $(y = y_{1,2,3}, \kappa = 0)$ as long as the roots $y_{1,2,3}$ of the above cubic in y are distinct, which is true as long as the discriminant Δ is not zero. In terms of the original variable μ we have the three spectral branches

$$\mu_{1,2,3} = iy_{1,2,3}\kappa + O(\kappa^2)$$

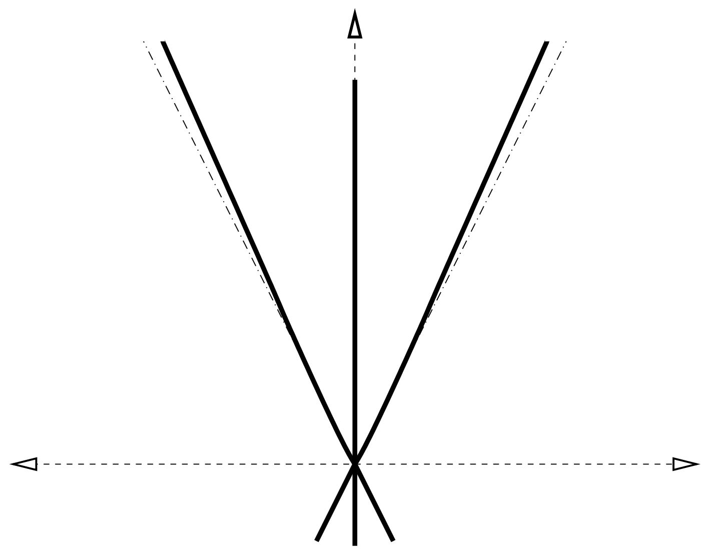

Figure 2.3: When $\Delta(f;u) < 0$, the local normal form of $\mathrm{spec}(\partial_x \mathcal{L}[u])$ consists of a segment of the imaginary axis union with two straight lines making equal angles with the imaginary axis. The non-horizontal dashed lines represent the actual spectrum, while the dark lines represent our approximation obtained through Lemma 4. Notice that these lines intersect at the origin, corresponding to the fact that 1 is an eigenvalue of $\mathbf{M}(0)$ with algebraic multiplicity three.

This cubic has three real roots when $\Delta > 0$, giving three branches of spectrum emerging from the origin tangent to the imaginary axis. It is clear from symmetry that these must in fact lie on the imaginary axis, giving a interval of spectrum of multiplicity three along the imaginary axis. In the case that the discriminant is negative there is one real root and two complex conjugate roots, giving one branch of spectrum along the imaginary axis and two branches emerging from the origin along (to leading order) straight lines. □

Remark 3. *First we note that* $\mathrm{tr}(\mathbf{M}_{\mu\mu}(0)) < 0$ *is a sufficient condition for modulational instability of the periodic wave.*

Secondly, notice that the analyticity of the spectrum is only dependent on the non-vanishing of $\mathrm{tr}(\mathbf{M}_{\mu\mu\mu}(0))$*, and is independent on whether* $\mathrm{tr}(\mathbf{M}_{\mu\mu}(0))$ *vanishes or not. In particular, this analyticity is broken if* $\mathrm{tr}(\mathbf{M}_{\mu\mu\mu}(0)) = 0$ *since then there is a slope in the Newton diagram greater than one, corresponding to at least one of the branches of spectrum bifurcating from the* $\mu = 0$ *state to admit a series in fractional powers of* κ*.*

Later we will give a formula for $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ *in terms of the Jacobian of a map, and we will see that vanishing of this quantity signals a change in the Jordan structure of the underlying linearized operator.*

Finally we note that this agrees with the result of Bottman and Deconinck[12], in which they considered cnoidal wave solutions to the KdV. Using the algebro-geometric techniques of Belokolos, Bobenko, Enol'skii, Its and Matveev[9] they explicitly computed the spectrum of the linearized operator and found that such solutions are always spectrally stable. Their results prove that an interval of the imaginary axis containing the origin is in the spectrum of the linearized operator with multiplicity three. Our results imply this is a generic phenomenon: either one has an interval of spectrum of multiplicity three about the origin, or one has three curves intersecting at the origin. For cnoidal solutions of the KdV, the discriminant Δ *is expressible in terms of elliptic functions in this case and must be positive (by the results in [12]), although we have not tried to show this explicitly.*

It is instructive to compare this theorem with that of Theorem 6. By Theorem 7, the sign of $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ has no effect on the spectral stability of the underlying periodic wave in a sufficiently small neighborhood of the origin. However, Theorem 6 guarantees the existence of unstable real spectrum if $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0)) > 0$. To reconcile these results, notice Proposition 1 implies there is no unstable real spectrum sufficiently close to the origin. Thus, the instability brought on by $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0)) > 0$ is not local to $\mu = 0$, and hence should not be detected by the quantity $\Delta(f; u)$.

Our next goal is to try to evaluate the "modulational stability index" $\Delta(f; u)$ as well as the finite wavelength orientation index $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ in terms of the conserved quantities of (2.1). As we will see, this can be done very explicitly. Notice that while we have chosen to express the coefficients as $\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))$ and $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$, which suggests that they arise at second order and third order in a perturbation calculation for small μ, these quantities can be expressed in terms of quantities which arise at first and second order in μ due to the action of the vector field $\frac{d}{dc}$. Furthermore, while all of the first order terms contribute to these expressions, only a few terms which are second order actually contribute - these are the terms which are associated to the minors of the off-diagonal piece of the unperturbed Jordan form. Again, due to the action of the vector field $\frac{d}{dc}$, these second order terms are explicitly computable via a single quadrature.

Theorem 8. *We have the following identities:*

$$\mathrm{tr}(\mathbf{M}_{\mu\mu})|_{\mu=0} = \{T,P\}_{E,c} + 2\{M,P\}_{a,E} \tag{2.33}$$

$$\mathrm{tr}(\mathbf{M}_{\mu\mu\mu})|_{\mu=0} = -\frac{3}{2}\{T,M,P\}_{a,E,c} \tag{2.34}$$

where T, M, P are the period, mass, and momentum of the underlying periodic traveling wave. Thus the modulational stability index has the following representation

$$\Delta = \frac{1}{2}\left(\{T,P\}_{E,c} + 2\{M,P\}_{a,E}\right)^3 - 3\left(\frac{3}{2}\{T,M,P\}_{a,E,c}\right)^2.$$

Proof. Let $w_i(x;\mu)$, $i = 1,2,3$, be three linearly independent solutions of (2.1), and let $\mathbf{W}(x,\mu)$ be the solution matrix with columns w_i. Expanding the above solutions in powers of μ as

$$w_i(x,\mu) = w_i^0(x) + \mu w_i^1(x) + \mu^2 w_i^2(x) + \mathcal{O}(|\mu|^3)$$

and substituting them into (2.16), the leading order equation becomes

$$\frac{d}{dx}w_i^0(x) = \mathbf{H}(x;0)w_i^0(x).$$

Using Proposition 2, we choose $w_i(x) = Y_i(x)$ where the vectors $Y_i(x)$ are defined in equation (2.23). The higher order terms in the above expansion yield

$$\frac{d}{dx}w_i^j(x) = \mathbf{H}(x;0)w_i^j(x) + V_i^{j-1}(x), \quad j \geq 1,$$

where $V_i^{j-1} = \left(0, 0, -(w_i^{j-1})_1\right)^t$ and $(v)_1$ denotes the first component of the vector v. Notice that for each of the higher order terms $j \geq 1$, we require $w_j^i(0) = 0$ in order to ensure that $\mathbf{W}(0,\mu) = \mathbf{U}(0,0)$ in a neighborhood of $\mu = 0$, where $\mathbf{U}(0,0)$ is defined in (2.27). In the case $j = 1$, the $i = 1$ equation is equivalent to the equation $L_0 w_1^1 = u_x$.

It follows again from Proposition 2 that we can choose

$$w_1^1(x) = \begin{pmatrix} -u_c \\ -u_{cx} \\ -u_{cxx} \end{pmatrix} + A \begin{pmatrix} u_a \\ u_a' \\ u_a'' \end{pmatrix} + B \begin{pmatrix} u_E \\ u_E' \\ u_E'' \end{pmatrix}$$

for some constants A and B. In order to compute A and B, we evaluate at $x = 0$ and require the resulting expression vanish. This yields to the system of equations

$$\begin{pmatrix} -\frac{\partial u_-}{\partial c} & \frac{\partial u_-}{\partial a} & \frac{\partial u_-}{\partial E} \\ -u_- + V''(u_-)\frac{\partial u_-}{\partial c} & 1 - V''(u_-)\frac{\partial u_-}{\partial a} & -V''(u_-)\frac{\partial u_-}{\partial E} \end{pmatrix} \begin{pmatrix} 1 \\ A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $A = u_-$ and B can be expressed as

$$B = \frac{\frac{\partial u_-}{\partial c} - u_- \frac{\partial u_-}{\partial a}}{\frac{\partial u_-}{\partial E}}.$$

Differentiating the relation $E - V(u_-; a, c) = 0$ implies $2V'(u_-; a, c)\frac{\partial u_-}{\partial_c} = u_-^2$ and $V'(u_-; a, c)\frac{\partial u_-}{\partial_a} = u_-$. Thus, $2B = -u_-^2$ and hence we can take

$$w_1^1(x) = \begin{pmatrix} -u_c \\ -u_{cx} \\ -u_{cxx} \end{pmatrix} + u_- \begin{pmatrix} u_a \\ u_a' \\ u_a'' \end{pmatrix} - \frac{u_-^2}{2} \begin{pmatrix} u_E \\ u_E' \\ u_E'' \end{pmatrix}.$$

Therefore, the following asymptotic expression of the matrix $\delta\,\mathbf{W}(\mu) := \mathbf{W}(T;\mu) - \mathbf{W}(0;\mu)$ is valid in a neighborhood of $\mu = 0$:

$$\begin{pmatrix} \mathcal{O}(|\mu|^2) & \mathcal{O}(|\mu|) & \mathcal{O}(|\mu|) \\ \mu V'(u_-)P(u_-) + \mathcal{O}(|\mu|^2) & V'(u_-)T_a + \mathcal{O}(|\mu|) & V'(u_-)T_E + \mathcal{O}(|\mu|) \\ \mathcal{O}(|\mu|^2) & \mathcal{O}(|\mu|) & \mathcal{O}(|\mu|) \end{pmatrix}$$

where $P(x) = -T_c + xT_a - \frac{x^2}{2}T_E$. In particular, notice that since $\mathbf{M}(\mu) = \delta\,\mathbf{W}(\mu) + \mathbf{I}$, it follows directly from this expansion that $D(\mu, 1) = \det(\delta\,\mathbf{W}(\mu))\det(\mathbf{W}(0))^{-1} = \mathcal{O}(\mu^3)$.

In order to compute the higher order terms in the above expression, we use variation of parameters as well as the identities $\{u_x, u\}_{x,E} = -1$ and $\{u, u_x\}_{x,a} = u$ to calculate

$$w_i^j(x) = \mathbf{W}(x,0) \int_0^x \mathbf{W}(z,0)^{-1} V_i^{j-1}(z) dz \tag{2.35}$$

$$= \begin{pmatrix} u_x \int_0^x (w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_a \int_0^x (w_i^{j-1})_1 dz + u_E \int_0^x (w_i^{j-1})_1 u\, dz \\ u_{xx} \int_0^x (w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_{ax} \int_0^x (w_i^{j-1})_1 dz + u_{Ex} \int_0^x (w_i^{j-1})_1 u\, dz \\ u_{xxx} \int_0^x (w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_{axx} \int_0^x (w_i^{j-1})_1 dz + u_{Exx} \int_0^x (w_i^{j-1})_1 u\, dz \end{pmatrix}$$

for w_i^1, $i = 2, 3$, and w_i^j for $i > 1$. Note that all of the $\mathcal{O}(\mu, \mu^2)$ terms in the above are necessary for the calculation, however we do not write them out explicitly. Noting that our choice of basis implies $\det(\mathbf{W}(0,\mu)) = -V'(u_-)\frac{\partial u_-}{\partial E} = -1$, a tedious yet straightforward calculation shows that

$$\begin{aligned} D(\mu, 1) &= -\det\left(\delta\, \mathbf{W}(\mu)\right) \\ &= -\frac{1}{2}\{T, M, P\}_{a,E,c}\, \mu^3 + \mathcal{O}(|\mu|^4), \end{aligned}$$

from which the expression for $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ follows by Theorem 6. Moreover, it follows from Lemma 4 and the fact $\mathbf{M}(\mu) = \delta\, \mathbf{W}(\mu)\, \mathbf{W}(0,0)^{-1} + \mathbf{I}$ and a rather tedious calculation that

$$\begin{aligned} \operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) &= -2\mu^{-2} \operatorname{tr}(\operatorname{cof}(\mathbf{M}(\mu) - I))\big|_{\mu=0} \\ &= \{T, P\}_{E,c} + 2\{M, P\}_{a,E} \end{aligned}$$

as claimed. □

Corollary 3. *$\{T, M, P\}_{a,E,c} < 0$ is a sufficient condition for the spectral instability of a periodic traveling wave solution of (2.1). Moreover, if $T_E > 0$ at $(a_0, E_0, c_0) \in \Omega$, the periodic wave traveling wave solution $u(\,\cdot\,; a_0, E_0, c_0)$ with $(a_0, E_0, c_0) \in \Omega$ is exponentially unstable to periodic perturbations if $\{T, M, P\}_{a,E,c} < 0$ at (a_0, E_0, c_0), and is spectrally stable to such perturbations if $\{T, M, P\}_{a,E,c} > 0$ at (a_0, E_0, c_0).*

Proof. This is now clear from Theorems 6 and 8, and the fact that the number of negative periodic eigenvalues of $\mathcal{L}[u]$ bound above the number of possible unstable periodic eigenvalues of the linearized operator $\partial_x \mathcal{L}[u]$ with positive real part. □

At this point we can make a connection to the stability theory for the solitary waves. A natural assertion is that the periodic traveling waves with $(a, E, c) \in \Omega$ of sufficiently large period should have the same stability properties as the limiting homoclinic (solitary) wave. This is in fact true for a large class of dispersive equations. Indeed, the work of Gardner [30] shows that the linearized operator about such a periodic wave has spectrum supported in the unstable half-space whenever the solitary wave has an unstable eigenvalue. In particular, it is known that $\partial_x \mathcal{L}[u]$ has spectrum supported in the unstable half-space in the case of the gKdV with $f(u) = u^{p+1}$ if $p > 4$. We now present a slight extension of this result in this special case of the gKdV with power-law nonlinearity. We begin with the following technical lemma.

Lemma 5. *If $(a, E, 1) \in \Omega$, then $M_a(a, E, 1) < 0$ for a and E sufficiently small.*

Proof. Notice that it is enough to show that $M_a(a, 0, 1) < 0$ for a sufficiently small. This provides a great simplification since taking $E = 0$ implies $V(0; a, 1) = 0$, and so we know one of the classical turning points of the Hamiltonian system (2.5) corresponding to this solution. Now, after some rescaling $M(a, 0, 1)$ can be expressed in the form

$$M^*(a, 0, 1) = \int_0^{r(a)} \frac{\sqrt{u}}{\sqrt{a + u - u^{p+1}}} \, du$$

where $r(a)$ is the smallest positive root of $a + u - u^{p+1} = 0$, and $M^*(a, 0, 1) = C^* M(a, 0, 1)$ for some $C^* > 0$. Now, r is a smooth function for a sufficiently small and satisfies

$$r(a) = 1 + \frac{a}{p+1} + O(a^2).$$

The main idea is to now rescale the above integral to be over a fixed domain, and then show the integrand is a decreasing function of a on this new domain.

Defining the new variable $v = r(a)^{-1}u$ gives

$$M^*(a,0,1) = \int_0^1 \frac{\sqrt{v}}{\sqrt{\frac{a}{r(a)^3} + \frac{v}{r(a)^2} - r(a)^{p-2}v^{p+1}}}\, dv.$$

Now, using the above expansion for $r(a)$ implies that

$$\frac{a}{r(a)^3} + \frac{v}{r(a)^2} - r(a)^{p-2}v^{p+1} = v - v^{p+2} + a\left(1 - \frac{2}{p+1}\, v - \frac{p-2}{p+1}\, v^{p+1}\right) + \mathcal{O}(a^2).$$

Since the $\mathcal{O}(a)$ term in this expression is positive on the open interval $(0,1)$, it follows that the rescaled integrand is a decreasing function of a. Hence, $M_a(a,0,1) < 0$ for a small enough. □

The proof of our extension of Gardner's result is now straightforward. This is the content of the following corollary of Theorem 8 and Corollary 3.

Corollary 4. *In the case of power-law nonlinearity and wavespeed $c > 0$, there are always unstable periodic traveling waves in a neighborhood of the solitary wave ($a = E = 0$) if $p > 4$. Moreover, such long wavelength periodic waves are modulationally unstable if $p > 4$ and are modulationally stable if $p < 4$.*

Proof. First, note that the scaling invariance in equation (2.4) implies the periodic solution $u(x;a,E,c)$ satisfies

$$u(x;a,E,c) = c^{1/p}u\left(c^{1/2}x; \frac{a}{c^{1+1/p}}, \frac{E}{c^{1+2/p}}, 1\right).$$

This allows us to compute the quantities T_c, M_c, P_c explicitly as follows:

$$
\begin{aligned}
T_c &= -\frac{1}{2c}T - \frac{a(p+1)}{pc}T_a - \frac{E(p+2)}{pc}T_E \\
M_c &= \left(\frac{1}{pc} - \frac{1}{2c}\right) M + \left(T_c + \frac{T}{2c}\right) u_- - \frac{a(p+1)}{pc}\left(M_a - T_a u_-\right) \\
&\quad - \frac{E(2+p)}{pc}\left(M_E - T_E u_-\right) \\
P_c &= \left(\frac{2}{pc} - \frac{1}{2c}\right) P + \left(T_c + \frac{T}{2c}\right) u_-^2 - \frac{a(p+1)}{pc}\left(P_a - T_a u_-^2\right) \\
&\quad - \frac{E(2+p)}{pc}\left(P_E - T_E u_-^2\right),
\end{aligned}
$$

where the equation for T_c follows from equation (2.8). Since we know that $P_E = 2T_c$ and $P_a = 2M_c$, the above identities serve to express the last row and column of the matrix

$$
\begin{pmatrix} T_a & T_E & T_c \\ M_a & M_E & M_c \\ P_a & P_E & P_c \end{pmatrix}
$$

essentially in terms of the entries in the upper left 2×2 block.

Now, when a and E are small there are two turning points r_1, r_2 in the neighborhood of the origin and a third turning point r_3 which is bounded away from the origin. In the solitary wave limit $a, E \to 0$ we have $r_1 - r_2 = O(\sqrt{a^2 - 2E})$. From this, it follows that the period satisfies the asymptotic relation

$$
T(a, E, 1) = \mathcal{O}\left(\ln(a^2 - 2E)\right). \tag{2.36}
$$

To see this, notice that if p is an odd integer there are three roots of the equation $E - V(u; a, 1)$=0: two roots $r_{1,2}$ are in a neighborhood of zero and the third root r_3 is positive and bounded away from zero. Thus, we can write $E - V(u; a, 1) = (r - r_1)(r - r_2)(r_3 - r)Q(r)$ where $Q(r)$ is positive on the set $[r_1, r_3]$. The period can then be expressed as

$$\begin{aligned}
T(a,E,1) &= \sqrt{2}\int_{r_2}^{r_3} \frac{dr}{\sqrt{(r-r_1)(r-r_2)(r_3-r)Q(r)}} \\
&= \sqrt{2}\int_{r_2-r_1}^{r_2-r_1+\delta} \frac{dr}{\sqrt{r(r-(r_2-r_1))(r_3-r_1-r)Q(r+r_1)}} \\
&+ \sqrt{2}\int_{r_2-r_1+\delta}^{r_3-r_1} \frac{dr}{\sqrt{r(r-(r_2-r_1))(r_3-r_1-r)Q(r+r_1)}}
\end{aligned}$$

The integral over the set $(r_2-r_1+\delta, r_3-r_1)$ is clearly $\mathcal{O}(1)$ as $(a,E)\to(0,0)$. For the other integral, notice that

$$(r_3-r_1-r)Q(r+r_1) > 0$$

on the set $[r_2-r_1, r_2-r_1+\delta]$. Thus, in the limit as $(a,E)\to\infty$ we have

$$\begin{aligned}
T(a,E,1) &\sim \int_{r_2-r_1}^{r_2-r_1+\delta} \frac{dr}{\sqrt{r(r-(r_2-r_1))}} \\
&= -4\ln(4(r_2-r_1)) + 2\ln(2(\sqrt{r_2-r_1+\delta}+\sqrt{\delta})) \\
&\sim -\ln(r_2-r_1)
\end{aligned}$$

from which (2.36) follows in the case when p is an odd integer: the case when p is not odd is handled similarly.

By analogous calculations, we have the asymptotics

$$\begin{aligned}
M(a,E,1) &= \mathcal{O}(1) \\
P(a,E,1) &= \mathcal{O}(1) \\
T_a(a,E,1) &= \mathcal{O}\left(\frac{a}{a^2-2E}\right) = M_E(a,E,1) \\
T_E(a,E,1) &= \mathcal{O}\left(\frac{1}{a^2-2E}\right) \\
M_a(a,E,1) &= \mathcal{O}\left(\frac{a^2}{a^2-2E}\right) + \mathcal{O}(\ln(a^2-2E))
\end{aligned}$$

for small (a,E). Thus the asymptotically largest minor of $\{T,M,P\}_{a,E,c}$ is $-T_E M_a P_c$, and hence the above scaling gives the asymptotic relation

$$\{T,M,P\}_{a,E,c} \sim -T_E M_a \left(\frac{2}{pc} - \frac{1}{2c}\right) P$$

as $a, E \to 0$. Now, it follows from Lemma 10 and Lemma 5 that $T_E(a,E,1) > 0$ and $M_a(a,E,1) < 0$ for a and E sufficiently small such that $(a,E,1) \in \Omega$. Thus the orientation index $\{T,M,P\}_{a,E,c}$ is negative for $p > 4$ and a, E sufficiently small (in other words sufficiently close to the solitary wave) and positive for $p < 4$ and a, E sufficiently small. This also follows, of course, from Gardner's long-wavelength theory[30] but it provides a good check for the present theory.

To prove the second claim, notice the above asymptotics implies

$$\mathrm{tr}(\mathbf{M}_{\mu\mu}(0)) \quad \sim \quad \frac{1}{2}\, T_E \left(\frac{2}{pc} - \frac{1}{2c}\right) P$$

in the limit as $a, E \to 0$. Hence, it follows that $\mathrm{sign}\Delta(a,E,c) = \mathrm{sign}(4-p)$ for a, E sufficiently small such that $(a,E,c) \in \Omega$. Thus, such periodic traveling waves of sufficiently long period are also modulationally unstable for $p > 4$, and are modulationally stable for $p < 4$. □

It is worth noting that the instability mechanism detected by the discriminant Δ is not present in the solitary wave case: in the solitary wave limit the bands of spectrum connected to the origin collapse to the origin. Moreover, this instability does *not* appear to follow from Gardner's calculation: Gardner shows that the point eigenvalue of the solitary wave opens into a small loop of spectrum, predicting the real eigenvalue detected by $\{T,M,P\}_{a,E,c}$, but the modulational instability detected by Δ is not detected. This suggests the heuristic that periodic solutions should go unstable before the solitary waves. The small amplitude stability calculation of Hărăguş and Kapitula for the generalized KdV equation amounts to a calculation of this discriminant in that limiting case, and their proof that the small amplitude waves go unstable at $p = 2$ is the first result we are aware of along these lines.

We believe that a small amplitude analysis of $\Delta(a,E,c)$ should be possible. It follows by a simple calculation that $\Delta = 0$ at the stationary solution: see the proof of Lemma 15 for the methods involved. By expanding near by solutions in terms of amplitude instead of the energy E, we believe the first non-zero term of the discriminant should be proportional to a polynomial which switches signs at $p = 2$, thus recovering the small amplitude result of Hărăguş and Kapitula [35]. We have not as yet carried out such an analysis.

Using the identities between the gradients of the conserved quantities of (2.1) restricted to the manifold of periodic traveling wave solutions, i.e. using (2.13), we now have a sufficient criterion for the existence of a non-trivial intersection of $\text{spec}(\partial_x \mathcal{L}[u])$ with the real axis in terms of the conserved quantities M, P and H of the gKdV flow (assuming $E \neq 0$), as well as a necessary and sufficient condition for understanding the normal form of the spectrum in a neighborhood of the origin. It is a rather striking fact that both of these indices can be expressed entirely in terms of the conserved quantities of the flow. The monodromy itself depends on the classical turning points $u_\pm(a, E, c)$ of the traveling wave, as well as various functions and derivatives of this quantity, but the indices themselves only depend on the gradients of the conserved quantities. This is, of course, the Whitham philosophy, but the above analysis constitutes one of the only cases we are aware of (other than the integrable calculations, which are very special) which make this connection rigorous.

In the next section we outline the connections of this calculation to a calculation based more directly on the linearized operator. While not strictly necessary, we believe this calculation is useful since it clarifies the way in which various bifurcations can occur.

2.7 Local Analysis of $\text{spec}(\partial_x \mathcal{L}[u])$ via the Floquet-Bloch Decomposition

In this section we sketch an approach to the modulational instability problem working directly with the linearized operator rather than with the Evans function. While these two approaches are presumably equivalent the former seems less straightforward than the latter. In particular it is not clear how one might derive the orientation index in this way, and the calculation of the modulational stability index gives a quantity which seems much less transparent. Nevertheless we present an outline of this calculation (omitting some details) since it does give some insight into the results of the previous section. It should also be noted that, historically, this approach has favored such calculations. Indeed, many of the small amplitude stability results for nonlinear PDE hinge on the use of a Floquet-Bloch decomposition of the linearized operator, and then applying a perturbation to study the stability of the solutions bifurcating from the constant state.

As mentioned several times before, one of the main difficulties in the spectral theory of linear differential operators with periodic coefficients on L^2 is the fact that the spectrum is purely continuous, admitting no isolated eigenvalues of finite multiplicity. Thus, many of the classical methods concerning spectral theory do not apply. However, difficulty can be circumvented by applying a Floquet-Bloch decomposition of the spectral problem, i.e. by considering the operators

$$J_\gamma := e^{-i\gamma x/T}\partial_x e^{i\gamma x/T}, \;\; \mathcal{L}_\gamma[u] := e^{-i\gamma x/T}\mathcal{L}[u]e^{i\gamma x/T}$$

on $L^2_{\rm per}([0,T])$. By performing such a decomposition, the spectral problem $\partial_x\mathcal{L}[u]v = \mu v$ on $L^2(\mathbb{R})$ is replaced with a *family* of spectral problems for the operators $\{J_\gamma\mathcal{L}_\gamma[u]\}_{\gamma\in[-\pi,\pi]}$ considered on the Hilbert space $L^2_{\rm per}([0,T];\mathbb{C})$ such that $\text{spec}(\partial_x\mathcal{L}[u])$ is precisely the union of the spectrum of the operators $J_\gamma\mathcal{L}_\gamma[u]$ considered on $L^2_{\rm per}([0,T])$: through a slight abuse of notation, we denote this as $\text{spec}(J_\gamma\mathcal{L}_\gamma[u])$[3]. Moreover, each of the operators $J_\gamma\mathcal{L}_\gamma[u]$ has a compact resolvent when acting on $L^2_{\rm per}([0,T])$ and hence its spectrum consists only of eigenvalues of finite-multiplicity. In this sense, such a decomposition greatly simplifies the original spectral problem. However, one must now work with a family of spectral problems, which carries its own hardships.

The outline for this section is as follows: first, we implement the Floquet-Bloch decomposition to the linearized spectral problem corresponding to a periodic traveling wave solution of the gKdV (2.1). We then prove from this operator view point that

[3]Recall that $\text{spec}(A)$ has thus far denoted the $L^2(\mathbb{R})$ spectrum of an operator A. Throughout this section, it will be obvious from context when we mean $L^2(\mathbb{R})$ spectrum of $L^2_{\rm per}([0,T])$ spectrum.

the spectrum bifurcates from the $\mu = 0$ state analytically in the Floquet parameter by using the Weierstrass perpetration theorem to narrow down the class of possible bifurcations, and then appealing to the Fredholm alternative to deduce the desired analyticity. Finally, we use the Fredholm alternative to develop a second modulational instability index in terms of a solvability condition for an over-determined system of equations. This index, while presumably equivalent to the one previously derived, seems to have a much more complicated structure than that presented in Theorem 8. However, this approach has the advantage of not explicitly utilizing the symmetry from Lemma 1 and hence may be applicable to more situations.

2.7.1 Floquet-Bloch Decomposition

We begin by detailing some of the basic properties of the Floquet-Bloch decomposition applied to the spectral problem $\partial_x \mathcal{L}[u]v = \mu v$ considered on $L^2(\mathbb{R})$.

From Floquet theory, we know any L^∞ eigenfunction $v(x)$ of the linearized operator $\partial_x \mathcal{L}[u]$ must satisfy

$$v(x+T) = e^{i\gamma} v(x)$$

for some $\gamma \in [-\pi, \pi]$. The quantity $e^{i\gamma}$ is known as the Floquet multiplier of the eigenfunction v, while the number γ is the corresponding Floquet parameter. Notice that the Floquet exponent is only defined up to an additive multiple of 2π, while the Floquet multiplier is always unique. It follows any eigenfunction $v(x)$ can be represented in the form $v(x) = e^{i\gamma x/T} P(x)$ where P is a T-periodic function. The fact that $\partial_x \mathcal{L}[u] v(x) = \mu v(x)$ for some $\mu \in \mathbb{C}$ implies

$$e^{i\gamma x/T} J_\gamma \mathcal{L}_\gamma[u] P(x) = \mu e^{i\gamma x/T} P(x)$$

where $J_\gamma = \left(\partial_x + i\frac{\gamma}{T}\right)$ and $\mathcal{L}_\gamma[u] = -\left(\partial_x + i\frac{\gamma}{T}\right)^2 - f'(u) + c$. The operators $J_\gamma \mathcal{L}_\gamma[u]$ are known as the Bloch operator corresponding to the original spectral problem. This suggests fixing a $\gamma \in [-\pi, \pi]$ and considering the eigenvalue problem for the operator $J_\gamma \mathcal{L}_\gamma[u]$ on the Hilbert space $\mathcal{H} = L^2_{\text{per}}([0,T]; \mathbb{C})$. This procedure is known as a Bloch, or Floquet-Bloch, decomposition of the eigenvalue problem (2.16) and we consider the Bloch operators as acting on $\mathcal{H}$. Notice for $\gamma \neq 0$ the operators $J_\gamma \mathcal{L}_\gamma[u]$ acting on the space $\mathcal{H}$ are closed with compactly embedded domain $H^3(\mathbb{R}/T\mathbb{Z};\mathbb{C})$. It follows that the Bloch operators have compact resolvent and hence their spectra consists of only point spectra with finite algebraic multiplicities. Moreover, one has

$$\text{spec}(\partial_x \mathcal{L}[u]) = \bigcup_{\gamma \in [-\pi,\pi]} \text{spec}\left(J_\gamma \mathcal{L}_\gamma[u]\right).$$

Thus, this decomposition reduces the problem of locating the continuous spectrum of the operator $\partial_x \mathcal{L}[u]$ on L^2 to the problem of determining the discrete spectrum of a one parameter family of operators $\{J_\gamma \mathcal{L}_\gamma[u]\}_{\gamma \in [-\pi,\pi]}$ on $\mathcal{H}$. Our first goal is to understand the nature of the spectrum of the operator $J_0 \mathcal{L}_0[u]$ at the origin. Notice in particular that for $\gamma_1, \gamma_2 \in [-\pi, \pi]$, $J_{\gamma_1} \mathcal{L}_{\gamma_1}[u]$ is a compact perturbation of $J_{\gamma_2} \mathcal{L}_{\gamma_2}[u]$, and hence routine calculations prove the above parametrization of $\text{spec}(J_\gamma \mathcal{L}_\gamma[u])$ is in fact continuous.

One of our fist goals is to prove that the operator $J_0 \mathcal{L}_0[u]$ acting on the space $\mathcal{H}$ has an eigenvalue at the origin of multiplicity three. As we are interested in the modulational instability of the underlying periodic traveling wave, we consider the family of operators $J_\gamma \mathcal{L}_\gamma[u]$ for $|\gamma| \ll 1$, treating each one as a small perturbation of $J_0 \mathcal{L}_0[u]$, with our end goal being to study how the spectrum bifurcates from the $\gamma = 0$ state. We begin with analyzing the generalized periodic null space of the operator $\partial_x \mathcal{L}[u]$, denoted $N_g(\partial_x \mathcal{L}[u]) = \bigcup_{n=1}^\infty N((\partial_x \mathcal{L}[u])^n)$.

Proposition 4. *Suppose that the Jacobians $\{T, M\}_{a,E}$ and $\{T, P\}_{a,E}$ do not simultaneously vanish, and that $\{T, M, P\}_{a,E,c} \neq 0$. Then zero is an eigenvalue of the operator $\partial_x \mathcal{L}[u] = J_0 \mathcal{L}_0[u]$ considered on $\mathcal{H}$ of algebraic multiplicity three and geometric multi-*

plicity two. In the case $\{T,M\}_{a,E} \neq 0$*, define the functions*

$$\begin{aligned} \phi_0 &= \{T,u\}_{a,E}, & \psi_0 &= 1, \\ \phi_1 &= \{T,M\}_{a,E}\, u_x, & \psi_1 &= \int_0^x \phi_2(s)ds, \\ \phi_2 &= \{u,T,M\}_{a,E,c} & \psi_2 &= \{T,M\}_{E,c} + \{T,M\}_{a,E}u. \end{aligned}$$

Then the set $\{\phi_j\}_{j=1}^3$ *provides a basis for* $N_g(\partial_x \mathcal{L}[u])$ *and, in particular, we have the relations*

$$\begin{aligned} \partial_x \mathcal{L}[u]\phi_0 &= 0 & \mathcal{L}[u]\partial_x\psi_0 &= 0 \\ \partial_x \mathcal{L}[u]\phi_1 &= 0 & \mathcal{L}[u]\partial_x\psi_1 &= -\psi_2 \\ \partial_x \mathcal{L}[u]\phi_2 &= -\phi_1 & \mathcal{L}[u]\partial_x\psi_2 &= 0. \end{aligned}$$

Proof. The constants above are chosen for convenience, and the functions above are not normalized. For instance, ϕ_2 can be any multiple of u_x and similarly ψ_0 any constant. Also, the ordering is chosen so that $\langle \phi_j, \psi_k \rangle = 0$ for $i \neq k$. Notice this proposition does not follow directly from Proposition 2 since the functions u_a, u_E and u_c are not in general T-periodic, and thus do not belong to $\mathcal{H}$.

First observe that (2.28) implies ϕ_0 and ϕ_1 are T-periodic and belong to $N(J_\gamma \mathcal{L}_\gamma[u])$. Since Lemma 10 implies that $T_E > 0$, it follows from Theorem 9 from Chapter 3 that $\mu = 0$ is a simple eigenvalue of $\mathcal{L}_0[u]$ acting on $\mathcal{H}$. Since There is a linear combination of u_a and u_E which is non-trivial and belongs to $\mathcal{H}$, it follows that $N(\partial_x \mathcal{L}[u]) = \text{span}\{\phi_0, \phi_1\}$. Moreover, the fact that the monodromy at the origin is the identity plus a rank one perturbation suggests that there is another non-trivial linear combination involving u_c which can be chosen to be periodic. Specifically we define

$$\phi_2 = \begin{vmatrix} u_a & T_a & \int_0^T u_a dx \\ u_E & T_E & \int_0^T u_E dx \\ u_c & T_c & \int_0^T u_c dx \end{vmatrix} = \{u,T,M\}_{a,E,c}$$

and note it is clear from (2.28) that $\phi_2 \in \mathcal{H}$ and $J_0\mathcal{L}_0[u]\phi_2 = -\phi_1$ as claimed. Thus, if $\{T, M\}_{a,E} \neq 0$, ϕ_2 gives a function in $N((J_0\mathcal{L}_0[u])^2) - N(J_0\mathcal{L}_0[u])$.

Similar arguments show that ψ_0 and ψ_2 are belong to $N(\mathcal{L}[u]\partial_x)$, and are linearly independent provided that $\{T, M\}_{a,E} \neq 0$. Moreover, a it is clear from construction that $\psi_1 \in \mathcal{H}$ and a straightforward computation shows that ψ_1 belongs to $N((\mathcal{L}_0[u]J_0)^2) - N(\mathcal{L}_0[u]J_0)$ as claimed.

In order to complete the proof, we must now show these three functions comprise the entire generalized null space of $J_0\mathcal{L}_0[u]$ on $\mathcal{H}$. To this end, we prove that neither of the functions ϕ_0, ϕ_2 belong to the range of $J_0\mathcal{L}_0[u]$ by appealing to the Fredholm alternative. Notice the equation $J_0\mathcal{L}_0[u]v = \phi_0$ has a solution in $\mathcal{H}$ if and only if the following solvability conditions are simultaneously satisfied:

$$\begin{aligned}\langle 1, \phi_0\rangle &= \{T, M\}_{a,E} = 0, \quad \text{and}\\ \langle u, \phi_0\rangle &= \frac{1}{2}\{T, P\}_{a,E} = 0.\end{aligned}$$

Thus, if either $\{T, M\}_{a,E}$ or $\{T, P\}_{a,E}$ are non-zero, then $N((J_0\mathcal{L}_0[u])^2) - N(J_0\mathcal{L}_0[u]) = \text{span}\{\phi_2\}$. Similarly, $N((J_0\mathcal{L}_0[u])^3) - N((J_0\mathcal{L}_0[u])^2) \neq \emptyset$ if and only if the equation $L_0 v = \phi_2$ has a solution in $\mathcal{H}$, i.e. if and only if

$$\langle u, \phi_2\rangle = \frac{1}{2}\{T, M, P\}_{a,E,c} = 0,$$

which completes the proof.

Notice that a similar construction in the case $\{T, M\}_{a,E} = 0$ but $\{T, P\}_{a,E} \neq 0$ the basis

$$\begin{aligned}\widetilde{\phi}_0 &= \{T, u\}_{a,E}, & \widetilde{\psi}_0 &= 1,\\ \widetilde{\phi}_1 &= \{T, P\}_{a,E}\, u_x, & \widetilde{\psi}_1 &= \int_0^x \widetilde{\phi}_2(s)ds,\\ \widetilde{\phi}_2 &= \{u, T, P\}_{a,E,c} & \widetilde{\psi}_2 &= u.\end{aligned}$$

Similar arguments as above prove the set $\{\widetilde{\phi}_0, \widetilde{\phi}_1, \widetilde{\phi}_2\}$ is indeed a basis for the generalized periodic null-space of $\partial_x \mathcal{L}[u]$ if $\{T, P\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are non-zero. However, we will assume throughout the rest of this chapter that $\{T, M\}_{a,E} \neq 0$. In Chapter 3 we will prove that $\{T, M\}_{a,E} > 0$ in the case of the KdV equation, and the asymptotics from the previous section show the same conclusion sufficiently close, but beneath, the solitary (homoclinic) wave. □

Remark 4. *It is worth remarking in some detail on the physical significance of these conditions and the relationship to the Whitham modulation theory. Obviously* (a, E) *are constants of integration arising in the ordinary differential equation defining the traveling wave for a fixed wave-speed* c*, and* T, M, P *are constants of the PDE evolution. In particular, the constants* (a, E, c) *parameterize the manifold of periodic traveling wave solutions of* (2.1)*. One of the main ideas of the Whitham modulation theory is to locally parameterize the wave by the constants of motion. The non-vanishing of the Jacobians is exactly what allows one to do this. Non-vanishing of* $\{T, M, P\}_{a,E,c}$ *is equivalent to demanding that locally the map* $(a, E, c) \mapsto (T, M, P)$ *have a unique* C^1 *inverse - in other words the conserved quantities* (T, M, P) *are good local coordinates for the family of traveling waves. Similarly non-vanishing of one of* $\{T, M\}_{a,E}$ *and* $\{T, P\}_{a,E}$ *is, at least for periodic waves below the separatrix, equivalent to demanding that the matrix*

$$\begin{pmatrix} T_a & M_a & P_a \\ T_E & M_E & P_E \end{pmatrix}$$

have full rank, which is equivalent to demanding that the map $(a, E) \mapsto (T, M, P)$ *(at fixed* c*) have a unique* C^1 *inverse - in other words two of the conserved quantities give a smooth parametrization of the family of traveling waves of fixed wave-speed. As long as* $E \neq 0$ *we can use the identities developed in the appendix to eliminate* T *in favor of* H*. Thus in the case* $E \neq 0$ *(which* does not *include the solitary wave) the null-space being two dimensional is equivalent to two of the conserved quantities* (M, P, H) *giving a* C^1 *parametrization of the traveling wave solutions at constant wavespeed, and the space* $N((J_0\mathcal{L}_0[u])^2) - N(J_0\mathcal{L}_0[u])$ *being one dimensional is equivalent to the three conserved quantities* (M, P, H) *giving a* C^1 *parametrization of the full family of traveling waves.*

Notice it follows the vanishing of $\{T, M, P\}_{a,E,c}$ is connected with a change in the Jordan structure of the linearized operator $J_0\mathcal{L}_0[u]$ considered on $\mathcal{H}$: $\{T, M, P\}_{a,E,c} \neq 0$ ensures the existence of a non-trivial Jordan piece in the generalized null space of of dimension exactly one. Moreover, it guarantees that the variations in the constants associated to the family of traveling wave solutions by reducing (2.1) to quadrature are enough to generate the entire generalized periodic null space of the operator $J_0\mathcal{L}_0[u]$. Henceforth, we shall assume $\{T, M, P\}_{a,E,c} \neq 0$ and that $\{T, M\}_{a,E} \neq 0$ - by our above remarks, trivial modifications are necessary if $\{T, M\}_{a,E}$ vanishes but $\{T, P\}_{a,E}$ does not.

2.7.2 Analyticity of Eigenvalues Bifurcating from $\mu = 0$

Our next goal is to consider the operator $J_\gamma\mathcal{L}_\gamma[u]$ for small γ, treating it as a small perturbation of $J_0\mathcal{L}_0[u]$. To this end, notice that if we define $L_0 := J_0\mathcal{L}_0[u]$, $L_1 := \mathcal{L}_0[u] - 2\partial_x^2$, and $L_2 := -3\partial_x$, it follows that

$$J_\gamma\mathcal{L}_\gamma[u] = L_0 + \varepsilon L_1 + \varepsilon^2 L_2 - \varepsilon^3,$$

where ε is related to the Floquet exponent via $\varepsilon = \frac{i\gamma}{T}$. By Proposition 4, we know the operator L_0 has three periodic eigenvalues at the origin. Our present goal is to determine how these eigenvalues bifurcate from the $\gamma = 0$ state. In this section we only sketch the relevant details - for similar calculations see the papers of Ivey and Lafortune[38], or Kapitula, Kutz and Sanstede.[40]

Since the Hilbert space $\mathcal{H}$ consists of T-periodic functions, eigenvalues of $J_\gamma\mathcal{L}_\gamma[u]$ correspond to 1 being an eigenvalue of the monodromy operator $\Phi(T; \mu, \varepsilon)$ for to the eigenvalue problem $J_\gamma\mathcal{L}_\gamma[u]v = \mu v$. Thus, it is natural to introduce the following "modified" periodic Evans function

$$D_0(\mu, \varepsilon) = \det\left(\Phi(T; \mu, \varepsilon) - I\right).$$

Notice that $D_0(\mu, \varepsilon)$ is clearly an analytic function of the two complex variables μ and

ε. Our first goal then is to analyze the possible behavior of the solutions of $D_0(\mu, \varepsilon) = 0$ in a small neighborhood of $(0,0)$.

Lemma 6. *Let $F(x, y)$ be a complex valued function of two complex variables x and y which is analytic in a neighborhood of $(0,0) \in \mathbb{C}^2$. Moreover, suppose that $F(0,0) = F_x(0,0) = F_{xx}(0,0) = 0$, $F_{xxx}(0) \neq 0$, and $F_y(0,0) = 0$. Then for small y, the equation $F(x, y) = 0$ has three roots in a neighborhood of the origin. Moreover, these roots are given by $(x, y) = (f_j(y), y)$, $j = 1, 2, 3$, where the f_j satisfy one of the following conditions:*

(i) One function f_j can be expressed as a Puiseux series as $f_j(y) = \sum_{n=1}^{\infty} a_n^j y^{n/2}$ in a neighborhood of $y = 0$, where $a_1 \neq 0$.

(ii) Two of the functions f_j admits a Puiseux series representation of the form $f_j(y) = \sum_{n=2}^{\infty} a_n^j y^{n/3}$ in a neighborhood of $y = 0$, where $a_2 \neq 0$.

(iii) All three functions f_j are $\mathcal{O}(\varepsilon)$ and are analytic in y in a neighborhood of $y = 0$, i.e. they can be represented as $f_j(y) = \sum_{n=1}^{\infty} a_n^j y^n$ where $a_1 \neq 0$ assuming $F_{yyy}(0,0) \neq 0$.

In the case (iii), if $F_{yyy}(0,0) = 0$ then all three eigenvalues are analytic in ε, with two eigenvalues of order $\mathcal{O}(|\varepsilon|)$ and the remaining eigenvalue or order at least $\mathcal{O}(|\varepsilon|^2)$.

Proof. By the Weierstrass preparation theorem, the function $F(x, y)$ can be expressed as

$$F(x, y) = \left(x^3 + \eta_2(y)x^2 + \eta_1(y)x + \eta_0(y)\right) h(x, y)$$

for small x and y, where each η_j is analytic, and h is analytic satisfying $h(0,0) \neq 0$. It follows the three roots of $F(x, y)$ near $(0,0)$ are determine by the cubic polynomial $G(x, y) = x^3 + \eta_2(y)x^2 + \eta_1(y)x + \eta_0(y)$. By hypothesis, we have that $\eta_j(0) = 0$ for $j = 0, 1, 2$, $\eta_0'(0) = 0$, and $\eta_0'''(0) \neq 0$. The lemma now follows by analyzing the corresponding Newton diagram. □

We now wish to apply Lemma 6 to the equation $D_0(\mu, \varepsilon) = 0$, with $x = \mu$ and $y = \varepsilon$, and use the Fredholm alternative to show only possibility *(iii)* can occur. Notice that

Theorem 8 implies $\frac{\partial^k}{\partial \mu^k} D_0(\mu; 0) = 0$ for $k = 0, 1, 2$ and, moreover, $\frac{\partial^3}{\partial \mu^3} D_0(\mu; 0) \neq 0$ under the assumption $\{T, M, P\}_{a,E,c} \neq 0$. To apply Lemma 6 then, we need the following lemma.

Lemma 7. *We have* $\frac{\partial}{\partial \varepsilon} D_0(0, 0) = 0$.

Proof. This proof proceeds much like that of Theorem 8. Defining $W(x; \mu, \varepsilon)$ to be the solution matrix to the first order system corresponding to $J_\gamma \mathcal{L}_\gamma[u]v = \mu v$ written in the basis $Y_i(x)$ defined in (2.23), arguments similar to those above yield for small ε

$$\det\left(W(T; 0, \varepsilon) - W(0; 0, \varepsilon)\right) = \begin{pmatrix} \mathcal{O}(|\varepsilon|) & \mathcal{O}(|\varepsilon|) & \mathcal{O}(|\varepsilon|) \\ \mathcal{O}(|\varepsilon|) & V'(u_-)T_a + \mathcal{O}(|\varepsilon|) & V'(u_-)T_E + \mathcal{O}(|\varepsilon|) \\ \mathcal{O}(|\varepsilon|) & \mathcal{O}(|\varepsilon|) & \mathcal{O}(|\varepsilon|) \end{pmatrix},$$

and hence $D_0(0, \varepsilon) = \mathcal{O}(|\varepsilon|^2)$ as claimed. □

We are now in position to prove our main result of this section. By the above work, we can apply lemma 6 to the equation $D_0(\mu, \varepsilon) = 0$. The next theorem uses the Fredholm alternative to discount possibilities *(i)* and *(ii)* from Lemma 6, and establish the analyticity of the eigenvalues near $\mu = 0$. Basically this amounts to checking that (generically) the null-space of the linearized operator has the same Jordan structure as the monodromy map at the origin.

Theorem 9. *For small ε, the linear operator $J_\gamma \mathcal{L}_\gamma[u]$ has three eigenvalues which bifurcate from $\mu = 0$ and are analytic in ε.*

Proof. The idea of the proof is to systematically discount possibilities *(i)* and *(ii)* from lemma 6, thus leaving only the third possibility. First, suppose case *(i)* holds. It follows from the Dunford calculus that we can expand the eigenvalues and eigenfunctions as

$$\begin{cases} \mu = \varepsilon^{1/2}\nu_1 + \varepsilon\nu_2 + \mathcal{O}(|\varepsilon|^{3/2}), \\ v = f_0 + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \mathcal{O}(|\varepsilon|^{3/2}). \end{cases}$$

We will show the assumption that $\{T, M, P\}_{a,E,c} \neq 0$ implies $\nu_1 = 0$, which yields the desired contradiction. Using the above expansions of v, μ and $J_\gamma \mathcal{L}_\gamma[\phi]$ in terms of ε, the leading order equation becomes $L_0 f_0 = 0$. Thus, $f_0 = b_0\phi_0 + b_1\phi_1$ for some $b_0,\ b_1 \in \mathbb{C}$. Continuing, the $\mathcal{O}(|\varepsilon|^{1/2})$ equation turns out to be $L_0 f_1 = \nu_1 f_0$. Suppose $\nu_1 \neq 0$. By the Fredholm alternative, this equation is solvable in $\mathcal{H}$ if and only if $b_0\phi_0 + b_1\phi_1 \perp N(L_0^\dagger)$. Clearly, $\phi_1 \perp N(L_0^\dagger)$ since $\phi_1 \in \text{Range}(L_0)$. Moreover, by Lemma 4 $\phi_0 \notin N(L_0^\dagger)^\perp$ and hence we must have $b_0 = 0$ and, with out loss of generality, we take $b_1 = 1$. It follows that f_1 must satisfy the equation

$$L_0 f_1 = \nu_1 \phi_1,$$

i.e. $f_1 = \nu_1\phi_2 + b_2\phi_0 + b_3\phi_1$ for some constants $b_2,\ b_3 \in \mathbb{C}$.

Continuing in this fashion, the $\mathcal{O}(|\varepsilon|)$ equation becomes

$$L_0 f_2 = \nu_1 f_1 + \nu_2 f_0 - L_1 f_0.$$

By the Fredholm alternative, this is solvable if and only if

$$\begin{aligned} \langle \psi_0, \nu_1 f_1 + \nu_2 f_0 - L_1 f_0 \rangle &= 0 \text{ and} \\ \langle \psi_2, \nu_1 f_1 + \nu_2 f_0 - L_1 f_0 \rangle &= 0. \end{aligned}$$

By above, f_0 is an odd function and since L_1 preserves parity, the solvability condition implies we must require $\langle \psi_0, f_1 \rangle = \langle \psi_2, f_1 \rangle = 0$. However, this is a contradiction since $\langle \psi_2, \phi_2 \rangle = \frac{1}{2}\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$ and hence it must be that $\nu_1 = 0$ as claimed. Thus, possibility *(i)* can not occur.

Next, assume case *(ii)* of lemma 6 holds. Then the Dunford calculus again implies the eigenvalues and eigenvectors can be expanded in a Puiseux series of the form

$$\begin{cases} \mu = \omega_1 \varepsilon^{2/3} + \omega_2 \varepsilon^{4/3} + \mathcal{O}(|\varepsilon|^2), \\ v = w_0 + \varepsilon^{2/3} w_1 + \varepsilon^{4/3} w_2 + \mathcal{O}(|\varepsilon|^2). \end{cases}$$

Our goal again is to prove the assumptions $\{T, M, P\}_{a,E,c} \neq 0$ and $\{T, M\}_{a,E} \neq 0$ imply $\omega_1 = 0$. Substituting these expansions into $J_\gamma \mathcal{L}_\gamma[u]v = \mu v$ as before, the leading order equation leads to $w_0 = a_0\phi_0 + a_1\phi_1$ and the $\mathcal{O}(|\varepsilon|^{2/3})$ equation implies $a_0 = 0$. Without loss of generality, we assume $a_1 = 1$, so that it follows that $w_1 = \omega_1\phi_2 + a_2\phi_0 + a_3\phi_1$. The solvability condition at $\mathcal{O}(|\varepsilon|^{4/3})$ implies that

$$-\omega_1^2 \langle \psi_2, \phi_2 \rangle = 0,$$

which implies $\omega_1 = 0$ as above. Thus, case *(ii)* of Lemma 6 can not occur leaving only case *(iii)*, which completes the proof. □

2.7.3 Perturbation Analysis of $\mathrm{spec}(J_\gamma \mathcal{L}_\gamma[u])$ near $(\mu, \gamma) = (0, 0)$

We are now set to derive a modulational stability index in from this operator theoretic approach. By Theorem 9, it follows that the eigenvalues and eigenvectors are analytic in ε, and hence admit a representation of the form

$$v = v_0 + v_1\varepsilon + v_2\varepsilon^2 + \mathcal{O}(|\varepsilon|^3),$$
$$\mu = \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \mathcal{O}(|\varepsilon|^3),$$

where $\lambda_1 \neq 0$ and v_0 is not identically zero. Moreover, our modulational instability theory will be based on the assumption that the three eigenvalues bifurcating from the $\mu = 0$ state are distinct: as in the previous section, we will derive a modulational instability index as the discriminant of a cubic polynomial which determines the first order piece of the bifurcating eigenvalue. If these leading order pieces are the same, then the problem requires further expansion in the parameter ε: we will not attempt to derive this theory here.

At this point, it is tempting to use the functionals $P_j := \langle \psi_j, \cdot \rangle$ to compute the matrix action of the operator $J_\gamma \mathcal{L}_\gamma[u]$ onto the corresponding spectral subspace associated with $N_g(L_0)$. This would convert the above eigenvalue problem for a fixed γ to the problem of solving the polynomial equation

$$\det\left[M_0 + \varepsilon M_1 + \varepsilon^2 M_2 + \mathcal{O}(\varepsilon^3) - \lambda P\right] = 0,$$

at $\mathcal{O}(\varepsilon^2)$, where $M_k = \{P_i L_k \phi_j\}_{i,j}$ and $P = \{P_i \phi_j\}_{i,j}$. Although this approach has been used to determine stability in the case where the underlying periodic waves are small (see [28] and [35]), this approach is flawed in the current case since, as shown below, the eigenvector v has a non-trivial projection onto $N_g(L_0)^{\perp}$ of size $\mathcal{O}(\varepsilon)$. Since we have no information about the range of such a projection, it is unlikely that one can determine the nature of the spectrum near $\mu = 0$ by computing the matrix action of the operator $J_\gamma \mathcal{L}_\gamma[u]$ on $\mathcal{H}$ for a general periodic solution of (2.1). Instead, we proceed below by developing a perturbation theory for such a degenerate eigenvalue problem based on the Fredholm alternative.

Substituting the analytic representation of the eigenvector and eigenvalue into the equation $J_\gamma \mathcal{L}_\gamma[u]v = \mu v$, the leading order equation implies $v_0 \in N(L_0)$, i.e. $v_0 = c_0\phi_0 + c_1\phi_1$ for some $c_0,\ c_1 \in \mathbb{C}$. At $\mathcal{O}(|\varepsilon|)$, we get the equation $L_0 v_1 = (\lambda_1 - L_1)v_0$, equipped with the corresponding solvability conditions

$$\begin{aligned} 0 = \langle \psi_0, L_0 v_1 \rangle &= \lambda_1 c_0 \langle \psi_0, \phi_0 \rangle - c_0 \langle \psi_0, L_1 \phi_0 \rangle - c_1 \langle \psi_0, L_1 \phi_1 \rangle, \text{ and} \\ 0 = \langle \psi_2, L_0 v_1 \rangle &= -c_0 \langle \psi_2, L_1 \phi_0 \rangle - c_1 \langle \psi_2, L_1 \phi_1 \rangle. \end{aligned}$$

It follows that we must require $c_0 = 0$. Indeed, from the parity relation $\langle \psi_i, L_k \phi_j \rangle = 0$ if $i + j + k = 0 \text{ mod}(2)$, and the relations $\langle \psi_0, \phi_0 \rangle = \{T, M\}_{a,E} \neq 0$ and $\langle \psi_0, L_1 \phi_0 \rangle = T_E \neq 0$, we either have $c_0 = 0$ or all three eigenvalues bifurcating from $\mu = 0$ have the same leading order non-zero real part, which is not allowed by our assumption that the three branches of spectrum are distinct. With out loss of generality, we then set $c_1 = 1$ and fix the normalization

$$\langle \psi_1, v \rangle = \langle \psi_1, v_0 \rangle = -\frac{1}{2}\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$$

for all ε. It follows that $v_0 = \phi_1$ and hence v_1 satisfies the equation

$$L_0 v_1 = (\lambda_1 - L_1)\phi_1.$$

Notice that $L_1 v_0 = -2\{T,M\}_{a,E} u_{xxx}$ does not belong to $N_g(L_0)$, and hence the eigenfunction v has a non-trivial projection onto $N_g(L_0)^{\perp}$ of size $\mathcal{O}(\varepsilon)$, as claimed above.

We now define L_0^{-1} on $R(L_0)$ with the requirement that $R(L_0^{-1})$ is orthogonal to $span\{\psi_0, \psi_1\}$. This requirement ensures that $L_0^{-1} f$ is well-defined and unique for all $f \in R(L_0)$. In particular, it allows us to compute the projection of $L_0^{-1} f$ onto $N(L_0)$ for each $f \in R(L_0)$. In order to express the explicit dependence of v_1 on λ_1, we now write

$$v_1 = L_0^{-1} (\lambda_1 - L_1) \phi_1 + c_2 \phi_0 + c_3 \phi_1 \tag{2.37}$$

for some c_1, $c_3 \in \mathbb{C}$. The above normalization condition implies $\langle \psi_1, v_1 \rangle = 0$, i.e.

$$0 = \left\langle \psi_1, L_0^{-1} (\lambda_1 - L_1) \phi_1 \right\rangle + c_3 \left\langle \psi_1, \phi_1 \right\rangle .$$

It follows $c_3 = 0$ by the definition of L_0^{-1} and the fact that

$$\langle \psi_1, \phi_1 \rangle = -\{T,M\}_{a,E} \{T,M,P\}_{a,E,c} \neq 0.$$

Continuing, the $\mathcal{O}(|\varepsilon|^2)$ equation is

$$L_0 v_2 = -L_1 v_1 - L_2 v_0 + \lambda_1 v_1 + \lambda_2 v_0$$

with corresponding solvability conditions

$$\begin{aligned} 0 &= -\left\langle \psi_0, L_1 v_1 \right\rangle - \left\langle \psi_0, L_2 v_0 \right\rangle + \lambda_1 \left\langle \psi_0, v_1 \right\rangle , \text{ and} \\ 0 &= -\left\langle \psi_2, L_1 v_1 \right\rangle - \left\langle \psi_2, L_2 v_0 \right\rangle + \lambda_1 \left\langle \psi_2, v_1 \right\rangle . \end{aligned}$$

Using the explicit dependence of v_1 on λ_1 and c_2, it follows that we can express the above solvability conditions as

$$\begin{aligned} P_1(\lambda_1) + \widetilde{P}_1(\lambda_1) c_2 &= 0 \text{ and} \\ P_2(\lambda_1) - P_0(\lambda_1) c_2 &= 0. \end{aligned}$$

where P_1 and $\widetilde{P}_1$ are linear polynomials, P_0 is a constant, and P_2 is a quadratic polynomial. As this is an over determined system of linear equations for c_2, the consistency condition

$$P(\lambda_1) := P_0(\lambda_1)P_1(\lambda_1) + \widetilde{P}_1(\lambda_1)P_2(\lambda_1) = 0$$

must hold. In particular, this expresses λ_1 as a root of a cubic polynomial with real coefficients. Since ε is purely imaginary, modulational stability follows if and only if $P(\lambda)$ has three real roots, and hence it must be that $\Delta(f;u)$ is a positive multiple of the discriminant of the cubic polynomial $P(\lambda)$. Notice that one can explicitly calculate $P(\lambda)$ for a general non-linearity using just the definitions of the ϕ_j and ψ_j, except for the inner products $\langle \psi_0, L_1 L_0^{-1} L_1 \phi_1 \rangle$ and $\langle \psi_2, L_1 L_0^{-1} L_1 \phi_1 \rangle$: however, these expressions are quite daunting and do not seem to contribute to the overall understanding of the structure of the modulational instability index. It follows that we can explicitly write down the compatibility condition $P(\lambda_1) = 0$ only in terms of the underlying periodic solution u and terms built up out of the generalized null spaces of L_0 and $L_0^\dagger$ acting on $L^2_{\text{per}}([0,T])$. Since the roots of this polynomial determine the structure of $\text{spec}(J_\gamma \mathcal{L}_\gamma[u])$ in a neighborhood of the origin, we have proven the following theorem.

Theorem 10. *The periodic solution $u = u(x; a, E, c)$ of* (2.1) *is spectrally unstable in a neighborhood of the origin if and only if the discriminant $\Delta(a, E, c)$ of the real cubic polynomial $P(\lambda)$ is positive. Recall that the discriminant of a cubic $P(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d$ is given by $\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$*

The above result gives a second characterization of the modulational stability of periodic solutions to the generalized Korteveg-de Vries equation with power law nonlinearity since it is expressed entirely in terms of T, M, P, H and their derivatives, which in turn can be written as functions of a, E, c via integral type formulae. (These are hyperelliptic integrals in the case that p is rational). The formulae remain, however, somewhat daunting. Since this detects the same instability that the Evans function based criterion does this quantity must have the same sign as the discriminant derived in that section, although we have not been able to show this.

To close this section, we now compute the quantities $\left\langle \psi_0, L_1 L_0^{-1} L_1 \phi_1 \right\rangle$ and $\left\langle \psi_2, L_1 L_0^{-1} L_1 \phi_1 \right\rangle$. The first of these can be calculated regardless of the nonlinearity, but we must restrict to power-law nonlinearity for the computation of the second. The complexity of these expressions shows why describing the modulational instability index in this way seems unfeasible.

Proposition 5. $\left\langle \psi_0, L_1 L_0^{-1} L_1 \phi_1 \right\rangle = -T\{T,K\}_{a,E}$.

Proof. Define an operator $\xi : \{g \in L^2(\mathbb{R}/T\mathbb{Z}) : \langle g \rangle \neq 0\} \to L^2(\mathbb{R}/T\mathbb{Z})$ by

$$\xi(g) = x - \frac{T}{\langle g \rangle} \int_0^x g(s)ds.$$

Then a straight forward computation shows that $L_0^{\dagger}\xi(\phi_0) = f'(u) - c + \frac{T_E T}{\{T, \langle u \rangle\}_{a,E}}$. It follows that

$$\begin{aligned} \left\langle \psi_0, L_1 L_0^{-1} L_1 \phi_1 \right\rangle &= 2\{T,M\}_{a,E} \left\langle \left(f'(u) - c\right), L_0^{-1} u_{xxx} \right\rangle \\ &= 2\{T,M\}_{a,E} \left\langle L_0^{\dagger}\xi(\phi_0), L_0^{-1} u_{xxx} \right\rangle \\ &= T \left\langle \phi_0, u_{xx} \right\rangle \\ &= -T\{T,K\}_{a,E} \end{aligned}$$

as claimed. ☐

While the above expression holds for an arbitrary nonlinearity, we have found a closed form expression for $\left\langle \psi_2, L_1 L_0^{-1} L_0 \phi_1 \right\rangle$ only in the case of power non-linearities. From the evaluation of the modulational instability index via Evans function techniques, it should be that this inner product is calculable in the general case as well, although we have yet to be able to do this.

Proposition 6. *In the case of a power nonlinearity $f(x) = x^{p+1}$, we have*

$$\begin{aligned} \left\langle \psi_2, L_1 L_0^{-1} L_1 \phi_1 \right\rangle &= -T\{T,M\}_{E,c}\{T,K\}_{a,E} \\ &+ \frac{2-p}{p}\{T,M\}_{a,E} \left(M\{T,K\}_{a,E} - 2\{T,M\}_{a,E} K\right) \\ &+ 2c\{T,M\}_{a,E}\{T,M,K\}_{a,E,c}. \end{aligned}$$

Proof. Notice that in the case of power-law nonlinearity, one has

$$\begin{aligned}\left\langle \psi_2, L_1 L_0^{-1} L_1 \phi_1 \right\rangle &= -T\{T,M\}_{E,c}\{T,K\}_{a,E} \\ &+ \{T,M\}_{a,E}\left((2-p)\left\langle u^{p+1}, L_0^{-1}L_1\phi_1\right\rangle - 2c\left\langle u, L_0^{-1}L_1\phi_1\right\rangle\right),\end{aligned}$$

and hence we must evaluate $\left\langle u, L_0^{-1}L_1\phi_1\right\rangle$ and $\left\langle u^{p+1}, L_0^{-1}L_1\phi_1\right\rangle$. First, from the definition of v_1 in equation (2.37) it follows that

$$\begin{aligned}\langle \psi_2, v_1\rangle &= \lambda_1\left\langle \psi_2, L_0^{-1}\phi_1\right\rangle - \left\langle \psi_2, L_0^{-1}L_1\phi_1\right\rangle \\ &= -\frac{1}{2}\lambda_1\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} - \{T,M\}_{a,E}\left\langle u, L_0^{-1}L_1\phi_1\right\rangle\end{aligned}$$

Moreover, using the fact that $\psi_2 = L_0^{\dagger}\psi_1$ gives

$$\begin{aligned}\langle \psi_2, v_1\rangle &= \langle \psi_1, (\lambda_1 - L_1)\phi_1\rangle \\ &= \lambda_1\langle\psi_1,\phi_1\rangle + 2\{T,M\}_{a,E}\langle\psi_1, u_{xxx}\rangle \\ &= -\frac{1}{2}\lambda_1\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} - 2\{T,M\}_{a,E}\langle\phi_2, u_{xx}\rangle \\ &= -\frac{1}{2}\lambda_1\{T,M\}_{a,E}\{T,M,P\}_{a,E,c} + \{T,M\}_{a,E}\{T,M,K\}_{a,E,c}\end{aligned}$$

and hence $\left\langle u, L_0^{-1}L_1\phi_1\right\rangle = -\{T,M,K\}_{a,E,c}$.

Next, let the functional ξ be as in Lemma 5 and notice that

$$L_0^{\dagger}\xi(u) = f'(u) - c - \frac{T}{M}\left(pu^{p+1} + a\right).$$

It follows that

$$\begin{aligned}-\frac{Tp}{M}\left\langle u^{p+1}, L_0^{-1}L_1\phi_1\right\rangle &= \left\langle L_0^{\dagger}\xi(u) - (f'(u)-c), L_0^{-1}L_1\phi_1\right\rangle \\ &= \langle\xi(u), L_1\phi_1\rangle + 2\{T,M\}_{a,E}\left\langle (f'(u)-c), L_0^{-1}u_{xxx}\right\rangle \\ &= \frac{2T\{T,M\}_{a,E}T}{M}K - T\{T,K\}_{a,E}\end{aligned}$$

which completes the proof. □

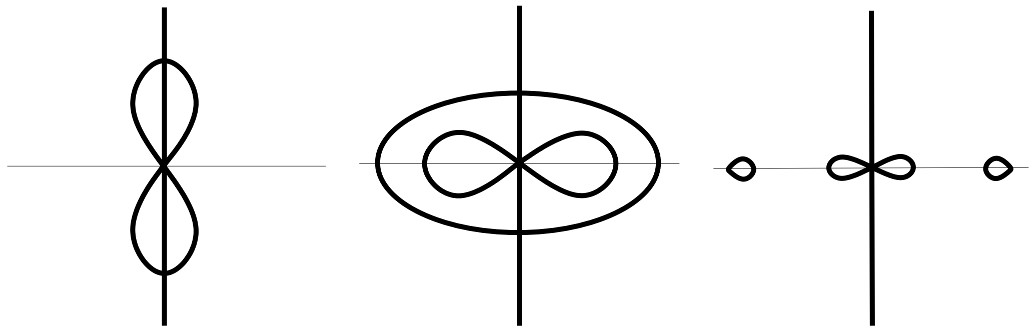

Figure 2.4: Cartoon of the spectrum of the linearization of gKdV about a periodic traveling wave for $p = 5, a = 0, E = 0$ and three different values of the period (ordered by increasing period). Thanks to Mariana Hărăguş and Todd Kapitula for supplying these numerical simulations.

2.8 Concluding Remarks

We'd like to consider a concrete example to illustrate our results. We have chosen to consider the power law gKdV with $p = 5$. In this case the solitary wave is unstable and hence (by Gardner's result, which we have checked in this case using our methods) periodic waves of sufficiently long period are also unstable. Hărăguş and Kapitula[35] have done some very nice experiments on this case using the SpectruW[20] package, which they have been kind enough to share. For clarity we have drawn figures representing the spectra they computed numerically, rather than reproducing their figures.

The first figure shows a cartoon of the spectrum for short periods - in other words small amplitude periodic waves. The modulational instability index $\Delta < 0$ indicating a modulational instability, while the orientation index $\{T, M, P\}_{a,E,c} > 0$. The latter indicates that the number of eigenvalues on the real axis away from the origin is even. In this case there are none. The spectrum near the axis looks like a union of three straight lines. Globally the spectrum looks like the union of the imaginary axis with a figure eight shaped curve.

As the period increases one sees spectra which resemble the second figure, where there is a modulational instability together with a pair of eigenvalues along the real axis. In this case we are still in the case $\Delta < 0$, indicating a modulational instability, and $\{T, M, P\}_{a,E,c} > 0$ indicating an even number of eigenvalues along the positive real axis. The fact that these two very different spectral pictures have the same orientation and modulational instability indices shows that these quantities alone are not enough to say qualitatively what the spectral picture looks like, even in this very simple problem with only one free parameter (the period).

As the period increases still further one sees spectral pictures which resemble the third picture. As in the previous figure there is an ∞ shaped curve of spectrum connected to the origin indicating a modulational instability ($\Delta < 0$) as well as a circle of spectrum out on the real axis. This circle is that predicted by Gardner in his paper. As the period increases and the periodic solution approaches the solitary wave the circle collapses to a point and the ∞ curve collapses to the origin. The size of both of these features is exponentially small in the period. In the paper of Kapitula and Hărăguş the ∞ curve is not visible at the scale of the graph, but it is visible in numerics they performed for smaller values of period.

Since there is an odd number of eigenvalues on the real axis in this case (one periodic, two antiperiodic) the orientation index must now be negative $\{T, M, P\}_{a,E,c} < 0$. The general mechanism by which this must occur is clear: a periodic eigenvalue moves down the real axis, collides with the origin (changing the Jordan structure of the null-space of the linearized operator, which is again signalled by the vanishing of $\{T, M, P\}_{a,E,c}$) and moves off along the real axis. However the exact way in which this occurs is not quite clear. It is somewhat puzzling that the Evans function based calculation gives a substantially simpler criteria for the existence of a modulational instability than one based on a direct analysis of the linearized operator. It must be true that the two discriminants we've derived always have the same sign, as the predict the same phenomenon, but we have been unable to see this directly from the formulae. Often when apparently unconnected quantities share a sign this sign has a topological or geometric interpretation (for example as a Krein signature), so this may well be the case here. Such an interpretation would be very interesting.

CHAPTER 3

Orbital Stability of Periodic Solutions of the gKdV

In this chapter, we study the orbital stability for a four-parameter family of periodic stationary traveling wave solutions to the generalized Korteweg-de Vries (gKdV) equation

$$u_t = u_{xxx} + f(u)_x.$$

In particular, we derive sufficient conditions for such a solution to be orbitally stable (in a suitable sense) in terms of the Hessian of the classical action of the corresponding traveling wave ordinary differential equation restricted to the manifold of periodic traveling wave solution. We show this condition is equivalent to the solution being spectrally stable with respect to perturbations of the same period in the case when $f(u) = u^2$ (the Korteweg-de Vries equation) and in neighborhoods of the homoclinic and equilibrium solutions if $f(u) = u^{p+1}$ for some $p \geq 1$.

3.1 Introduction

This chapter concerns the stability analysis of periodic traveling wave solutions of the generalized Korteweg-de Vries (gKdV) equation

$$u_t = u_{xxx} + f(u)_x \tag{3.1}$$

where f is a sufficiently smooth non-linearity satisfying certain convexity assumptions. Probably the most famous equation among this family is given by $f(u) = u^2$, in which case (3.1) corresponds to the Korteweg-de Vries (KdV) equation. The KdV serves as an approximate description of small amplitude waves propagating in a weakly dispersive media. Other choices of the nonlinearity f arise in various applications, such as internal waves and plasmas. Thus, to ensure general application of the forthcoming theory we find it beneficial to consider general nonlinearities in (3.1).

It is well known that the gKdV equation admits traveling wave solutions of the form

$$u(x,t) = u_c(x+ct), \quad x \in \mathbb{R}, \quad t \in \mathbb{R} \tag{3.2}$$

for wave speeds $c > 0$. Historically, there has been much interest in the stability of traveling solitary waves of the form (3.2) where the profile u_c decays exponentially to zero as its argument becomes unbounded. Such waves were initially discovered by Scott Russell in the case of the KdV where the traveling wave is termed a soliton. While (3.1) does not in general possess exact "soliton" solutions, which requires complete integrability of the partial differential equation, exponentially decaying traveling wave solutions still exist. Moreover, the stability of such solitary waves is well understood and dates back to the pioneering work of Benjamin [7], which was then further developed by Bona [10], Grillakis [31], Grillakis, Shatah and Strauss [32, 33], Bona, Souganides and Strauss [11], Pego and Weinstein [56, 57], Weinstein[62, 63], and many others. In this theory, it is shown that traveling solitary waves of (3.1) are orbitally stable if the solitary wave stability index

$$\frac{\partial}{\partial c}\int_{-\infty}^{\infty} u_c^2 \, dx \tag{3.3}$$

is positive, and is orbitally unstable if this index is negative. In the case where (3.1) has a power-law nonlinearity $f(u) = u^{p+1}$, the sign of this stability index is positive if $p < 4$ and is negative if $p > 4$. Moreover, in [56, 57] it was shown that the mechanism for this instability is as follows: Linearizing the traveling wave partial differential equation

$$u_t = u_{xxx} + f(u)_x - cu_x, \tag{3.4}$$

which is satisfied by the traveling solitary wave profile, about the solution u_c and taking the Laplace transform in time leads to a spectral problem of the form

$$\partial_x \mathcal{L}[u_c]v = \mu v$$

considered on the real Hilbert space $L^2(\mathbb{R})$, where $\mathcal{L}[u_c]$ is a second order self adjoint differential operator with asymptotically constant coefficients. The authors then make a detailed study of the Evans function $D(\mu)$, which is an analytic function such that if ψ is a solution of (3.4) satisfying $\psi(x) \sim e^{\omega x}$ as $x \to \infty$, then $\psi(x) \sim D(\mu)e^{\omega x}$ as $x \to -\infty$: in essence, $D(\mu)$ plays the role of a transmission coefficient familiar from quantum scattering theory. This approach motivated by the fact that for $\text{Re}(\mu) > 0$ the vanishing of $D(\mu)$ implies that μ is an L^2 eigenvalue of the linearized operator $\partial_x \mathcal{L}[u_c]$, and conversely. Pego and Weinstein were able to use this machinery to prove that the Evans function satisfies

$$\lim_{\mu \to +\infty} \text{sign}(D(\mu)) > 0$$

as well as the asymptotic relation

$$D(\mu) = C_1 \left(\frac{\partial}{\partial c} \int_{-\infty}^{\infty} u_c(x)^2 dx \right) \mu^2 + o(|\mu|^2)$$

in a neighborhood of $\mu = 0$, for some positive constant C_1. Thus, in the case when the solitary wave stability index is negative, it follows by the continuity of $D(\mu)$ for $\mu \in \mathbb{R}^+$ that $D(\mu) < 0$ for small positive μ and hence $D(\mu)$ must have a positive root, thus proving exponential instability of the underlying traveling solitary wave in this case.

In this paper, however, we are concerned with traveling wave solutions of (3.1) of the form (3.2), where this time we require the profile u_c be a periodic function of its argument. In contrast to the traveling solitary wave theory, relatively little is known concerning the stability of periodic traveling waves of nonlinear dispersive equations such as the gKdV. Existing results usually come two types: spectral stability with respect to localized or bounded perturbations, and orbital (nonlinear) stability with respect to periodic perturbations. Most spectral stability results seem to rely on a Floquet-Bloch decomposition of the linearized operator and a detailed analysis of the resulting family of spectral problems, or else perturbation techniques which analyze modulational instability (spectrum near the origin).

There is a fairly substantial amount of literature devoted to the stability of the cnoidal solutions of the KdV

$$u(x,t) = u_0 + 12k^2\kappa^2 \operatorname{cn}^2\left(\kappa\left(x - x_0 + \left(8k^2\kappa^2 - 4\kappa^2 + u_0\right)t\right), k\right),$$

where $k \in [0,1)$ and $\kappa,\ x_0$, and u_0 are real constants. Such cnoidal solutions of the KdV have been studied by McKean [50], and more recently in papers by Pava, Bona, and Scialom [2] and by Bottman and Deconinck [12]. The results in [50] uses the complete integrability of the KdV in his study of the periodic initial value problem in order to show nonlinear stability of the cnoidal solutions to perturbations of the same period. Also using the machinery of complete integrability, in [12] the spectrum of the linearized operator on the Hilbert space $L^2(\mathbb{R})$ is explicitly computed and shown to be confined to the imaginary axis. In particular, it follows that cnoidal solutions of the KdV are spectrally stable to perturbations of the same period, and more generally, perturbations with periods which are integer multiples of the period of the cnoidal wave.

Returning to the generalized KdV equation (3.1), spectral stability results have recently been obtained by Hărăguş and Kapitula [35] where the spectral stability of small amplitude periodic traveling wave solutions of (3.1) with $f(u) = u^{p+1}$ was studied. By using a Floquet-Bloch decomposition of the linearized spectral problem, the authors found that such solutions are spectrally stable if $p \in [1,2)$ and exhibit a modulational instability if $p > 2$. In particular, they found that such solutions are always spectrally stable to perturbations of the same period: in section 3.5, we will verify and extend this result through the use of the periodic Evans function. Recall from chapter 2 that we have already derived a stability index in a manner quite similar to the solitary wave theory outlined above such that the negativity of this index implies exponential instability of the periodic traveling wave with respect to perturbations of the same period. The relevant results of this analysis will be briefly reviewed in section 3.3.

It seems natural to consider the role this periodic instability index derived in [16] plays in the nonlinear stability of the periodic traveling wave. As mentioned above, the analogue of this index controls the nonlinear stability in the solitary wave context. Thus, one would like the periodic traveling wave to be nonlinearly stable whenever the aforementioned periodic stability index positive. While we are able to show this is true in certain cases, we find that two other quantities, which are essentially not present in the spectral stability theory[1] nor the solitary wave theory, play a role in the nonlinear stability. This is the content of our main theorem, which is stated below [2].

Theorem 11. *Let $u(x+c_0t)$ be a periodic traveling wave solution of* (3.1)*, corresponding to an $(a_0, E_0, c_0) \in \Omega$. Moreover, assume the principle minors of the matrix*

$$D^2_{E,a,c}K(a,E,c) = \begin{pmatrix} T_E & T_a & T_c \\ M_E & M_a & M_c \\ P_E & P_a & P_c \end{pmatrix}, \tag{3.5}$$

satisfy $d_1 = T_E > 0$, $d_2 = T_E M_a - M_E T_a < 0$ and $d_3 = \det(D^2_{E,a,c}K(a,E,c)) < 0$ at (a_0, E_0, c_0), where $K(a,E,c)$ is the classical action of the ODE governing the traveling waves. Then there exists $C_0 > 0$ and $\varepsilon > 0$ such that for all $\phi_0 \in X$ with $\|\phi_0\|_X < \varepsilon$, the solution $\phi(x,t)$ of (3.1) *with initial data $\phi(x,0) = u(x) + \phi(x)$ satisfies*

$$\inf_{\xi \in \mathbb{R}} \|\phi(\cdot,t) - u(x + c_0 t + \xi)\|_X \leq C_0 \|\phi_0\|_X$$

for all $t > 0$.

Remark 5. *Throughout this paper "orbital stability" will always mean orbital stability with respect to periodic perturbations, i.e. perturbations of the same period as the underlying wave.*

Moreover, recall the from Lemma 10 from chapter 2, the quantity T_E is positive on Ω for a large class of nonlinearities. Thus, in many cases one must only determine the signs of the quantities d_2 and d_3 in the above theorem in order to conclude orbital

[1]This is not quite correct. They are present, but their signs do not play into the spectral stability theory. See section 3.3 for more details.

[2]For the definition of the real Hilbert space X, see section 3.4.

stability.

Remark 6. *Notice that Theorem 11 provides no information in the case when either T_E is negative or when $\{T, M\}_{a,E}$ is negative. In the previous chapter, we showed the condition $d_3 > 0$ implies spectral instability of the periodic wave to perturbations of the same period: this work will be reviewed in the next section. Moreover, we saw the condition $T_E > 0$ was used to infer that any unstable T-periodic eigenvalues of the linearized operator are real: if $T_E < 0$, we seemingly no longer have this restriction. The quantity $\{T, M\}_{a,E}$ is not present in the spectral stability calculation, and hence may provide a mechanism for orbital instability in the presence of spectral stability.*

Recently, Deconinck and Kapitula [19] have proven results on the orbital stability of periodic solutions of the gKdV by relating the stability to the number of negative eigenvalues of two associated operators: the restriction of the operator $\mathcal{L}[u]$ to a class of mean free T-periodic functions, and another quantity which ends up being precisely d_3. Their methods are very different from those presented in this chapter, and by mixing the two methods together an extension of the work here may be possible to determine when stability is possible if either either T_E or $\{T, M\}_{a,E}$, or both, are negative.

The outline for this paper is as follows. In section 3.2, we will recall the recent results of chapter 2 concerning the spectral stability of periodic traveling wave solutions of (3.1) with respect to perturbations of the same period. The resulting instability index will play an important role throughout the rest of the paper. Section 3.3 is devoted to the proof of Theorem 11. Finally, two applications of our theory are described in sections 3.4 and 3.5 in the case of a power-law nonlinearity $f(u) = u^{p+1}$ for $p \geq 1$. In section 3.4, we study the orbital stability of periodic traveling wave solutions of (3.1) in neighborhoods of the homoclinic and equilibrium solutions. Section 3.5 is devoted to the application of our theory to the case of the KdV. In particular, it is shown that such solutions are orbitally stable if and only if they are spectrally stable to perturbations of the same period as the underlying wave.

3.2 Spectral Stability Analysis

In this section, we recall the relevant results of chapter 2 on the spectral stability of periodic-traveling wave solutions of the gKdV. Suppose that $u = u(\,\cdot\,; a, E, c) \in C^3(\mathbb{R};\mathbb{R})$ is a T-periodic solution of the traveling wave ODE

$$u_{xxx} + f(u)_x - cu_x = 0. \tag{3.6}$$

Linearizing (3.1) about this solution and taking the Laplace transform in time leads to the spectral problem

$$\partial_x \mathcal{L}[u]v = \mu v \tag{3.7}$$

considered on $L^2(\mathbb{R};\mathbb{R})$, where $\mathcal{L}[u] := -\partial_x^2 - f'(u) + c$ is a closed symmetric linear operator with periodic coefficients. In particular, since u is bounded it follows that $\mathcal{L}[u]$ is in fact a self-adjoint operator on $L^2(\mathbb{R})$ with densely defined domain $C^\infty(\mathbb{R})$. Notice that considering (3.7) on $L^2(\mathbb{R})$ corresponds to considering the spectral stability with respect to localized perturbations[3], and as a result the spectrum $\operatorname{spec}(\partial_x\mathcal{L}[u])$ is purely continuous. Moreover, the Hamiltonian nature of (3.7) implies that such a solution is spectrally stable if and only if $\operatorname{spec}(\partial_x\mathcal{L}[u]) \subset \mathbb{R}i$.

In order to study the spectrum of the operator $\partial_x\mathcal{L}[u]$ we note that (3.7) can be written as first order system of the form $\Phi_x = \mathbf{H}(x,\mu)\Phi$. We define the monodromy matrix to be the corresponding matrix solution with initial condition $\Phi(0) = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix It follows that $\mu \in \operatorname{spec}(\partial_x\mathcal{L}[u])$ if and only if there exists a non-trivial bounded function ψ such that $\partial_x\mathcal{L}[u]\psi = \mu\psi$ or, equivalently, if there exists a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that the periodic Evans function

$$D(\mu,\lambda) = \det\left(\mathbf{M}(\mu) - \lambda\mathbf{I}\right)$$

vanishes: see section 1.2 of the introduction. In particular we see that $D(\mu, 1)$ detects spectra which corresponds to perturbations which are T-periodic. perturbations of the

[3]One could also study the spectral stability with respect to uniformly bounded perturbations, but by standard results in Floquet theory the resulting theories are equivalent.

same period. To study such instabilities, we recall the following result.

Lemma 8. *The function $D(\mu,1)$ satisfies the following properties:*

1. *$D(\mu,1)$ is an odd function of μ.*
2. *The limit $\lim_{\mu\to\infty}\operatorname{sign}(D(\mu,1))$ exists and is negative.*
3. *The asymptotic relation*

$$D(\mu,1) = -\frac{1}{2}\{T,M,P\}_{a,E,c}\,\mu^3 + \mathcal{O}(|\mu|^4).$$

holds in a neighborhood of $\mu = 0$.

The main idea is that the integrability of the ODE (3.6) governing the traveling waves immediately allows direct computation of the tangent space of the manifold of traveling wave solutions at $\mu = 0$. As such, the calculation is undoubtedly related to the multi-symplectic formalism of Bridges: see [14]. It follows that if $\{T,M,P\}_{a,E,c}$ is negative then the number of positive roots of $D(\mu,1)$ is odd and hence one has exponential instability of the underlying periodic traveling wave. Moreover, we will show in Lemma 10 that $T_E > 0$ implies $\mathcal{L}[u]$ has exactly one negative eigenvalue. It follows from that the linearized operator $\partial_x\mathcal{L}[u]$ has at most one unstable eigenvalue with positive real part, counting multiplicities (see Theorem 3.1 of [57]) if $T_E > 0$. Since the spectrum of $\partial_x\mathcal{L}[u]$ is symmetric about the real and imaginary axis, it follows that all unstable periodic eigenvalues of the linearized operator must be real. This proves the following theorem, which is the main theorem of chapter 2 relating to spectral stability to periodic perturbations.

Theorem 12. *Let $u(x;a_0,E_0,c_0)$ be a periodic traveling wave solution of (2.1), and suppose that T_E is positive and $\{T,M,P\}_{a,E,c}$ is non-zero at (a_0,E_0,c_0). Then the solution is spectrally stable to perturbations of the same period if and only if $\{T,M,P\}_{a,E,c}$ is positive at (a_0,E_0,c_0).*

In the next section, we will show that if $T_E < 0$ the operator $\mathcal{L}[u]$ has two negative eigenvalues. Thus, even if $\{T,M,P\}_{a,E,c} > 0$ in this case, there is no way of proving from these methods whether the number of periodic eigenvalues of $\partial_x \mathcal{L}[u]$ with positive real part is equal to zero or two. Moreover, by drawing a direct analogy with the solitary wave theory one would suspect if $T_E > 0$, then such solutions of (3.1) are nonlinearly stable if and only if $\{T, M, P\}_{a,E,c}$ is positive[4], i.e. if and only if it is spectrally stable to perturbations of the same period. However, this seems not to be true in general: the sign of the Jacobian $\{T, M\}_{a,E}$ also plays a role in the orbital stability analysis, even though it does not seem to play into the periodic spectral stability theory at all[5]. This stands in stark contrast to the solitary wave theory.

3.3 Orbital Stability

In this section, we prove our main theorem on the orbital stability of periodic traveling wave solutions of (3.1). Our proof follows the general method of Bona, Souganidis and Strauss [11], and that of Grillakis, Shatah and Strauss [32, 33]: the goal is to show that a given periodic traveling wave solution of the gKdV is a constrained minimizer of a particular augmented energy functional. To this end, the majority of the work is dedicated to proving an appropriate coercive estimate on the augmented energy functional. Once this estimate is established, the orbital stability proof is straightforward.

Throughout this section, we assume we have a T-periodic traveling wave solution $u(x; a_0, E_0, c_0)$ of equation (3.1), i.e. we assume u satisfies

$$\frac{1}{2}u_x^2 + F(u) - \frac{c_0}{2}u^2 - a_0 u = E_0 \tag{3.8}$$

with $(a_0, E_0, c_0) \in \Omega$ and $T = T(a_0, E_0, c_0)$. Moreover, we assume the non-linearity f present in (3.1) is such that the Cauchy problem for (3.4) is globally well-posed in a real Hilbert space X of real valued T periodic functions defined on $\mathbb{R}$, which we equip

[4] In sections 5 and 6, we study cases where this is indeed the case.

[5] One could suspect that $\{T, M\}_{a,E}$ changes sign only when $\{T, M, P\}_{a,E,c}$ does, but this is shown not be the case in Corollary 5

with the standard $L^2([0,T])$ inner product

$$\langle g, h \rangle := \int_0^T g(x)h(x)dx$$

for all $g, h \in X$, and we identify the dual space X^* through the pairing

$$\langle g, h \rangle_* = \int_0^T g(x)h(x)dx$$

for all $g \in X^*$ and $h \in X$. In particular, notice that $L^2([0,T])$ is required to be a subspace of X. For example, if $f(u) = u^3/3$, corresponding to the modified Korteweg-de Vries equation, then the Cauchy problem for (3.4) is globally well-posed in the space

$$H^s_{\text{per}}([0,T];\mathbb{R}) = \{g \in H^s([0,T];\mathbb{R}) : g(x+T) = g(x) \text{ } a.e.\}$$

for all $s \geq \frac{1}{2}$, where we identify the dual space with $H^{-s}_{\text{per}}([0,T];\mathbb{R})$ through the above pairing (see [18] for proof). Moreover, due to the structure of the gKdV, we make the natural assumption that the evolution of (3.4) in the space X is invariant under a one parameter group of isometries G corresponding to spatial translation. Thus, G can be identified with the real line $\mathbb{R}$ acting on the space X through the unitary representation

$$(R_\xi g)(x) = g(x+\xi)$$

for all $g \in X$ and $\xi \in G$. Since the details of our proof works regardless of the form of the non-linearity f, we make the above additional assumptions on the nonlinearity and make no other references to the exact structure of the space X nor f.

In view of the symmetry group G, we now describe precisely what we mean by orbital stability. We define the G-orbit generated by u to be

$$\mathcal{O}_u := \{R_\xi u : \xi \in G\}.$$

Now, suppose we have initial data $\phi_0 \in X$ which is close to the orbit $\mathcal{O}_\varepsilon$. By orbital stability, we mean that if $\phi(\cdot,t) \in X$ is the unique solution with initial data ϕ_0, then $\phi(\cdot,t)$ is close to the orbit of u for all $t > 0$. More precisely, we introduce a semi-distance ρ defined on the space X by

$$\rho(g,h) = \inf_{\xi \in G} \|g - R_\xi h\|_X,$$

and use this to define an ε-neighborhood of the orbit $\mathcal{O}_u$ by

$$\mathcal{U}_\varepsilon := \{\phi \in X : \rho(u,\phi) < \varepsilon\}.$$

The main result of this section is the following reformulation of Theorem 11.

Proposition 7. *Let $u(x) = u(x; a_0, E_0, c_0)$ solve (3.8) and suppose the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive at (a_0, E_0, c_0). Then there exists positive constants C_0, ε_0 such that if $\phi_0 \in X$ satisfies $\rho(\phi_0, u) < \varepsilon$ for some $\varepsilon < \varepsilon_0$, then the solution $\phi(x,t)$ of (3.1) with initial data ϕ_0 satisfies $\rho(\phi(\cdot,t), u) \leq C_0\varepsilon$.*

Remark 7. *Notice that Theorem 12 implies a periodic solution $u(x; a_0, E_0, c_0)$ of (2.5) is an exponentially unstable solution of (3.1) if $\{T, M, P\}_{a,E,c}$ is negative at (a_0, E_0, c_0). Thus, the positivity of this Jacobian is a necessary condition for nonlinear stability.*

Remark 8. *Since the gKdV is a conservative system, one expects that the orbital stability in Theorem 7 is the strongest one could prove: in non-conservative systems (which might contain resistive terms such as friction) it would be natural to ask whether the solution is asymptotically stable, that is, if the perturbed solution actually converges back to the original solution. With out the presence of a resistive or forcing term, however, such a stability result for the gKdV seems unlikely, unless one studies asymptotic stability in possibly weighted L^p spaces as in the solitary wave theory (see [55]).*

Before we prove Proposition 7 we wish to shed some light on the hypotheses. Recall that the classical action $K(a, E, c)$ of the periodic traveling wave satisfies

$$D_{a,E,c}K(a,E,c) = \left(M(a,E,c), T(a,E,c), \frac{1}{2}P(a,E,c) \right).$$

As a result, we can write its Hessian as

$$D^2_{a,E,c}K(a,E,c) = \begin{pmatrix} M_a & M_E & M_c \\ T_a & T_E & T_c \\ P_a & P_E & P_c \end{pmatrix}.$$

Proposition 7 thus states that if $(a_0, E_0, c_0) \in \Omega$, the corresponding periodic traveling wave solutions of (3.1) is orbitally stable if the principle minor determinants of $D^2_{a,E,c}K(a,E,c)$ satisfy $d_1 = T_E > 0$, $d_2 = \{M,T\}_{a,E} < 0$, and $d_3 = \{M,T,P\}_{a,E,c} < 0$. It is clear that a necessary condition for this claim is that the Hessian $D^2_{a,E,c}K(a,E,c)$ is invertible with precisely one negative eigenvalue. However, this is clearly not sufficient.

We now proceed with the proof of Proposition 7. We define the following functionals on the space X, which correspond to the "energy", "mass" and "momentum" respectively:

$$\begin{aligned} \mathcal{E}(\phi) &:= \int_0^T \left(\frac{1}{2}\phi_x(x)^2 - F(\phi(x))\right) dx \\ \mathcal{M}(\phi) &:= \int_0^T \phi(x)\, dx \\ \mathcal{P}(\phi) &:= \frac{1}{2}\int_0^T \phi(x)^2 dx. \end{aligned}$$

These functionals represent conserved quantities of the flow generated by (3.1). In particular, if $\phi(x,t)$ is a solution of (3.1) of period T, then the quantities $\mathcal{E}(\phi(\cdot,t))$, $\mathcal{M}(\phi(\cdot,t))$, and $\mathcal{P}(\phi(\cdot,t))$ are constants in time. Also, notice that $\mathcal{E}(u) = H(a_0,E_0,c_0)$, $\mathcal{M}(u) = M(a_0,E_0,c_0)$, and $\mathcal{P}(u) = P(a_0,E_0,c_0)$ where H, M and P are defined in (2.9)-(2.11).

Remark 9. *Throughout the remainder of this paper, the symbols M and P will denote the functionals $\mathcal{M}$ and $\mathcal{P}$ restricted to the manifold of periodic traveling wave solutions of* (3.1) *with* $(a,E,c) \in \Omega$.

Remark 10. *Calculations in similar vein have been carried out recently in the special cases of cnoidal solutions of the KdV [2], as well as for traveling wave solutions* *of the modified KdV arising from* (3.1) *with* $f(u) = u^3$ *[1]. In each of these cases, however, it was assumed that* $a = 0$*, or equivalently that* $M(a, E, c) = 0$*. While this is always possible for the KdV (due to Galilean invariance), this is not possible for general nonlinearities without restricting your admissible class of traveling wave solutions, i.e. restricting* Ω*. As we are interested in deriving universal conditions for stability of traveling wave solutions of* (3.1)*, we are forced to work with all three functionals defined above.*

It is easily verified that $\mathcal{E}$, $\mathcal{M}$ and $\mathcal{P}$ are smooth functionals on X, whose first derivatives are smooth maps from X to X^* defined by

$$\mathcal{E}'(\phi) = -\phi_{xx} - f(\phi), \quad \mathcal{M}'(\phi) = 1, \quad \mathcal{P}'(\phi) = \phi.$$

If we now define an augmented energy functional on the space X by

$$\mathcal{E}_0(\phi) := \mathcal{E}(\phi) + c_0\mathcal{P}(\phi) + a_0\mathcal{M}(\phi) + E_0 T \tag{3.9}$$

it follows from (3.8) that $\mathcal{E}_0(u) = 0$ and $\mathcal{E}_0'(u) = 0$. Hence, u is a critical point of the functional $\mathcal{E}_0$.

Remark 11. *Notice that the added factor of* E_0T *on the right hand side of* (3.9) *is not technically needed for our calculation. However, we point out that (formally) if we consider variations in* $\mathcal{E}_0$ *in the period we obtain*

$$\begin{aligned}
\frac{\partial}{\partial T}\mathcal{E}_0(\phi)\big|_{\phi=u} &= \frac{1}{2}u_x^2(T) - F(u(T)) + au(T) + E + \left\langle \mathcal{E}_0'(u), \frac{\partial u}{\partial T} \right\rangle \\
&= \frac{1}{2}u_x^2(T) + \frac{1}{2}u_x^2(T) + \left\langle \mathcal{E}_0'(u), \frac{\partial u}{\partial T} \right\rangle \\
&= 0
\end{aligned}$$

since $u_x(T) = 0$ *and* $\mathcal{E}_0'(u) = 0$*. Hence* u *is also (formally) a critical point of the modified energy with respect to variations in the period. It would be very interesting to try to make this calculation rigorous and to see if it allows one to extend orbital stability* *results to include perturbations with period* close *to the period of the underlying periodic wave. Again, this is all very formal and we will make no attempt at such a theory here.*

To determine the nature of this critical point, we consider its second derivative $\mathcal{E}_0''$, which is a smooth map from X to $\mathcal{L}(X, X^*)$ defined by

$$\mathcal{E}_0''(\phi) = -\phi_{xx} - f'(\phi) + c_0.$$

This formula immediately follows by noticing the second derivatives of the mass, momentum, and energy functionals are smooth maps from X to $\mathcal{L}(X, X^*)$ defined by

$$\mathcal{E}''(\phi) = -\partial_x^2 - f'(\phi), \;\; \mathcal{M}''(\phi) = 0, \;\; \mathcal{P}''(\phi) = 1.$$

In particular, notice the second derivative of the augmented energy functional $\mathcal{E}_0$ at the critical point u is precisely linear operator $\mathcal{L}[u]$ arising from linearizing (3.4) with wave speed c_0 about u. It follows from the comments in the previous section that $\mathcal{E}_0''(u)$ is a self-adjoint linear operator on $L^2_{\text{per}}([0,T];\mathbb{R})$ with compact resolvent. In order to classify u as a critical point of $\mathcal{E}_0$, we must understand the nature of the spectrum of the second variation $\mathcal{L}[u]$: in particular, we need to know the number of negative eigenvalues. This is handled in the following lemma.

Lemma 9. *The spectrum of the operator $\mathcal{L}[u]$ considered on the space $L^2_{\text{per}}([0,T])$ satisfies the following trichotomy:*

(i) If $T_E > 0$, then $\mathcal{L}[u]$ has exactly one negative eigenvalue, a simple eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.

(ii) If $T_E = 0$, then $\mathcal{L}[u]$ has exactly one negative eigenvalue, a double eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.

(iii) If $T_E < 0$, then $\mathcal{L}[u]$ has exactly two negative eigenvalues, a simple eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.

Proof. This is essentially a consequence of the translation invariance of (3.1) and the Strum-Liouville oscillation theorem. Indeed, notice that for any $\xi \in G$ the function $R_\xi u$

is a stationary solution of (2.6) with wave speed c_0 and $a = a_0$. Differentiating this relation with respect to ξ and evaluating at $\xi = 0$ implies that $\mathcal{L}[u]u_x = 0$. Moreover, since u is radially increasing on $[0,T]$ from its local minimum there, u_x is periodic with the same period as u and hence $u_x \in L^2_{\text{per}}([0,T])$. This proves that zero is always a periodic eigenvalue of $\mathcal{L}[u]$ as claimed. To see there is exactly one negative eigenvalue, notice that since u is T-periodic with precisely one local critical point on $(0,T)$, its derivative u_x must have precisely one sign change over its period. By standard Strum-Liouville theory applied to the periodic problem (see Theorem 2.14 in [49]), it follows that zero must be the either the second or third[6] eigenvalue of $\mathcal{L}[u]$ considered on the space $L^2_{\text{per}}(\mathbb{R})$.

Next, we show that zero is a simple eigenvalue of $\mathcal{L}[u]$ on the space $L^2_{\text{per}}([0,T])$ if and only if $T_E \neq 0$. To this end, notice that the periodic traveling wave solutions of (2.6) are invariant under changes in the energy parameter E associated with the Hamiltonian ODE (2.5). As above, it follows that $\mathcal{L}[u]u_E = 0$. We must now determine whether the function u_E belongs to the space $L^2_{\text{per}}([0,T])$. Since it is clearly smooth, we must only check whether it is periodic with the same period as the underlying wave u. To this end, we use u_x and u_E as a basis to compute the monodromy matrix $\mathbf{m}(0)$ corresponding to the equation $\mathcal{L}[u]v = 0$. Notice that differentiating the relation $E = V(u_-; a, c)$ with respect to E and evaluating at (a_0, E_0, c_0) gives $\frac{\partial u_-}{\partial E} V'(u_-; a_0, c_0) = 1$, and hence $\frac{\partial u_-}{\partial E}$ is non-zero at (a_0, E_0, c_0). Defining $y_1(x) = \left(\frac{du_-}{dE}\right)^{-1} u_E$ and $y_2(x) = -\left(V'(u_-; a_0, c_0)\right)^{-1} u_x(x)$, it follows from direct calculation that

$$y_1(0) = 1, \quad y_2(0) = 0,$$
$$y_1'(0) = 0, \quad y_2'(0) = 1.$$

Thus, it follows by calculating $u_E(T)$ by the chain rule that we have

$$\mathbf{m}(0) = \begin{pmatrix} 1 & T_E \\ 0 & 1 \end{pmatrix},$$

[6] Clearly, we mean with respect to the natural ordering on $\mathbb{R}$.

where again we have used the fact that $V'(u_-;a,c)\frac{\partial u_-}{\partial E}=1$. Thus, it follows that zero is a simple eigenvalue of $\mathcal{L}[u]$ if and only if $T_E \neq 0$, and that the multiplicity will be two in the case $T_E = 0$.

Finally, to determine whether $\mu = 0$ is the second or third eigenvalue of $\mathcal{L}[u]$, we note that from the results of [16] we have

$$\operatorname{sign}(T_E) = \operatorname{sign}\left(\operatorname{tr}(\,\mathbf{m}_\mu(0))\right). \tag{3.10}$$

By the oscillation theorem Theorem 5 from chapter 2, it follows that $\mu = 0$ is the second periodic eigenvalue of $\mathcal{L}[u]$ if and only if $T_E \geq 0$, and the third periodic eigenvalue if and only if $T_E < 0$. This completes the proof. □

Remark 12. *If one considers $2T$-periodic orbits of* (3.6) *(say with a power-law nonlinearity) which are outside the separatrix, then the coefficients of the operator $\mathcal{L}[u]$ are T-periodic. In particular, u_x is an anti-periodic eigenvalue of the operator $\partial_x\mathcal{L}[u]$ in the space $X_2 := L^2_{\text{per}}([0,T])$ and hence there are at least two negative eigenvalues. Since our methods require the operator $\partial_x\mathcal{L}[u]$ acting on the space $L^2_{\text{per}}([0,2T])$ to have exactly one negative eigenvalue, this explains why we only consider periodic orbits which do not bound (non-trivial) homoclinic orbits in the definition of Ω.*

In the solitary wave case, the spectrum of the operator $\mathcal{L}[u]$ *always* satisfies (i) in the above trichotomy. Since E is not restricted to be zero in the periodic context, it is not surprising that such a non-trivial trichotomy might exist. The next lemma shows that for a large class of nonlinearities, the period is indeed an increasing function of E within the region Ω.

Lemma 10. *Let $(a_0,E_0,c_0)\in\Omega$ and $u = u(\,\cdot\,;a_0,E_0,c_0)$ denote the corresponding periodic solution of* (2.5) *with wave speed c_0 and period $T = T(a_0,E_0,c_0)$. If the nonlinearity f in* (3.1) *is such that $f'(u)$ is co-periodic with u, then $T_E > 0$ at (a_0,E_0,c_0).*

Proof. If $f'(u)$ is co-periodic with u, then the operator $\mathcal{L}[u]$ is a Hill operator with even potential with period $T = T(a_0,E_0,c_0)$. Thus, u_x is a periodic eigenvalue of

$\mathcal{L}[u]$ which satisfies Dirichlet boundary conditions. Moreover, since u_x changes signs once over $(0, T)$, it follows that zero is either the second or third periodic eigenvalue of $\mathcal{L}[u]$. Since the first periodic eigenvalue must be even, and hence must satisfy Neumann boundary conditions, it follows by Dirichlet-Neumann bracketing [58] that zero must be the second periodic eigenvalue of $\mathcal{L}[u]$. Using the notation of Lemma 9, it follows that $\mathrm{tr}(\mathbf{m}_\mu(0)) > 0$ and hence $T_E > 0$ by equation (3.10). $\square$

Remark 13. *In particular, it follows that if $f(u) = u^{p+1}$ for some $p \geq 1$ then the spectrum of $\mathcal{L}[u]$ will satisfy (i) in Lemma 9.*

Also, in the case of a power-law nonlinearity an alternative proof of Lemma 10 is provided by a theorem of Schaaf [60]: the details of this calculation are carried out in [16].

Throughout the rest of the paper, unless otherwise stated, we will assume that $T_E > 0$ at (a_0, E_0, c_0) and hence zero is a simple eigenvalue of the operator $\mathcal{L}[u]$ considered on the space $L^2_{\mathrm{per}}(\mathbb{R})$. In particular, we assume that the map $E \to T(E, a_0, c_0)$ does not have a critical point at E_0. It follows that we can define the spectral projections Π_-, Π_0 and Π_+ onto the negative, zero, and positive subspaces of the operator $\mathcal{L}[u]$ (respectively) via the Dunford calculus. Thus, any $\phi \in X$ can be decomposed as a linear combination of u_x, an element in the positive subspace of $\mathcal{L}[u]$, and χ, where χ is the unique positive eigenfunction of $\mathcal{L}[u]$ with $\|\chi\|_{L^2([0,T])} = 1$ which satisfies

$$\langle \mathcal{L}[u]\chi, \chi \rangle = -\lambda^2$$

for some $\lambda > 0$. From the above definition of χ it follows that χ is the eigenfunction corresponding to the unique negative eigenvalue $-\lambda^2$ of $\mathcal{L}[u]$.

From Lemma 9, we know that u is a degenerate saddle point of the functional $\mathcal{E}_0$ on X, with one unstable direction and one neutral direction. In order to get rid of the unstable direction, we note that the evolution of (3.1) does not occur on the entire space X, but on the co-dimension two submanifold defined by

$$\Sigma_0 := \{\phi \in X : \mathcal{M}(\phi) = M(a_0, E_0, c_0),\ \mathcal{P}(\phi) = P(a_0, E_0, c_0)\}.$$

It is clear that Σ_0 is indeed a smooth submanifold of X in a neighborhood of the group orbit $\mathcal{O}_u$. Moreover, the entire orbit $\mathcal{O}_u$ is contained in Σ_0. The main technical result needed for this section is that the functional $\mathcal{E}_0$ is coercive on Σ_0 with respect to the semi-distance ρ, which is the content of the following proposition.

Proposition 8. *If each of the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive, then there exists positive constants C_1, δ which depend on (a_0, E_0, c_0) such that*

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq C_1 \rho(\phi, u)^2$$

for all $\phi \in \Sigma_0$ such that $\rho(\phi, u) < \delta$.

The proof of Proposition 8 is broken down into three lemmas which analyze the quadratic form induced by the self adjoint operator $\mathcal{L}[u]$. To begin, we define a function ϕ_0 by

$$\phi_0(x) := \{u(x; a, E, c), T(a, E, c), M(a, E, c)\}_{a,E,c}\big|_{(a_0,E_0,c_0)}.$$

It follows from a straightforward calculation (see Proposition 4 from chapter 2) that $\phi_0 \in X$ and

$$\mathcal{L}[u]\phi_0 = -\{T, M\}_{E,c} - \{T, M\}_{a,E} u,$$

where the right hand side is evaluated at (a_0, E_0, c_0). This function plays a large role in the spectral stability theory for periodic traveling wave solutions[7] of (3.1) outlined in section 3.2. In particular, we have $\partial_x \mathcal{L}[u]\phi_0 = -\{T, M\}_{a,E} u_x$, and hence, assuming $\{T, M\}_{a,E} \neq 0$ at (a_0, E_0, c_0), ϕ_0 is in the generalized periodic null space of the linearized operator $\partial_x \mathcal{L}[u]$. Also, our assumption that $\{T, M, P\}_{a,E,c}$ is non-zero at (a_0, E_0, c_0)

[7] Actually, the function u_c plays a large role in our analysis via the periodic Evans function. However, since u_c is not in general T-periodic due to the dependence of the period on the wave speed, we work here with its periodic analogue ϕ_0.

implies that ϕ_0 does not belong to the set

$$\mathcal{T}_0 = \{\phi \in X : \langle u, \phi \rangle = \langle 1, \phi \rangle = 0\},$$

which is precisely the tangent space in X to Σ_0 at u. Indeed, while $\langle 1, \phi_0 \rangle = 0$ by construction, the inner product $\langle u, \phi_0 \rangle = \{T, M, P\}_{a,E,c}$ does not vanish by hypothesis. Using the spectral resolution of the operator $\mathcal{L}[u]$, we begin the proof of Proposition 8 with the following lemma.

Lemma 11. *Assume that the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive. Then*

$$\langle \mathcal{L}[u]\phi, \phi \rangle > 0.$$

for every $\phi \in \mathcal{T}_0$ which is orthogonal to the periodic null space of $\mathcal{L}[u]$.

Proof. The proof is essentially found in [11]. First, suppose that $T_E > 0$ and note that by Lemma 9 we can write

$$\phi_0 = \alpha\chi + \beta u_x + p$$
$$\phi = A\chi + \widetilde{p}$$

for some constants α, β, A, and functions p and $\widetilde{p}$ belonging to the positive subspace of $\mathcal{L}[u]$. By assumption the quantity

$$\langle \mathcal{L}[u]\phi_0, \phi_0 \rangle = -\{T, M\}_{a,E}\{T, M, P\}_{a,E,c} \tag{3.11}$$

is negative, and hence the above decomposition of ϕ_0 implies that

$$0 > \left\langle -\lambda^2\alpha\chi + \mathcal{L}[u]p, \alpha\chi + \beta u_x + p \right\rangle = -\lambda^2\alpha^2 + \langle \mathcal{L}[u]p, p \rangle, \tag{3.12}$$

which gives an upper bound on the positive number $\langle \mathcal{L}[u]p, p \rangle$. Similarly, the assumption that $\phi \in \mathcal{T}_0$ along with the above decomposition of ϕ implies

$$0 = \langle \mathcal{L}[u]\phi_0, \phi \rangle = -\lambda^2 A\alpha + \langle \mathcal{L}[u]p, \widetilde{p} \rangle \, . \tag{3.13}$$

Therefore, a simple application of Cauchy-Schwarz implies

$$\begin{aligned}
\langle \mathcal{L}[u]\phi, \phi \rangle &= -\lambda^2 A^2 + \langle \mathcal{L}[u]\widetilde{p}, \widetilde{p} \rangle \\
&\geq -\lambda^2 A^2 + \frac{\langle \mathcal{L}[u]\widetilde{p}, p \rangle^2}{\langle \mathcal{L}p, p \rangle} \\
&> -\lambda^2 A^2 + \frac{\left(\lambda^2 \alpha A\right)^2}{\lambda^2 \alpha^2} \\
&= 0
\end{aligned}$$

as claimed. □

Remark 14. *In the above proof, the positivity of the quantities $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ was never used: only the product was required to be positive. However, we show in Corollary 5 that the former is always positive if the latter is negative.*

Also, if $\{T, P\}_{E,c} \neq 0$, then one can repeat the above proof with the function ϕ_0 replaced by $\widetilde{\phi}_0 = \{u, T, P\}_{a,E,c}$. Then equation (3.11) would be replaced with $\left\langle \mathcal{L}[u]\widetilde{\phi}_0, \widetilde{\phi}_0 \right\rangle = \{T, P\}_{E,c}\{T, M, P\}_{a,E,c}$, which we would have to assume to be negative. It follows that $\operatorname{sign}\left(\{T, M\}_{a,E}\right) = -\operatorname{sign}\left(\{T, P\}_{E,c}\right)$ so long as $\{T, M, P\}_{a,E,c} \neq 0$. In particular, in the case of a power-law nonlinearity, $P_c < 0$ implies $\{T, M\}_{a,E} > 0$ if $\{T, M, P\}_{a,E,c} \neq 0$. It is unknown if $\{T, M, P\}_{a,E,c}$ is negative in this case.

Our strategy in proving Proposition 8 is to find a particular set of translates of a given $\phi \in \mathcal{U}_\varepsilon$ for which the inequality holds. To this end, we find that for each $\phi \in \mathcal{U}_\varepsilon$ with ε sufficiently small, there exists a set of translates of ϕ which are orthogonal to the periodic-null space of $\mathcal{L}[u]$. This is the content of the following lemma.

Lemma 12. *There exists an $\varepsilon > 0$ and a unique C^1 map $\alpha : \mathcal{U}_\varepsilon \to \mathbb{R}$ such that for all $\phi \in \mathcal{U}_\varepsilon$, the function $\phi\left(\cdot + \alpha(\phi)\right)$ is orthogonal to u_x.*

The proof is presented in [11], and is an easy result of the implicit function theorem.

Indeed, if we define the functional $\eta : X \times \mathbb{R} \to \mathbb{R}$ by

$$\eta(\phi, \alpha) = \int_0^T \phi(x+\alpha) u_x(x) dx,$$

then $\frac{\partial}{\partial \alpha}\eta(\phi,\alpha)\big|_{(\phi,\alpha)=(u,0)} = \int_0^T u_x^2 dx > 0$ and hence the lemma follows by the implicit function theorem and the fact that by translation invariance the function α can be uniquely extended to $\mathcal{U}_\varepsilon$ for $\varepsilon > 0$ sufficiently small.

We now complete the proof of Proposition 8 by proving the following lemma.

Lemma 13. *If each of the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive, there exists positive constants $\widetilde{C}$ and ε such that*

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \widetilde{C} \|\phi(\cdot + \alpha(\phi)) - u\|_X^2$$

for all $\phi \in \mathcal{U}_\varepsilon \cap \Sigma_0$.

Proof. Let $\varepsilon > 0$ be small enough such that Lemma 12 holds. Fix $\phi \in \mathcal{U}_\varepsilon \cap \Sigma_0$ and write

$$\phi(\cdot + \alpha(\phi)) = (1+\gamma)u + \left(\beta - \frac{\gamma \langle u \rangle}{T}\right) + y$$

where $y \in \mathcal{T}_0$. Moreover, define $v = \phi(\cdot + \alpha(\phi)) - u$ and note that by replacing u with $R_\xi u$ if necessary we can assume that $\|v\|_X < \varepsilon$. By Taylors theorem, we have

$$M(a_0, E_0, c_0) = \mathcal{M}(\phi) = M(a_0, E_0, c_0) + \langle 1, v \rangle + \mathcal{O}\left(\|v\|_X^2\right).$$

Since $\langle 1, v \rangle = \beta T$, it follows that $\beta = \mathcal{O}\left(\|v\|_X^2\right)$. Similarly, we have

$$P(a_0, E_0, c_0) = P(a_0, E_0, c_0) + \langle u, v \rangle + \mathcal{O}\left(\|v\|_X^2\right).$$

Moreover, defining $\langle g \rangle = \int_0^T g(x) dx$ for $g \in L^1_{\text{per}}([0,T];\mathbb{R})$, a direct calculation yields

$$\langle u, v \rangle = \gamma \left(\|u\|_{L^2([0,T])}^2 - \frac{\langle u \rangle^2}{T} \right) + \beta \langle u \rangle.$$

Since $\langle u\rangle^2 < T\|u\|^2_{L^2([0,T])}$ by Jensen's inequality, it follows that $\gamma = \mathcal{O}\left(\|v\|_X^2\right)$.

Now, by Taylor's theorem and the translation invariance of $\mathcal{E}_0$, we have

$$\begin{aligned}
\mathcal{E}_0(\phi) &= \mathcal{E}_0\left(\phi(\cdot + \alpha(\phi))\right)\\
&= \mathcal{E}_0(u) + \left\langle \mathcal{E}_0'(u), v\right\rangle + \frac{1}{2}\left\langle \mathcal{E}_0''(u)v, v\right\rangle + o\left(\|v\|_X^2\right)\\
&= \mathcal{E}_0(u) + \frac{1}{2}\left\langle \mathcal{L}[u]v, v\right\rangle + o\left(\|v\|_X^2\right).
\end{aligned}$$

Hence, by the previous estimates on γ and β, it follows that

$$\begin{aligned}
\mathcal{E}_0(\phi) - \mathcal{E}_0(u) &= \frac{1}{2}\left\langle \mathcal{L}[u]v, v\right\rangle + o\left(\|v\|_X^2\right)\\
&= \frac{1}{2}\left\langle \mathcal{L}y, y\right\rangle + o\left(\|v\|_X^2\right).
\end{aligned}$$

Since $y \in \mathcal{T}_0$ and $\langle y, u_x\rangle = 0$ by Lemma 12, it follows from Lemma 11 that

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \frac{C_1}{2}\|y\|^2 + o\left(\|v\|_X^2\right).$$

Finally, the estimates

$$\begin{aligned}
\|y\|_X &= \left\| v - \gamma u - \beta - \frac{\gamma\langle u\rangle}{T}\right\|_X\\
&\geq \left| \|v\|_X - \left\| \gamma u - \beta - \frac{\gamma\langle u\rangle}{T}\right\|_X \right|\\
&\geq \|v\|_X - \mathcal{O}\left(\|v\|_X^2\right).
\end{aligned}$$

prove that $\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \frac{C_1}{4}\|v\|_X^2$ for $\|v\|_X$ sufficiently small. □

Proposition 8 now clearly follows by Lemma 13 and the definition of the semi-distance ρ. It is now straightforward to complete the proof of Proposition 7.

Proof of Proposition 7: We now deviate from the methods of [11], [32] and [33], and rather follow the direct method of [27]. Let $\delta > 0$ be such that Proposition 8 holds, and let $\varepsilon \in (0, \delta)$. Assume $\phi_0 \in X$ satisfies $\rho(\phi_0, u) \leq \varepsilon$ for some small $\varepsilon > 0$. By replacing ϕ_0 with $R_\xi\phi_0$ if needed, we may assume that $\|\phi_0 - u\|_X \leq \varepsilon$. Since u is a critical point of the functional $\mathcal{E}_0$, it is clear that we have

$$\mathcal{E}_0(\phi_0) - \mathcal{E}_0(u) \leq C_1\varepsilon^2$$

for some positive constant C_1. Now, notice that if $\phi_0 \in \Sigma_0$, then the unique solution $\phi(\cdot, t)$ of (3.1) with initial data ϕ_0 satisfies $\phi(\cdot, t) \in \Sigma_0$ for all $t > 0$. Thus, Proposition 8 implies there exists a $C_2 > 0$ such that $\rho(\phi(\cdot, t), u) \leq C_2\varepsilon$ for all $t > 0$. Thus $\phi(\cdot, t) \in \mathcal{U}_\varepsilon$ for all $t > 0$, which proves Proposition 7 in this case.

If $\phi_0 \notin \Sigma_0$, then we claim we can vary the constants (a, E, c) slightly in order to effectively reduce this case to the previous one. Indeed, notice that since we have assumed $\{T, M, P\}_{a,E,c} \neq 0$ at (a_0, E_0, c_0), it follows that the map

$$(a, E, c) \mapsto (T(u(\,\cdot\,; a, E, c)), M(u(\,\cdot\,; a, E, c)), P(u(\,\cdot\,; a, E, c)))$$

is a diffeomorphism from a neighborhood of (a_0, E_0, c_0) onto a neighborhood of $(T, M(a_0, E_0, c_0), P(a_0, E_0, c_0))$. In particular, we can find constants a, E, and c with $|a| + |E| + |c| = \mathcal{O}(\varepsilon)$ such that the function

$$\widetilde{u} = \widetilde{u}(\,\cdot\,; a_0 + a, E_0 + E, c_0 + c)$$

solves (3.1), belongs to the space X, and satisfies

$$M(a_0 + a, E_0 + E, c_0 + a) = \mathcal{M}(\phi_0)$$
$$P(a_0 + a, E_0 + E, c_0 + c) = \mathcal{P}(\phi_0).$$

Defining a new augmented energy functional on X by

$$\widetilde{\mathcal{E}}(\phi) = \mathcal{E}_0(\phi) + c\mathcal{P}(\phi) + a\mathcal{M}(\phi) + ET,$$

it follows as before that

$$\widetilde{\mathcal{E}}(\phi(\cdot,t)) - \widetilde{\mathcal{E}}(\widetilde{u}) \geq C_3\rho(\phi(\cdot,t),\widetilde{u})^2$$

for some $C_3 > 0$ as long as $\rho(\phi(\cdot,t),\widetilde{u})$ is sufficiently small. Since $\widetilde{u}$ is a critical point of the functional $\widetilde{\mathcal{E}}$ we have

$$C_3\rho(\phi(\cdot,t),\widetilde{u})^2 \leq \widetilde{\mathcal{E}}(\phi_0) - \widetilde{\mathcal{E}}(\widetilde{u}) \leq C_4\|\phi_0 - \widetilde{u}\|_X^2$$

for some $C_4 > 0$. Moreover, it follows by the triangle inequality that

$$\|\phi_0 - \widetilde{u}\|_X \leq \|\phi_0 - u\|_X + \|u - \widetilde{u}\|_X \leq C_5\varepsilon$$

for some $C_5 > 0$ and hence there is a $C_6 > 0$ such that

$$\rho(\phi(\cdot,t),u) \leq \rho(\phi(\cdot,t),\widetilde{u}) + \|\widetilde{u} - u\|_X \leq C_6\varepsilon$$

for all $t > 0$. The proof of Proposition 7, and hence Theorem 11, is now complete. □

We would like to point out an interesting artifact of the above proof. Notice that the step at which the sign of the quantities $\{T,M,P\}_{a,E,c}$ and $\{T,M\}_{a,E}$ came into play was in the proof of Lemma 11, from which we have the following corollary.

Corollary 5. *On the set Ω, the quantity $\{T,M\}_{a,E}$ is positive whenever $\{T,M,P\}_{a,E,c}$ is negative and T_E is positive.*

Proof. This is an easy consequence of Theorem 12 and equation (3.11). Indeed, if $\{T,M\}_{a,E}$ and $\{T,M,P\}_{a,E,c}$ were both negative for some $(a_0,E_0,c_0) \in \Omega$, then by the proof of Lemma 11 we could conclude that $\langle \mathcal{L}[u(\,\cdot\,;a_0,E_0,c_0)]\phi,\phi\rangle > 0$ for all $\phi \in \mathcal{T}_0$ which are orthogonal to u_x. Since this is the only time in which the signs of these quantities arise, it follows that Proposition 7 would hold thus contradicting Theorem 12. □

It follows that we have a geometric theory of the orbital stability of periodic traveling wave solutions of (3.1) to perturbations of the same period as the underlying periodic wave. In the next two sections, we consider specific examples and limiting cases where the signs of these quantities can be calculated. First, we consider periodic traveling wave solutions sufficiently close to an equilibrium solution (a local minimum of the effective potential) or to the bounding homoclinic orbit (the separatrix solution). By considering power-law nonlinearities in each of these cases, we give necessary and sufficient[8] conditions for the orbital stability of such solutions. Secondly, we consider the KdV and prove that all periodic traveling wave solutions are orbitally stable to perturbations of the same periodic as the underlying periodic wave.

3.4 Analysis Near Homoclinic and Equilibrium Solutions

In this section, we use the theory from section 3.3 in order to prove general results about the stability of periodic traveling wave solutions of (3.1) in two distinguished limits: as one approaches the solitary wave, i.e. $(a, E, c) \in \Omega$ and consider the limit $T(a, E, c) \to \infty$ for fixed a, c, as well as in a neighborhood of the equilibrium solution, i.e. near a non-degenerate local minimum of the effective potential $V(u; a, c)$. Throughout this section, we will consider only power-law nonlinearities.

We begin with considering stability near the solitary wave. Our main result in this limit is that the quantities T_E and $\{T, M\}_{a,E}$ are positive for $(a_0, E_0, c_0) \in \Omega$ with sufficiently large period. Hence, the orbital stability of such a solution in this limit is determined completely by the periodic spectral stability index $\{T, M, P\}_{a,E,c}$, which in turn is controlled by the sign of the solitary wave stability index (3.3). This is the content of the following theorem.

Theorem 13. *In the case of a power-law nonlinearity, i.e.* $f(u) = u^{p+1}$ *with* $p \geq 1$*, a periodic traveling wave solution of* (3.1) *of sufficiently large period and* $(a, E, c) \in \Omega$ *is orbitally stable if* $p < 4$ *and exponentially unstable to perturbations of the same period as the underlying wave if* $p > 4$.

[8] Except in the exceptional case of being near the homoclinic orbit for $p = 4$.

Proof. By the work in chapter 2, the quantity M_a is negative for such $(a, E, c) \in \Omega$. Moreover, since we are working with a power-law nonlinearity, the periodic traveling wave solutions satisfy the scaling relation

$$u(x; a, E, c) = c^{1/p} u \left(c^{1/2} x; \frac{a}{c^{1+1/p}}, \frac{E}{c^{1+2/p}}, 1 \right)$$

from which we get the asymptotic relation

$$\{T, M, P\}_{a,E,c} \sim -T_E M_a \left(\frac{2}{pc} - \frac{1}{2c} \right) P$$

as $\Omega \ni (a, E, c) \to (0, 0, c)$ for a fixed wave speed. Since $\{T, M\}_{a,E} = M_E^2 - T_E M_a$, it follows from Lemma 10 and Theorem 12 that the solutions $u(x; a, E, c)$ with $(a, E, c) \in \Omega$ of sufficiently large period are orbitally stable if $p < 4$ and exponentially unstable to periodic perturbations if $p > 4$. □

Next, we consider periodic traveling wave solutions near the equilibrium solution. We will use the methods of this paper to prove that such solutions are orbitally stable to periodic perturbations, provided that a is sufficiently small. To begin, we fix a wave speed $c > 0$ and consider (3.1) with a power-law nonlinearity $f(u) = u^{p+1}$ with $p \geq 1$. Recall that $T_E > 0$ by Lemma 10, and hence it suffices to prove that $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are both positive near the equilibrium solution. By continuity, it is enough to evaluate both these indices at the equilibrium and to show they are both positive there. This is the content of the following lemma.

Lemma 14. *Consider* (3.1) *with a power-law nonlinearity* $f(u) = u^{p+1}$ *for* $p \geq 1$*. Then the quantity* M_a *is negative for all* $(a_0, E_0, c_0) \in \Omega$ *such that* $|a|$ *is sufficiently small and the corresponding solution* $u(\,\cdot\,; a_0, E_0, c_0)$ *is sufficiently close to the equilibrium solution*[9]*.*

Proof. First, denote the equilibrium solution as $u_{a,c}$ and let $E^*(a, c) = V(u_{a,c}; a, c)$. It

[9] In the case of the KdV ($p = 1$), M_a is negative in a deleted neighborhood of the equilibrium solution.

follows that

$$\lim_{E \searrow E^*} T(a,E,c) = \frac{2\pi}{\sqrt{cp}}$$

and that the equilibrium solution admits the expansion

$$u_{a,c} = c^{1/p}\left(1 + \frac{a}{p}\right) + \mathcal{O}(a^2).$$

Now, solutions near the equilibrium $u_{a,c}$ can each be written as $u(x;a,E,c) = P_{a,E,c}(k_{a,E,c}x)$, where $k_{a,E,c}T(a,E,c) = 2\pi$ and $P_{a,E,c}$ is a 2π periodic solution of the ordinary differential equation

$$k_{a,E,c}^2 v'' + v^{p+1} - c^{1+1/p}a = 0$$

such that

$$P_{a,E^*,c} = u_{a,c}, \quad k_{a,E^*,c}^2 = (p+1)u_{a,c}^p - c.$$

Straightforward computations give the expansions

$$\begin{aligned} P_{a,E,c}(z) &= u_{a,c} + \mathcal{O}(\sqrt{E-E^*}\,(1+a^2)), \\ k_{a,E,c}^2 &= cp + (p+1)ca + \mathcal{O}((E-E^*) + a^2). \end{aligned}$$

Thus, the mass $M(a,E,c)$ can be expanded as

$$\begin{aligned} M(a,E,c) &= \int_0^{2\pi/k_{a,E,c}} P_{a,E,c}(k_{a,E,c}z)dz \\ &= \frac{1}{k_{a,E,c}}\int_0^{2\pi} \widetilde{P}_{a,E,c}(z)dz \\ &= \frac{2\pi}{\sqrt{cp}}\left(1 + \frac{(1-p)a}{2p}\right) + \mathcal{O}(\sqrt{E-E^*} + a^2) \end{aligned}$$

It follows that

$$\frac{\partial}{\partial a} M(a,E,c)\big|_{(0,E^*,c)} = \frac{\pi(1-p)}{p\sqrt{cp}}$$

which is negative for $p > 1$.

The case $p = 1$, which corresponds to the KdV equation, will be discussed in the next section. There we will show that although M_a vanishes *at* the equilibrium solution, it is indeed negative for *nearby* periodic traveling waves with the same wave speed c, i.e.

$$\frac{\partial^2}{\partial E \partial a} M(a, E, c)\big|_{(0,E^*,c)} < 0.$$

$\square$

Next, we must determine the sign of the periodic spectral stability index $\{T, M, P\}_{a,E,c}$. Although it follows from Theorem 4.4 in [35] that this index must be positive [10], we present an independent proof based on the periodic Evans function methods of chapter 2. To this end, we recall out that the Hamiltonian structure of the linearized operator $\partial_x \mathcal{L}[u]$ we have the identity

$$\{T, M, P\}_{a,E,c} = -\frac{2}{3}\operatorname{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right),$$

where $\mathbf{M}(\mu)$ is the corresponding monodromy operator (see Theorem 8 of chapter 2 for details). Thus, it is sufficient to show that $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ is negative near the equilibrium solution. This is the content of the next lemma.

Lemma 15. *Consider* (3.1)*, and suppose u_0 is a non-degenerate local minima of the corresponding effective potential $V(u; a, c)$. Then* $\operatorname{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) < 0$ *at u_0.*

Proof. The key point is that if we write the spectral problem $\partial_x \mathcal{L}[u]v = \mu v$ as a first order system of the form $\Phi_x = \mathbf{H}(x, \mu)\Phi$ by the usual procedure, then the matrix $\mathbf{H}(x, \mu)$ reduces to the constant matrix

$$\mathbf{H}(\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & -V''(u_0; a, c) & 0 \end{pmatrix}$$

at the equilibrium solution u_0. Thus, the corresponding monodromy operator at u_0 can be expressed as $\mathbf{M}(\mu) = \exp\left(\mathbf{H}(\mu)T_0\right)$, where $T_0 = \frac{2\pi}{\sqrt{p}}$. Thus, in order to calculate the

[10] They prove the spectrum of the linearized operator $\partial_x \mathcal{L}[u]$ intersects the real axis only at $\mu = 0$ for such small amplitude solutions.

function $\mathrm{tr}(\mathbf{M}(\mu))$, it is sufficient to calculate the eigenvalues of the constant matrix $\mathbf{H}(\mu)$.

Now, the periodic Evans function corresponding to the constant coefficient system induced by $\mathbf{H}(\mu)$ can be written as

$$D_0(\mu,\lambda) = \det\left(\mathbf{H}(\mu) - \lambda\mathbf{I}\right) = -\lambda^3 - V''(u_0;a,c)\lambda - \mu.$$

In particular, notice that $\frac{\partial}{\partial\lambda}D_0(\mu,\lambda) = -\lambda^2 - V''(u_0;a,c)$. Since $V''(u_0;a,c) > 0$ it follows that the function $D_0(\mu,\,\cdot\,)$ will have precisely one real root for each $\mu\in\mathbb{R}$. This distinguished root is given by the formula

$$\gamma_1(\mu) = \underbrace{\frac{\left(\frac{2}{3}\right)^{1/3}V''(u_0)}{\left(9\mu+\sqrt{3}\sqrt{27\mu^2+4V''(u_0)^3}\right)^{1/3}}}_{=:\alpha(\mu)} + \underbrace{\left(-\frac{\left(9\mu+\sqrt{3}\sqrt{27\mu^2+4V''(u_0)^3}\right)^{1/3}}{2^{1/3}3^{2/3}}\right)}_{=:\beta(\mu)}.$$

Defining $\omega = \exp(2\pi i/3)$ to be the principle third root of unity, the two complex eigenvalues of $\mathbf{H}(\mu)$ can be written as $\gamma_2(\mu) = \omega\alpha(\mu) + \overline{\omega}\beta(\mu)$ and $\gamma_3(\mu) = \overline{\omega}\alpha(\mu) + \omega\beta(\mu)$, and hence

$$\mathrm{tr}\left(\mathbf{M}(\mu)\right) = \exp\left(\gamma_1(\mu)T_0\right) + \exp\left(\gamma_2(\mu)T_0\right) + \exp\left(\gamma_3(\mu)T_0\right).$$

Now, a straightforward, yet tedious, calculation using the facts that $1+\omega+\overline{\omega}=0$ and $\omega^2=\overline{\omega}$ implies that

$$\mathrm{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) = 9T_0^2\left(\alpha''(0)\beta'(0)+\alpha'(0)\beta''(0)\right) + 3T_0^3\left(\alpha'(0)^3+\beta'(0)^3\right).$$

Moreover, from the definitions of α and β we have

$$\alpha'(0) = -\frac{1}{2V''(u_0)} = \beta'(0), \quad \text{and} \quad \alpha''(0) = \frac{\sqrt{3}}{4V''(u_0)^{5/2}} = -\beta''(0).$$

Therefore, we have the equality

$$\operatorname{tr}\left(\left(\mathbf{M}_{\mu\mu\mu}(0)\right) = -\frac{6\pi^3}{V''(u_0)^{9/2}},\right.$$

which is clearly negative. □

Remark 15. *Notice that by similar methods, one could obtain a similar expression for* $\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))$ *and show the modulational instability index from [16] always vanishes at the equilibrium solutions of the traveling wave ordinary differential equation 3.6.*

Therefore, it follows that in the case of a power-nonlinearity and solutions sufficiently close to a non-degenerate minima of the effective potential, each of the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are all positive. Therefore, Theorem 11 immediately yields the following result.

Theorem 14. *Consider equation* (3.1) *with a power-law nonlinearity* $f(u) = u^{p+1}$ *for* $p \geq 1$. *Then the periodic traveling wave solutions* $u(x; a, E, c)$ *with* $(a, E, c) \in \Omega$ *and* $a^2 + (E - E^*)^2$ *sufficiently small are orbitally stable in the sense of Theorem 11.*

3.5 The Korteweg-de Vries Equation

In this section, we will apply the general theory from section 3.3 in order to prove that periodic traveling wave solutions of (3.1) with $f(u) = u^2$ and $c > 0$ are orbitally stable with respect to periodic perturbations if and only if they are spectrally stable to such perturbations. To this end, we notice that solutions of the KdV equation

$$u_t = u_{xxx} + (u^2)_x - cu_x \tag{3.14}$$

are invariant under the scaling transformation

$$u(x,t) \mapsto c\, u(\sqrt{c}x, c^{3/2}t),$$

and hence, by (2.7), the stationary periodic traveling wave solutions satisfy the identity

$$u(x;a,E,c) = c\,u\left(c^{1/2}x; \frac{a}{c}, \frac{E}{c^3}, 1\right).$$

Thus, by scaling we may always assume that $c = 1$ in (3.14). Moreover, we may always assume that $a = 0$ due to the Galilean invariance of the KdV. Therefore, it is sufficient to determine the stability of periodic traveling wave solutions of (3.14) of the form $u(x;0,E,1)$. In order to do so, we need the following easily proved lemma.

Lemma 16. *Let μ be a (Borel) probability measure on some interval $I \subset \mathbb{R}$, and let $f, g : I \to \mathbb{R}$ be bounded and measurable functions. Then*

$$\int_I f(x)g(x)d\mu - \left(\int_I f(x)d\mu\right)\left(\int_I g(x)d\mu\right) = \frac{1}{2}\int_{I\times I}\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)d\mu_x d\mu_y. \tag{3.15}$$

In particular, if both f and g are strictly increasing or strictly decreasing, and if the support of μ is not reduced to a single point, then

$$\int_I f(x)g(x)d\mu > \left(\int_I f(x)d\mu\right)\left(\int_I g(x)d\mu\right).$$

The proof of this lemma is a trivial result of Fubini's theorem, as one can see by writing the left hand side of (3.15) as an iterated integral and simplifying the resulting expression. Now, recall from Lemma 10 that $T_E > 0$ for periodic traveling wave solutions of (3.14). To conclude orbital stability, we must identify the signs of the Jacobians $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$. The main technical result we need for this section is the following lemma, which uses Lemma 16 to guarantee the sign of the quantity in (3.11) is completely determined by the Jacobian $\{T, M, P\}_{a,E,c}$.

Lemma 17. *If $f(u) = u^2$ in (3.1), then $\{T, M\}_{a,E} > 0$ for all $(a_0, E_0, c_0) \in \Omega$ which do not correspond to the unique equilibrium solution.*

Proof. First, notice that since $D_{a,E,c}K(a,E,c) = (M,T,P)$, it follows that $\{T, M\}_{a,E} = M_E^2 - T_E M_a$, and hence by Lemma 10 it is enough to prove that $M_a < 0$. Moreover, by our above remarks it is enough to consider the case $c = 1$ and $a = 0$. It follows for f given as above, we can find functions u_1, u_2, u_3 which depend smoothly on (a, E, c) within the domain Ω such that

$$3\,(E - V(u;0,1)) = (u - u_1)(u - u_2)(u_3 - u).$$

Notice that the assumption that we are not at the equilibrium solution implies that the roots u_i are distinct, and moreover that $V'(u_i;0,1) \neq 0$. Since $E - V(u_i;0,1) = 0$ on Ω, it follows that

$$\frac{\partial u_i}{\partial a} = \frac{u_i}{V'(u_i;0,1)}.$$

Since $u_1 < 0$ and $u_2,\ u_3 > 0$, we have

$$\frac{\partial u_1}{\partial_a} < 0, \quad \frac{\partial u_2}{\partial_a} < 0, \quad \text{and} \quad \frac{\partial u_3}{\partial_a} > 0. \tag{3.16}$$

Moreover, since $u_1 + u_2 + u_3 = \frac{3c}{2}$ we have the relation

$$\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a} = -\frac{\partial u_1}{\partial a} > 0 \tag{3.17}$$

on Ω.

Now, by making the change of variables $u \mapsto s(\theta) = u_2 \cos^2(\theta) + u_2 \sin^2(\theta)$, we have $du = 2\sqrt{(u-u_2)(u_3-u)}d\theta$ and hence we may express the mass of $u(x;a,E,c)$ as

$$\begin{aligned} M(a,E,c) &= \sqrt{2}\int_{u_2}^{u_3} \frac{u\ du}{\sqrt{E - V(u;a,c)}} \\ &= 2\sqrt{6}\int_0^{\pi/2} \frac{s(\theta)\ d\theta}{\sqrt{s(\theta) - u_1}}. \end{aligned} \tag{3.18}$$

Notice we suppress the dependence of $s(\theta)$ on the parameters (a, E, c). Defining $\sigma(\theta) = \sqrt{s(\theta) - u_1}$, a straightforward computation using (3.17) shows that the derivative of the integrand in (3.18) with respect to the parameter a can be expressed as

$$\begin{aligned}
\frac{\partial}{\partial a}\left(\frac{s(\theta)}{\sqrt{s(\theta)-u_1}}\right) &= \frac{\partial u_2}{\partial a}\left(\frac{\cos^2(\theta)}{2\sigma(\theta)}\right) + \frac{\partial u_3}{\partial a}\left(\frac{\sin^2(\theta)}{2\sigma(\theta)}\right) \\
&\quad - \left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a}\right)\left(\frac{s(\theta)}{2\sigma(\theta)^3}\right) - \frac{u_1}{2\sigma(\theta)^3}\left(\frac{\partial u_2}{\partial a}\cos^2(\theta) + \frac{\partial u_3}{\partial a}\sin^2(\theta)\right) \\
&= \frac{\partial u_2}{\partial a}\left(\frac{\cos^2(\theta)-\sin^2(\theta)}{2\sigma(\theta)}\right) + \left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a}\right)\left(\frac{\sin^2(\theta)}{2\sigma(\theta)^3}\right) \\
&\quad - \left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a}\right)\frac{s(\theta)}{2\sigma(\theta)^3} - u_1\frac{\partial u_2}{\partial a}\left(\frac{\cos^2(\theta)-\sin^2(\theta)}{2\sigma(\theta)^3}\right) \\
&\quad - u_1\left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a}\right)\left(\frac{\sin^2(\theta)}{2\sigma(\theta)^3}\right).
\end{aligned}$$

With a little more algebra, this may be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial a}\left(\frac{s(\theta)}{\sqrt{s(\theta)-u_1}}\right) &= \frac{\partial u_2}{\partial a}\left(\frac{\cos^2(\theta)-\sin^2(\theta)}{2\sigma(\theta)}\right) - u_1\frac{\partial u_2}{\partial a}\left(\frac{\cos^2(\theta)-\sin^2(\theta)}{2\sigma(\theta)^3}\right) \\
&\quad - \left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a}\right)\left(\frac{s(\theta)\cos^2(\theta) - u_1\left(\cos^2(\theta)-\sin^2(\theta)\right)}{2\sigma(\theta)^3}\right).
\end{aligned}$$

Since the functions $\cos^2(\theta)-\sin^2(\theta)$ and $\sigma(\theta)^{-1}$ are strictly decreasing on the interval $(0,\pi/2)$, it follows from Lemma 16 that

$$\int_0^{\pi/2}\frac{\cos^2(\theta)-\sin^2(\theta)}{\sigma^m(\theta)}\,d\theta > 0$$

for any $m>0$. Evaluating the above expression at $(a,E,c)=(0,E,1)\in\Omega$ implies that $s(\theta)>0$ for all $\theta\in(0,\pi/2)$, and hence (3.16) and (3.17) imply that

$$\int_0^{\pi/2}\frac{\partial}{\partial a}\left(\frac{s(\theta)}{\sqrt{s(\theta)-u_1}}\right)d\theta < 0$$

at $(0,E,1)$, from which the lemma follows. □

Therefore, our main theorem on the stability of periodic traveling wave solutions of the Korteweg-de Vries equation follows by Theorem 11, Theorem 12, and Lemma 17.

Theorem 15. *Let* $(a_0,E_0,c_0)\in\Omega$ *and assume that* $\{T,M,P\}_{a,E,c}\neq 0$ *at* (a_0,E_0,c_0). *Then the corresponding periodic solution of* (3.6) *is an orbitally stable solution of* (3.1) *if and only if the solution is spectrally stable to perturbations of the same period, i.e. if and only if* $\{T, M, P\}_{a,E,c} > 0$ *at* (a_0, E_0, c_0).

An interesting corollary of Theorem 15 applies to cnoidal wave solutions of the KdV. It was suggested by Benjamin [8] that such solutions should be stable to perturbations of the same period. This conjecture has indeed been proved both by using the complete integrability of the KdV [50] [12] and by variational methods as in the present paper [2]. In particular, in [12] it was shown that the cnoidal solutions of the KdV are spectrally stable to localized perturbations, and are linearly stable to perturbations with the same period as the underlying wave. Clearly then such solutions are spectrally stable with respect to periodic perturbations. Paired with Theorem 15, this provides another verification Benjamin's conjecture in the case where the cnoidal wave has positive wave speed.

Corollary 6. *The cnoidal wave solutions of* (3.1) *with* $f(u) = u^2$ *of the form*

$$u(x,t) = u_0 + 12k^2\kappa^2 \operatorname{cn}^2\left(\kappa\left(x - x_0 - \left(8k^2\kappa^2 - 4\kappa^2 + u_0\right)t\right)\right),$$

with $k \in [0,1)$ *and* κ*,* x_0*, and* u_0 *real constants, are orbitally stable in the sense of Theorem 11 if the wave speed* $8k^2\kappa^2 - 4\kappa^2 + u_0$ *is positive.*

3.6 Concluding Remarks

In this chapter, we extended the periodic spectral stability results of chapter 2 in order to determine sufficient conditions for the orbital stability of the four-parameter family of periodic traveling wave solutions of the generalized Korteweg-de Vries equation (3.1). By extending the methods of [11] to the periodic case, a new geometric condition was derived in terms of the conserved quantities of the gKdV flow restricted to the manifold of periodic traveling wave solution, and it was shown how this could be translated to a condition on the Hessian of the classical action of the ordinary differential equation governing the periodic traveling waves. As a byproduct of this theory, it was shown that such solutions of the KdV are orbitally stable to perturbations of the same period as the underlying wave if and only if they are spectrally stable to periodic perturbations of the same period.

There are several points in which this theory is still lacking. First off, it is not clear what happens in the case $T_E < 0$. In the solitary wave theory, the existence of two negative eigenvalues of the second variation $\mathcal{L}[u]$ indicates instability. Also, if $T_E > 0$ and $\{T, M\}_{a,E} < 0$ it is not clear whether this implies orbital instability, although we conjecture this is indeed the case. We would like to show in the case $\{T, M\}_{a,E}\{T, M, P\}_{a,E,c} < 0$ and $T_E > 0$ that there exists a 1-parameter family of functions in Σ_0 which contain the solution $u(x; a_0, E_0, c_0)$ such that the augmented energy functional $\mathcal{E}_0$ has a strict local maximum at (a_0, E_0, c_0). However, it is not clear how to do this in a reasonable manner: mainly, one must fix the period, mass, and momentum along this curve, and the derivative of this curve at (a_0, E_0, c_0) must also be in X. The existence of such an instability would stand in stark contrast to the solitary wave case, where the orbital stability is equivalent to the spectral instability (except possibly on the transition curve). However, it seems quite possible that such a situation arises due to the fact that the solitary waves are a co-dimension two subset of the family of traveling wave solutions.

CHAPTER 4

Transverse Instability Analysis of the gKdV

In this chapter, we consider the stability of periodic traveling wave solutions of the generalized KdV equation within higher dimensional models of shallow water waves. In particular, we study the stability of such solutions to long-wavelength perturbations in a transverse direction. We derive sufficient conditions for this instability in terms of the conserved quantities of the gKdV flow, much in the same way the modulational and finite wavelength indices were derived in previous chapters.

This notion of stability has been studied in some detail in the solitary wave setting. In particular, the first example of transverse instability of solitary waves is the instability of solitary wave solutions of the KdV within the Kadomtsev-Petviashvili (KP) equation

$$(u_t - u_{xxx} - uu_x)_x + u_{yy} = 0$$

It is clear that a y-independent solution of the KdV equation $u_t = uu_x + u_{xxx}$ solves the KP equation. Thus, a natural question is to ascertain the stability of such solitary wave solutions within the KP equation. As a first step to understanding this stability is to study the spectrum of the linearization of the KP equation about a KdV solitary wave. This question was first considered by Kadomtsev and Petviashvili in [39]. Here, the authors developed a perturbation theory for such calculations and successfully showed that the solitary wave solutions of the KdV are transversely unstable to perturbations of long-wavelength in the KP model.

In the periodic context, however, we find that our methods are insufficient to analyze such transverse instabilities within the KP model. Indeed, one can easily check that while the KP equation admits a four parameter family of y-independent traveling wave solutions, the corresponding periodic traveling waves constitute only a three parameter sub-family. Thus, the corresponding ODE governing the y-independent traveling wave solutions of the KP equation is not completely integrable, i.e. not reducible to quadrature, and hence we are unable to find a useful basis to calculate the tangent space of the manifold of periodic traveling wave solutions at the origin. This deficiency shows the weakness in the methods used in this thesis. Notice that in the solitary wave theory, this issue is avoided completely since upon the first integration the resulting constant must be zero by the boundary conditions at infinity: as we have seen throughout this thesis, there is no reason to force this constant to be zero in the periodic case.

The goal of this chapter is to consider the analogous question of transverse stability of periodic traveling wave solutions of the gKdV

$$u_t = u_{xxx} + (f(u))_x \tag{4.1}$$

to perturbations of long-wavelength within the Zakharov-Kuznetov (ZK) equation in two space dimensions:

$$u_t = u_{xxx} + (f(u))_x + u_{yyx}. \tag{4.2}$$

Clearly, if $u(x; a, E, c)$ is a y-independent periodic traveling wave solutions of (4.1), then u solves (4.2). We will use periodic Evans function techniques in order to derive sufficient conditions for a spectrally periodic traveling wave solutions of (4.1) to be spectrally stable to long-wavelength transverse perturbations in the ZK model. As an application of this theory, we will show that periodic traveling wave solutions of the KdV equation with $(a, E, c) \in \Omega$ are transversely unstable to long-wavelength perturbations in the ZK-model if the gKdV linearization has no non-zero real periodic eigenvalues.

4.1 Preliminaries

Throughout this chapter, we assume $(a_0, E_0, c_0) \in \Omega$ is such that $u(x; a_0, E_0, c_0)$ is a spectrally stable periodic traveling wave solution of the equation

$$u_t = u_{xxx} + (f(u))_x - cu_x. \tag{4.3}$$

In particular, we assume that the linearized operator $\partial_x \mathcal{L}[u]$ has no non-zero real periodic eigenvalues. By Corollary 3 of chapter 2, it follows that $\{T, M, P\}_{a,E,c}$ must be positive at (a_0, E_0, c_0). We wish to examine the spectral stability of u to long-wavelength perturbations in the framework of the traveling wave ZK equation

$$u_t = u_{xxx} + (f(u))_x - cu_x + u_{yyx}. \tag{4.4}$$

Notice that since u is a solution to (4.3), it is clearly a solution to (4.4) and hence it makes sense to discuss its spectral stability (in this section, spectral stability will refer to spectral stability in the ZK model).

Linearizing (4.4) around u yields

$$-v_t = \partial_x \left(\mathcal{L}[u] - \partial_y^2 \right) v$$

where $\mathcal{L}[u] = -\partial_x^2 - f'(u) + c$ is as in chapter 2. In particular, $\mathcal{L}[u]$ is a self adjoint second order differential operator on $L^2(\mathbb{R})$. Since this linearization is autonomous in time, we seek separated solutions of the form

$$v(x, y, t) = v(x) e^{-\mu t - iky}$$

where $\mu \in \mathbb{C}$, and $k \in \mathbb{R}$ is the transverse wave number of perturbation. This leads one to the (ordinary differential equation) spectral problem

$$\partial_x \left(\mathcal{L}[u] + k^2 \right) v = \mu v$$

considered on the real Hilbert space $L^2(\mathbb{R})$. Our goal is to study the spectrum of the operator $\partial_x\left(\mathcal{L}[u]+k^2\right)$ on $L^2(\mathbb{R})$ near the origin $(\mu,k)=(0,0)$. As in chapter 2, the spectrum is purely continuous and consists of piecewise smooth arcs. In particular, for a given $k\in\mathbb{R}$, $\mu\in\operatorname{spec}\left(\partial_x\left(\mathcal{L}[u]+k^2\right)\right)$ if and only if there exists a $\lambda\in S^1$ such that

$$D(\mu,k,\lambda)=\det\left(\mathbf{M}(\mu,k)-\lambda I\right)=0$$

where $\mathbf{M}(\mu,k)$ is the monodromy map corresponding to the first order system

$$Y_x=\mathbf{H}(x,\mu,k)Y,\quad Y(0,\mu,k)=\mathbf{I}, \tag{4.5}$$

where

$$\mathbf{H}(x,\mu,k)=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\mu-f''(u)u_x & c+k^2-f'(u) & 0\end{pmatrix}.$$

In chapter 2 we extensively studied the case when $k=0$, which corresponds to the generalized Korteweg-de Vries equation.. We found that $\mu=0$ is a periodic eigenvalue of multiplicity three. Instead of developing a perturbation theory in the Floquet parameter, from which we deduced the modulational instability index in chapter 2, we fix the Floquet parameter and develop a perturbation theory in the transverse wave number k. That is, we determine how the periodic eigenvalues of the linear operator $\partial_x\left(\mathcal{L}[u]+k^2\right)$ bifurcate from the $k=0$ state. Notice, in particular, that if u is spectrally unstable to periodic perturbations of the gKdV, then it is automatically unstable to transverse perturbations in the ZK-model. This justifies our assumption that $\{T,M,P\}_{a,E,c}$ is positive at (a_0,E_0,c_0).

4.2 Transverse Instability Results

From chapter 2, we know that $D(\mu,0,1)=\mathcal{O}(|\mu|^3)$, i.e. $\mu=0$ is a periodic eigenvalue of the operator $\partial_x\mathcal{L}[u]$ of multiplicity three. We expect these periodic eigenvalues to bifurcate from the origin as we consider $|k| \ll 1$. Thus, we are interested in determining the dominant balance of the equation $D(\mu, k, 1) = 0$ in a neighborhood of $(\mu, k) = (0, 0)$. From this, a sufficient condition for the transverse instability of periodic traveling wave solutions of the gKdV in the ZK-model will arise naturally.

We begin by studying the leading order asymptotics of the periodic Evans function $D(\mu, k, 1)$ in a neighborhood of the origin. This is the goal of the next lemma.

Lemma 18. *The equation $D(\mu, k, 1) = 0$ has the following local normal form in a neighborhood of the origin $(\mu, k) = (0, 0)$:*

$$-\frac{\mu^3}{2}\{T, M, P\}_{a,E,c} + 2\mu k^2 \{T, M\}_{a,E} \int_0^T u_x^2 dx + \mathcal{O}(4) = 0,$$

where $\mathcal{O}(4)$ denotes terms of order four and higher in the variables μ and k.

Proof. Recall from Theorem 8 that

$$D(\mu, 0, 1) = -\frac{\mu^3}{2}\{T, M, P\}_{a,E,c} + \mathcal{O}(|\mu|^4)$$

in a neighborhood of $\mu = 0$. Thus, we need only compute the $\mathcal{O}(\mu k^2)$ term in the above expansion. To this end, notice that we can write $\mathbf{H}(x, \mu, k) = \mathbf{H}(x, \mu) + \mathbf{H}_0(k)$, where $\mathbf{H}(\mu, k)$ is given as in (2.16) in Chapter 2 and

$$\mathbf{H}_0(k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k^2 & 0 \end{pmatrix}.$$

Our goal then follow the proof of Theorem 8 and use variation of parameters in order to determine the $\mathcal{O}(\mu k^2)$ term in the equation $D(\mu, k, 1)$.

To begin, let $\mathbf{W}(x, \mu, k)$ be a matrix solution of (4.5) such that

$$\mathbf{W}(x, 0, 0) = \begin{pmatrix} cu_x & cu_a & cu_E \\ cu_{xx} & cu_{ax} & cu_{Ex} \\ cu_{xxx} & cu_{axx} & cu_{Exx} \end{pmatrix}$$

and $\mathbf{W}(0,\mu,k) = \mathbf{W}(0,0,0)$ for all $\mu \in \mathbb{C}$ and $k \in \mathbb{R}$. The goal is to treat $\mathbf{W}(x,\mu,k)$ as a small perturbation of $\mathbf{W}(x,0,0)$ for $|(\mu,k)|_{\mathbb{C}\times\mathbb{R}} \ll 1$. Following the proof of Theorem 8 we define $\delta\,\mathbf{W}(\mu,k) = \mathbf{W}(x,\mu,k)|_{x=0}^{T}$ and notice that $\delta\,\mathbf{W}(\mu,k) = \delta\,\mathbf{W}(\mu) + k^2\,\mathbf{W}_0$, where $\delta\,\mathbf{W}(\mu)$ is from the proof of Theorem 8 and $\mathbf{W}_0$ is a matrix which is independent of μ and $\mathcal{O}(1)$ in the transverse wave number k. Since we clearly have

$$\mathbf{M}(\mu,k) = (\delta\,\mathbf{W}(\mu,k) + \mathbf{I})\;\mathbf{W}(0,0,0)^{-1},$$

we need only compute the k^2 variation of the first column of $\mathbf{W}(x,0,k)$. Using the variation of parameters formula given in (2.35) from chapter 2 yields

$$\mathbf{W}(T,0,0)\int_0^T \mathbf{W}(z,0,0)^{-1}\begin{pmatrix} 0 \\ 0 \\ u_{xx}(z)\end{pmatrix} dz = \begin{pmatrix} c\frac{\partial u_-}{\partial E}\int_0^T u_x^2 dx \\ * \\ (c-1-f'(u_-))\,\frac{\partial u_-}{\partial E}\int_0^T u_x^2 dx \end{pmatrix},$$

where the term $*$ is can be explicitly computed, but is not necessary at this order in the perturbation argument. Therefore, a straightforward calculation gives

$$\frac{1}{\mu k^2}\det\left(\delta\,\mathbf{W}(\mu,k)\right)\big|_{(\mu,k)=(0,0)} = -\{T,M\}_{a,E}\int_0^T u_x^2 dx.$$

Since $\det\left(\,\mathbf{W}(x,0,0)\right) = -1$, this completes the proof. □

Lemma 18 readily yields a necessary condition for the underlying periodic traveling wave solution of (4.3) to exhibit a modulational transverse instability in the ZK model.

Theorem 16. *If $\{T,M,P\}_{a,E,c} \neq 0$, then the spectrally stable periodic gKdV traveling wave $u(x;a_0,E_0,c_0)$ with $(a_0,E_0,c_0) \in \Omega$ is spectrally unstable to long-wavelength transverse perturbations in the ZK model if $\{T,M\}_{a,E} > 0$ at (a_0,E_0,c_0).*

Proof. By Lemma 18, there are three periodic eigenvalues in a neighborhood of the origin which are given by $\mu_0 = o(k)$ and

$$\mu_\pm = \pm|k|\sqrt{\frac{4\{T,M\}_{a,E}\int_0^T u_x^2 dx}{\{T,M,P\}_{a,E,c}}} + o(k)$$

Since $u(x; a, E, c)$ was assumed to be a spectrally stable solution of (4.1), we know from Corollary 3 that $\{T, M, P\}_{a,E,c} > 0$ and hence there will be two (non-zero) periodic eigenvalues off the imaginary axis in the neighborhood of the origin if $\{T, M\}_{a,E} > 0$. $\square$

We now point out a few interesting corollaries to Theorem 16. First off, the asymptotic analysis in a long-wavelength limit conducted in Chapter 2 implies that the sign of $\{T, M\}_{a,E} = M_E^2 - T_E M_a$ is positive for $(a, E, c) \in \Omega$ corresponding to periodic solutions of sufficiently long wavelength. Thus, by our analysis in Chapter 2, we immediately get the following corollary.

Corollary 7. *In the case of a power-law nonlinearity, the periodic traveling wave solutions $u(\,\cdot\,; a_0, E_0, c_0)$ of (4.1) with $(a_0, E_0, c_0) \in \Omega$ such that the period $T(a_0, E_0, c_0)$ is sufficiently large are spectrally unstable transverse perturbations in the ZK-model.*

Proof. Clearly, such a solution is unstable to long-wavelength transverse perturbations in the ZK model if $\{T, M, P\}_{a,E,c} > 0$, i.e. if $p < 4$. Moreover, if $p > 4$ then $\{T, M, P\}_{a,E,c} < 0$ and hence the function $D(\mu, 0, 1)$ has a non-zero real root $\mu_* > 0$, and hence by continuity the function $D(\mu, k, 1)$ will have a root near μ_* with positive real part for $|k| \ll 1$. Thus, although we are not guaranteed a long-wavelength transverse instability in this case, we still have spectral instability none the less. $\square$

Next we note that by Lemma 17 in Chapter 3, we know in the case of the KdV equation $\{T, M\}_{a,E} > 0$ for all $(a_0, E_0, c_0) \in \Omega$. Thus, we get the following corollary.

Corollary 8. *Periodic traveling wave solutions of the KdV with $(a_0, E_0, c_0) \in \Omega$ are spectrally unstable to transverse perturbations in the ZK-model if $\{T, M, P\}_{a,E,c} \neq 0$ at $(a_0, E_0, c_0) \in \Omega$.*

Proof. The proof is essentially the same as Corollary 7. Simply notice that if the periodic stability index $\{T, M, P\}_{a,E,c}$ is positive at (a_0, E_0, c_0), then the solution is spectrally unstable to long-wavelength transverse perturbations in the ZK model. Moreover, if $\{T, M, P\}_{a,E,c} < 0$, we know the function $D(\mu, k, 1)$ will have a root with positive real part for $|k| \ll 1$, which completes the proof. $\square$

CHAPTER 5

The Generalized Benjamin-Bona-Mahony and Camassa-Holm Equations

In this chapter, we show how the spectral stability techniques of chapter 2 and 4 can be extended to other third order non-linear dispersive equations. In particular, we consider the spectral stability of periodic traveling wave solutions of the generalized Benjamin-Bona-Mahony (gBBM) equation as well as the generalized Camassa-Holm (gCH) equation.

This analysis is very similar to that of chapter 2 and 4, and most of the analysis in this chapter is presented in the context of the gBBM equation. Our first observation is that the asymptotic behavior of the periodic Evans function in a neighborhood of the origin in the spectral plane yields two separate instability indices. One provides a necessary and sufficient condition for the existence of a modulational instability, assuming a non-degeneracy condition is met, while the other counts modulo 2 the number of positive periodic eigenvalues of the corresponding linearized operator. This second index is a generalization of the one which governs stability of the solitary wave: this is shown explicitly by studying long-wavelength asymptotics and recovering the well known results of Pego and Weinstein. As expected from our work in chapter 2, each of the above indices are expressible in terms of Jacobians of maps from a parameter space parameterizing the periodic traveling waves to the conserved quantities of the gBBM, and hence is geometric in nature. Moreover, the vanishing of the second index is shown to be equivalent to a change in the structure of the generalized periodic null-space of the linearized operator. Finally, we study the stability of such periodic traveling wave solutions of the gBBM to long-wavelength transverse perturbations, as well as show how the methods in this paper can be applied to periodic traveling wave solutions of the generalized Camassa-Holm equation.

5.1 Introduction and Preliminaries

In this chapter, we consider periodic standing wave solutions of the generalized Benjamin-Bona-Mahony (gBBM) equation

$$u_t - u_{xxt} + u_x + (f(u))_x = 0, \tag{5.1}$$

and the generalized Camassa-Holm equation

$$u_t - u_{xxt} = 2u_x u_{xx} + u u_{xxx} - (f(u)/2)_x - k u_x, \tag{5.2}$$

where $f(\cdot) \in C^2(\mathbb{R})$ is a prescribed nonlinearity satisfying suitable convexity assumptions. As in the case of the gKdV, these equations admit a four-parameter family of traveling wave solutions of the form $u(x,t) = u(x - ct)$ where the wave speed $c > 1$. However, unlike the analysis of chapter 2 and 4, the linearization about such a solution yields a non-local linear partial differential equation. The goal of this chapter is to study the spectrum of the associated non-local operators in appropriate Hilbert spaces in order to ascertain information about the stability of the periodic traveling wave solutions of (5.1) and (5.2) with respect to localized or uniformly bounded continuous perturbations.

Of particular interest is the case of a power nonlinearity $f(u) = u^{p+1}/(p+1)$. In the case $p = 1$, equation (5.1) is known as the Benjamin-Bona-Mahony (BBM) equation, or the regularized long-wave equation and (5.2) is known as the Camassa-Holm equation. Each of these equations are completely integrable, and have been derived as an alternative model to the Korteweg-de Vries equation as a description of gravity water waves in the long-wave regime. The BBM, as well as the gBBM, equation has received much attention over the years into the stability of solitary (L^2) type solutions. In particular, it is well known that solitary wave solutions of (5.1) are orbitally stable for all wave speeds $c > 1$ if $1 \leq p \leq 4$. Moreover, if $p > 4$, then there exists a critical wave speed $c = c_0(p)$ such that solitary traveling waves with $c > c_0(p)$ are orbitally stable, while those with $1 < c < c_0(p)$ are exponentially unstable due to the presence of a non-zero real eigenvalue of the linearized operator. There are relatively few spectral stability results for equation (5.1) in the periodic case when considering arbitrary L^2 perturbations of the underlying wave. In the well known work of Gardner [31], it is shown that periodic traveling wave solutions of (5.1) of sufficiently long wavelength are exponentially unstable whenever the limiting homoclinic orbit (solitary wave) is unstable. The mechanism behind this instability is the existence of a "loop" of spectrum in the neighborhood of any unstable eigenvalue of the limiting solitary wave.

The gCH equation, however, has only recently been derived and thus the stability theory is not as well developed. In contrast to the gKdV and gBBM equations, the derivative of continuous solutions to the gCH may experience point singularities, in which case the solution is called a Peakon or Cuspon (a pun on the term Soliton). In particular, a complete characterization of the traveling wave solutions of (5.2) has recently been derived in [46]: more will be said on this at the end of this chapter. Stability theory for the exponentially decaying solutions of (2.1) have been carried out by [48] in the case of peakon solutions. In the periodic context, Hărăguş recently carried out a detailed spectral stability analysis in the case of power-nonlinearity $f(u) = u^{p+1}/(p+1)$ for waves sufficiently close to the constant state $u = ((p+1)(c-1))^{1/p}$. The gCH equation is quite different from the previously mentioned gKdV and gBBM equations. In particular, it admits smooth traveling wave solutions (both periodic and solitary) which blow up in finite time. This may be expected, since the gCH equation was originally derived to describe wave-breaking phenomenon. Our results relating to this equation are not as explicit, but we are able to ascertain information about the stability spectrum in a neighborhood of the origin.

The outline for this chapter is as follows. Sections 5.2 through 5.5 concern only periodic traveling wave solutions of the gBBM equation. In section 5.2, we review some basic properties of the periodic traveling wave solutions and discuss the associated conserved quantities. In section 5.3, we study the nature of the periodic Evans function in a neighborhood of the origin, as well as for large real spectral values. We are able to determine a modulational instability index and a finite wavelength instability index as in chapter 2. In section 5.4, we study the asymptotics of these instability indices in a solitary wave limit, and show that the solitary wave theory (more precisely, the results of Gardner) are recaptured in this limit. In section 5.5, we study the transverse instability of the periodic traveling wave solutions of the gBBM within a model similar to that studied in chapter 4. Finally, we study the nature of the spectrum in a neighborhood of the origin and prove there are three branches of spectrum which bifurcate from the origin analytically in the associated Floquet parameter.

5.2 Properties of the Periodic Traveling Wave Solutions of gBBM

In this section, we review the basic properties of the periodic traveling wave solutions of the gBBM equation (5.1). As this basically parallels that of chapter 2, we only mention these results briefly and do not go into much detail.

To begin, notice that equation (5.1) admits stationary traveling wave solutions of the form $u(x,t) = u(x-ct)$ where $c > 1$ which satisfy the traveling wave ordinary differential equation

$$cu_{xxx} - (c-1)u_x + (f(u))_x = 0, \tag{5.3}$$

i.e. they are stationary solutions of (5.1) in the traveling coordinate frame defined by $x - ct$. Integrating (5.3) twice with respect to x yields the equations

$$\begin{aligned} cu_{xx} - (c-1)u + f(u) &= a \\ \frac{c}{2}u_x^2 - \left(\frac{c-1}{2}\right)u^2 + F(u) &= au + E \end{aligned} \tag{5.4}$$

where $F'(u) = f(u)$ and the parameters a and E are constants of motion. Hence the traveling wave solutions are reducible to quadrature and constitute a four-parameter family of solutions of (5.1). Notice that in the solitary wave case, the boundary conditions (exponential decay at $\pm\infty$) force $a = E = 0$, but there is no physical reason for this restriction in the periodic traveling wave case. It follows that the solitary waves can be considered a co-dimension two subset of the full four-parameter family of traveling wave solutions.

In order to guarantee the existence of periodic solutions of (5.3), we make the standard assumptions from chapter 2 on the non-linearity f. In particular, we assume the effective potential

$$V(u;a,c) = F(u) - \frac{(c-1)}{2}u^2 - au$$

has a non-degenerate local minimum. Moreover, we consider only those periodic orbits which are bounded by a homoclinic orbit in phase space, and do not themselves bound a homoclinic orbit (other than the equilibrium solution). This places a natural restriction on the parameter regime for our problem: we always assume we are within this open region of $\mathbb{R}^4$ (see Definition 1 of chapter 2 for more details on this assumption). As before, we can factor out one of four parameters defining the periodic traveling waves by modding out the translation invariance: this is usually done by requiring $u_x(0) = 0$ and $V'(u(0)) < 0$. It follows that such solutions are even and have a local maximum at $x = 0$. Moreover, this implies the roots $u_\pm$ of the equation $E = V(u;a,c)$ for (a,E,c) within this open region are C^1 functions of (a,E,c). Also, notice that $u(0) = u_-$ and $V(u;a,c) < E$ for $u \in (u_-, u_+)$. As is standard, one can use equation (5.4) to express the period of the periodic wave u as

$$T(a,E,c) = 2\sqrt{c}\int_{u_-}^{u_+} \frac{du}{\sqrt{2\left(E - V(u;a,c)\right)}}. \tag{5.5}$$

The above interval can be regularized at the square root branch points $u_\pm$ by the procedure discussed in chapter 2. Similarly, the mass and momentum of the periodic wave can be expressed as

$$\begin{aligned} M(a,E,c) &= 2\sqrt{c}\int_{u_-}^{u_+} \frac{u\,du}{\sqrt{2\left(E - V(u;a,c)\right)}} \\ P(a,E,c) &= 2\sqrt{c}\int_{u_-}^{u_+} \left(\frac{u^2}{\sqrt{2\left(E - V(u;a,c)\right)}} + \sqrt{2\left(E - V(u;a,c)\right)}\right) du \end{aligned}$$

where these integrals can be regularized at the square root branch points by the same procedure. In particular, by our assumptions on $u_{\pm}$, and hence on the non-linearity f, it follows that one can differentiate these functionals restricted to the periodic wave $u(x;a,E,c)$ with respect to the parameters (a,E,c). As in chapter 2, the gradients of these quantities will play an important role in the subsequent theory.

Notice that the quantities T, M, and P satisfy a number of identities. In particular, it is useful to consider the classical action

$$K(a,E,c) = \int_0^T u_x^2 dx = 2\sqrt{c}\int_{u_-}^{u_+} \sqrt{2\left(E - V(u;a,c)\right)}du$$

(which is not itself conserved) since this then acts as a generating function for the above conserved quantities. Indeed, it is clear the classical action satisfies the following relations:

$$\begin{aligned} K_E &= \frac{1}{c}T \\ K_a &= \frac{1}{c}M \\ K_c &= \frac{1}{2c}P. \end{aligned}$$

This establishes several useful identities between the gradients of these quantities. For example, it immediately follows that $T_a = M_E$ and $2T_c = P_E + \frac{1}{c}T$. These identities will be useful in the forthcoming analysis.

5.3 Linearization and The Periodic Evans Function

Throughout this section, we assume that $u(x;a,E,c)$ is a periodic traveling wave solutions of (5.1) satisfying the hypothesis outlined in the previous section. The behavior of infinitesimal perturbations of u is determined by the spectrum of the (non-local) linearized operator

$$\left(1-\partial_x^2\right)^{-1}\partial_x\left(-c\partial_x^2 + (c-1) - f'(u)\right).$$

In particular, by considering a solution to the partial differential equation (5.1) of the form

$$\psi(x,t) = u(x;a,E,c) + \varepsilon v(x,t) + \mathcal{O}(\varepsilon^2)$$

where $|\varepsilon| \ll 1$ is considered as a small perturbation parameter, it follows from substituting into (5.1) and collecting terms at $\mathcal{O}(\varepsilon)$ that v must satisfy the equation

$$\partial_x \mathcal{L} v = \mu \mathcal{D} v_t,$$

where $\mathcal{L} = -c\partial_x^2 + (c-1) - f'(u)$ and $\mathcal{D} = 1 - \partial_x^2$ By taking the Laplace transform in time, we are led to the spectral problem

$$\partial_x \mathcal{L} v = \mu \mathcal{D} v \tag{5.6}$$

on the real Hilbert space $L^2(\mathbb{R})$[1]. Throughout this paper, we consider the above operators as acting on $L^2(\mathbb{R})$ with domain $H^3(\mathbb{R})$ corresponding to spatially localized perturbations, or on $C_b(\mathbb{R})$ with domain $C^3(\mathbb{R})$ corresponding to bounded uniformly continuous perturbations. In both cases, the operator $\mathcal{L}$ is self adjoint and $\mathcal{D}$ is a positive operator and is hence invertible. In particular, notice that the operator $\mathcal{D}^{-1}\partial_x$ is skew-adjoint on $L^2(\mathbb{R})$. It follows that (5.6) can be written as

$$\mathcal{A} v = \mu v, \quad \mathcal{A} = \mathcal{D}^{-1}\partial_x \mathcal{L}.$$

It follows that the L^2 spectrum of the operator $\mathcal{A}$ is entirely essential. Since the operator $\mathcal{A}$ is a non-local operator with periodic coefficients, we recall basic definitions and results from Floquet theory.

Definition 6. *The monodromy operator* $\mathbf{M}(\mu)$ *is defined to be the period map*

$$\mathbf{M}(\mu) = \Phi(T;\mu)$$

[1] As in chapter 2, this corresponds to considering localized perturbations of the underlying periodic wave u. On could also study the stability to uniformly bounded continuous perturbations, but by Floquet theory these result in the same spectral stability theories.

where $\Phi(x;\mu)$ *satisfies the first order system*

$$\Phi(x;\mu)_x = \mathbf{H}(x,\mu)\Phi(x;\mu) \tag{5.7}$$

subject to the initial condition $\Phi(0) = \mathbf{I}$*, where* $\mathbf{I}$ *is the* 3×3 *identity matrix and*

$$\mathbf{H}(x,\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{c}\left(\mu + u_x f''(u)\right) & \frac{1}{c}\left(c - 1 - f'(u)\right) & \frac{\mu}{c} \end{pmatrix}.$$

Definition 7. *We say* $\mu \in \operatorname{spec}(\mathcal{A})$ *is there exists a non-trivial bounded function* ψ *such that* $\mathcal{A}\psi = \mu\psi$ *or, equivalently, if there exists a* $\lambda \in S^1$ *such that*

$$\det(\mathbf{M}(\mu) - \lambda\mathbf{I}) = 0.$$

Following Gardner we define the periodic Evans function $D : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ *to be*

$$D(\mu,\lambda) = \det(\mathbf{M}(\mu) - \lambda\mathbf{I}).$$

Finally, we say the periodic solution $u(x;a,E,c)$ *is spectrally stable if* $\operatorname{spec}(\mathcal{A})$ *does not intersect the open right half plane.*

Remark 16. *First, notice by the Hamiltonian nature of* (5.6)*,* $\operatorname{spec}(\mathcal{A})$ *is symmetric with respect to reflections about the real and imaginary axis. Thus, spectral stability occurs if and only if* $\operatorname{spec}(\mathcal{A}) \subset \mathbb{R}i$*.*

Secondly, since we are interested primarily with the roots of $D(\mu,\lambda)$ *for* λ *on the unit circle, we will frequently work with the function* $D(\mu, e^{i\kappa})$ *for* $\kappa \in \mathbb{R}/2\pi\mathbb{Z}$*, which is actually the function considered by Gardner.*

As we will see below, it follows from the integrable structure of (5.3) that the function $D(\mu,1)$ has a zero of multiplicity at least three at $\mu = 0$. As λ varies on S^1 there will be in general three branches $\mu_j(\kappa)$ of roots of $D(\mu_j(\kappa), e^{i\kappa})$ for κ small which bifurcate from the $\mu = 0$ state. Assuming these branches are analytic[2] in κ, it follows that a necessary condition for spectral stability is thus

$$\frac{\partial}{\partial \kappa}\mu_j(\kappa)\Big|_{\kappa=0} \in \mathbb{R}i. \tag{5.8}$$

This leads one to the use of perturbation methods in the study of the spectrum of $\mathcal{A}$ near the origin, exactly as in the analysis of chapter 2. Indeed, as we will see, the first order terms of a Taylor series expansion of the three branches $\mu_j(\kappa)$ can be encoded as roots of a homogeneous cubic polynomial, and hence spectral stability is determined by the sign of the associated cubic discriminant. Moreover, it follows by the Hamiltonian structure of (5.6) that in fact $\sigma(\mathcal{A}) \subset \mathbb{R}i$ if (5.8) holds and the branches are distinct.

We conclude this section by reviewing some basic global features of the spectrum of the linearized operator $\mathcal{A}$ which are useful in a local analysis near $\mu = 0$. We also state some important properties of the Evans function $D(\mu, \lambda)$ which are vital to the foregoing analysis.

Proposition 9. *The spectrum of the operator $\mathcal{A}$ has the following properties:*

(i) There are no isolated points of the spectrum. In particular, the spectrum consists of piecewise smooth arcs.

(ii) The entire imaginary axis is contained in the spectrum, i.e. $\mathbb{R}i \subset \operatorname{spec}(\mathcal{A})$.

Moreover, the Evans function $D(\mu, \lambda)$ satisfies the following:

(iii) $D(\mu, \lambda) = -\lambda^3 + a(\mu)\lambda^2 - a(-\mu)e^{\mu T/c}\lambda + e^{\mu T/c}$ with $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$.

(iv) The function $a(\mu)$ satisfies $a(0) = 3, a'(0) = \frac{T}{c}$.

Proof. The first claim follows that of Proposition 1 in chapter 2. We postpone the proof of (iv) until Lemma 19.

[2]In general, for each j, the theory of branching solutions of non-linear equations guarantees the existence of a natural number m_j such that $\mu_j(\cdot)$ is an analytic function of κ^{1/m_j}. As we will see in our case, the Hamiltonian nature of the linearized operator $\mathcal{A}$ assures that $m_j = 1$, and hence the roots are in fact analytic functions of the Floquet parameter.

Next, we prove claim (iii). First, notice that Abel's formula along with the fact that $\operatorname{tr}(\mathbf{H}(x,\mu)) = \frac{\mu}{c}$, where $\mathbf{H}(x,\mu)$ is as in (5.7), implies that $D(\mu,0) = e^{\mu T/c}$. As $\mathcal{A}v = \mu v$ is invariant under the transformation $x \mapsto -x$ and $\mu \mapsto -\mu$, we have as in the case of the gKdV that $\mathbf{M}(\mu) \sim \mathbf{M}(-\mu)^{-1}$. If we define $a(\mu)$ as above and $b(\mu)$ such that

$$\det[\mathbf{M}(\mu) - \lambda\mathbf{I}] = -\lambda^3 + a(\mu)\lambda^2 + b(\mu)\lambda + e^{\mu T/c},$$

it follows that

$$\begin{aligned}
\det[\mathbf{M}(\mu) - \lambda\mathbf{I}] &= \det[\mathbf{M}^{-1}(-\mu) - \lambda\mathbf{I}] \\
&= -\lambda^3 \det[\mathbf{M}^{-1}(-\mu)] \det[\mathbf{M}(-\mu) - \lambda^{-1}] \\
&= -e^{\mu T/c}\lambda^3 \left(-\lambda^{-3} + a(-\mu)\lambda^{-2} + b(-\mu)\lambda^{-1} + e^{-\mu T/c}\right) \\
&= -\lambda^3 - e^{\mu T/c} b(-\mu)\lambda^2 - e^{\mu T/c} a(-\mu)\lambda + e^{\mu T/c}.
\end{aligned}$$

Thus, it follows from (iii) that $b(\mu) = -e^{\mu T/c} a(-\mu)$ as claimed.

Claim (ii) follows from a symmetry argument. Since $a(\mu)$ is real on the real axis, it follows by Schwarz reflection that for $\mu \in \mathbb{R}i$ we have $a(\overline{\mu}) = \overline{a(\mu)}$. For $\mu \in \mathbb{R}i$ then the Evans function takes the form

$$D(\mu,\lambda) = -\lambda^3 + a(\mu)\lambda^2 - e^{\mu T/c}\overline{a(\mu)}\lambda + e^{\mu T/c}.$$

It follows that

$$D(\mu,\lambda) = -\lambda^3 e^{\mu T/c}\overline{D(\mu,\overline{\lambda}^{-1})}$$

so that the roots of $D(\mu,\lambda)$ for a fixed $\mu \in \mathbb{R}i$ are symmetric about the unit circle. Since there must be three such roots, one must live on the unit circle and hence $\mu \in \operatorname{spec}(\mathcal{A})$ as claimed. □

5.4 Local Analysis of the Period Map

In this section, we begin our study of the structure of the spectrum of the linearized operator $\mathcal{A}$ in a neighborhood of the origin. To this end, we study the asymptotic behavior of the Evans function in the limit $\mu \to 0$. We begin by proving that $D(0, e^{i\kappa})$ has a zero of multiplicity three at $\kappa = 0$ by explicitly calculating the Jordan normal form of $\mathbf{M}(0)$. It follows from this calculation that $\lambda = 1$ is an eigenvalue of algebraic multiplicity three and geometric multiplicity two. This fact reflects the following structure in the manifold of solutions to the ordinary differential equation defining the traveling waves: the traveling waves form a three parameter manifold, with traveling waves of constant period forming a two parameter submanifold. The two eigenvectors of the period map correspond to elements of the tangent plane to the submanifold of constant period solutions, while the third vector in the Jordan chain is associated to the normal to the constant period submanifold. By then treating $\mathbf{M}(\mu)$ as a small perturbation of $\mathbf{M}(0)$, we use a perturbation theory appropriate to the Jordan form of $\mathbf{M}(0)$ to determine the dominant balance of the equation $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0, 0)$. From this information, we extract a modulational instability index which governs the local normal form of the spectrum in a neighborhood of the origin. Finally, by a standard orientation index argument we derive a finite-wavelength instability index which yields sufficient conditions for non-trivial intersections of the spectrum with the real axis, immediately implying exponential instability of the underlying periodic wave.

5.4.1 Calculation of the Period Map at $\mu = 0$

We begin by studying the structure of the operator $\mathbf{M}(0)$, which corresponds to understanding the null space of the operator $\mathcal{A}$. Notice that since $\mathcal{D}$ is invertible, the null space of $\mathcal{A}$ is determined by the null space of $\partial_x \mathcal{L}$, which leads us to the following proposition.

Proposition 10. *Let $u = u(\,\cdot\,; a, E, c)$ be the solution of the traveling wave equation* (5.3) *satisfying $u(0; a, E, c) = u_-$ and $u_x(0; a, E, c) = 0$. A basis of solutions to the first* *order system*

$$Y_x = \mathbf{H}(x,0)Y$$

is given by

$$Y_1(x) = \begin{pmatrix} cu_x \\ cu_{xx} \\ cu_{xxx} \end{pmatrix}, \quad Y_2(x) = \begin{pmatrix} cu_a \\ cu_{ax} \\ cu_{axx} \end{pmatrix}, \quad Y_3(x) = \begin{pmatrix} cu_E \\ cu_{Ex} \\ cu_{Exx} \end{pmatrix},$$

where we have suppressed the dependence on $(x; a, E, c)$*. Moreover, a particular solution to the inhomogeneous problem*

$$Y_x = \mathbf{H}(x,0)Y + W$$

where $W = (0, 0, c\mathcal{D}u_x)$ *is given by*

$$Y_4(x) = \begin{pmatrix} -cu_c \\ -cu_{cx} \\ -cu_{cxx} \end{pmatrix}.$$

Proof. This is easily verified by differentiating (5.3) with respect x and the parameters a, E, and c. □

By the above proposition three linearly independent solutions of (5.7) are given by the vector functions Y_1, Y_2, and Y_3 above. We now wish to explicitly write down the solution matrix in this basis $\mathbf{U}(x,0)$ at $x = T$ and $x = 0$. Notice by hypothesis, for any a, E, c the solution u satisfies

$$u(0; a, E, c) = u_- = u(T; a, E, c) \tag{5.9}$$

$$u_x(0; a, E, c) = 0 = u_x(T, a, E, c) \tag{5.10}$$

$$u_{xx}(0; a, E, c) = -\frac{1}{c}V'(u_-; a, c) = u_{xx}(0; a, E, c) \tag{5.11}$$

and, moreover, it follows from (5.3) that $u_{xxx}(0; a, E, c) = 0$. Defining $\mathbf{U}(x, 0) = [Y_1(x), Y_2(x), Y_3(x)]$ it follows by differentiating the above relations that

$$\mathbf{U}(0,0) = \begin{pmatrix} 0 & c\frac{\partial u_-}{\partial a} & c\frac{\partial u_-}{\partial E} \\ -V'(u_-) & 0 & 0 \\ 0 & 1 - V''(u_-)\frac{\partial u_-}{\partial a} & -V''(u_-)\frac{\partial u_-}{\partial E} \end{pmatrix}. \tag{5.12}$$

Note that differentiating the relation $E = V'(u_-; a, c)$ with respect to E gives

$$\det(\mathbf{U}(0,0)) = -cV'(u_-)\frac{\partial u_-}{\partial E} = -c$$

so these solutions are linearly independent at $x = 0$, and hence for all x. Thus, we can compute the monodromy matrix $\mathbf{M}(\mu)$ by determining $\mathbf{U}(T, 0)$ and right composing with $\mathbf{U}(0, 0)^{-1}$.

The matrix $\mathbf{U}(T; 0)$ can now be calculated by using the chain rule to differentiate (5.9)-(5.11). For example, differentiating (5.9) with respect to a gives

$$\frac{\partial u}{\partial a}(T; a, E, c) + u_x(T; a, E, c)T_a(a, E, c) = \frac{\partial u_-}{\partial a}$$

from which it follows $u_a(T; a, E, c) = \frac{\partial u_-}{\partial a}$. Continuing in this manner yields

$$\mathbf{U}(T,0) = \mathbf{U}(0,0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & V'(u_-)T_a & V'(u_-)T_E \\ 0 & 0 & 0 \end{pmatrix}.$$

This proves that $\mathbf{U}(T, 0) - \mathbf{U}(0, 0)$ is a rank one matrix, which, as in chapter 2, naturally leads one to the following proposition.

Proposition 11. *There exists a basis in $\mathbb{R}^3$ such that the monodromy matrix $\mathbf{M}(\mu)$*

evaluated at $\mu = 0$ takes the following Jordan normal form:

$$\mathbf{M}(0) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix} \tag{5.13}$$

where $\sigma \neq 0$ as long as $D_{a,E}T(a,E,c) \neq 0$. In particular, the monodromy operator at $\mu = 0$ has a single eigenvalue $\lambda = 1$ with algebraic multiplicity three and geometric multiplicity two as long as the period is not a critical point with respect to the parameters a and E for a fixed wave speed c.

Remark 17. *Notice that by Lemma 10 applies in this case, and hence we are guaranteed that $T_E > 0$ for all power-nonlinearities, and more generally for all nonlinearities f such that $f(u)$ and u are co-periodic.*

5.4.2 Asymptotic Analysis of $D(\mu, e^{i\kappa})$ Near $(\mu, \kappa) = (0, 0)$

We now use Proposition 11 to compute an asymptotic expansion of the characteristic polynomial of $\mathbf{M}(\mu)$ in a neighborhood of $\mu = 0$. This is accomplished by treating $\mathbf{M}(\mu)$ as a small perturbation of the matrix $\mathbf{M}(0)$ constructed above.

Recall from Proposition 9 that the spectrum near $\mu = 0$ is continuous. By the analyticity of $\mathbf{M}(\mu)$ in a neighborhood of $\mu = 0$, we can expand $\mathbf{M}(\mu)$ for small μ as

$$\mathbf{M}(\mu) = \mathbf{M}(0) + \mu\,\mathbf{M}_\mu(0) + \frac{\mu^2}{2}\,\mathbf{M}_{\mu\mu}(0) + \mathcal{O}\left(|\mu|^3\right)$$

where $\mathbf{M}_\mu(0) = [M_{i,j}^{(1)}]$ and $\mathbf{M}_{\mu\mu}(0) = [M_{i,j}^{(2)}]$. If one makes a similarity transform $\widetilde{\mathbf{M}}(\mu) = \mathbf{V}^{-1}\mathbf{M}(\mu)\mathbf{V}$ so that $\widetilde{\mathbf{M}}(0)$ is in the Jordan normal form (5.13) then a direct calculation using the above second order expansion of $\widetilde{\mathbf{M}}(\mu)$ implies that in a neighborhood of $\mu = 0$, the characteristic polynomial can be expressed as

$$\begin{aligned} D(\mu, e^{i\kappa}) &= \det\left((\widetilde{\mathbf{M}}(\mu) - I) - (e^{i\kappa} - 1)\mathbf{I}\right) \\ &= -\eta^3 + \eta^2\left(\mu \operatorname{tr}\left(\widetilde{\mathbf{M}}_\mu(0)\right) + \frac{\mu^2}{2}\operatorname{tr}(\widetilde{\mathbf{M}}_{\mu\mu}(0))\right) \\ &- \eta\left(\mu \widetilde{M}^{(1)}_{3,2}\sigma + \mu^2\left(\frac{1}{2}\left(\operatorname{tr}(\widetilde{\mathbf{M}}_\mu)\right)^2 - \frac{1}{2}\operatorname{tr}(\widetilde{\mathbf{M}}^2_\mu) - \frac{\sigma}{2}\widetilde{M}^{(2)}_{3,2}\right)\right) \\ &- \sigma(\widetilde{M}^{(1)}_{1,1}\widetilde{M}^{(1)}_{3,2} - \widetilde{M}^{(1)}_{3,1}\widetilde{M}^{(1)}_{1,2})\mu^2 \\ &+ \mu^3\left(\det\left(\widetilde{\mathbf{M}}_\mu(0)\right) + \sigma S\right) + \mathcal{O}(4), \end{aligned} \tag{5.14}$$

where $\eta = e^{i\kappa} - 1$, S represents mixed terms from $\widetilde{\mathbf{M}}_\mu(0)$ and $\widetilde{\mathbf{M}}_{\mu\mu}(0)$, σ is as in Proposition 11, and the notation $\mathcal{O}(4)$ represents terms whose degree is four or higher. Notice there are no other μ^3 terms since $\mathbf{M}(0) - I$ has rank one. Our next goal is to determine the dominant balance of the equation $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0, 0)$. As in chapter 2, the symmetry $\mathbf{M}(\mu) \sim \mathbf{M}^{-1}(-\mu)$ causes a number of terms in (5.14) to vanish, which leads to an expansion in integer powers of μ. This is the content of the next lemma.

Lemma 19. *If $D_{\mu\mu\mu}(0,1) \neq 0$, the equation $D(\mu, e^{i\kappa}) = 0$ has the following normal form in a neighborhood of $(\mu, \kappa) = (0, 0)$:*

$$-(i\kappa)^3 + \frac{(i\kappa)^2 \mu T}{c} + \frac{i\kappa\mu^2}{2}\left(\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) - \left(\frac{T}{c}^2\right)\right) + \frac{\mu^3}{6}D_{\mu\mu\mu}(0,1) + \mathcal{O}(4) = 0. \tag{5.15}$$

whose Newton diagram is depicted in Figure 5.1, where $\mathcal{O}(4)$ represents terms of degree four and higher.

Proof. Define the function b in a neighborhood of $\mu = 0$ by

$$\det[(\mathbf{M}(\mu) - \mathbf{I}) - (e^{i\kappa} - 1)\mathbf{I}] = -\eta^3 + (a(\mu) - 3)\eta^2 + b(\mu)\eta + D(\mu, 1). \tag{5.16}$$

where $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$ and $\eta = e^{i\kappa} - 1$. Notice in particular that $\eta = i\kappa + \mathcal{O}\left(\kappa^2\right)$ in

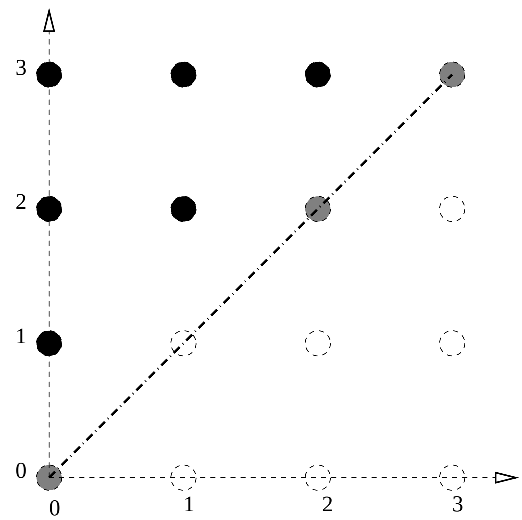

Figure 5.1: The Newton diagram corresponding to the asymptotic expansion of $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu, \kappa) = (0, 0)$ is shown to $O(|\mu|^3)$. Terms associated to open circles with dashed boundary are shown to vanish due to the natural symmetries inherent in (5.6). The open circles with dark boundary are non-vanishing terms which are a part of the lower convex hull, and hence contribute to the dominant balance. The closed dark circles lie above the lower convex hull and thus do not contribute to the leading order asymptotics.

a neighborhood of $\kappa = 0$. By (5.14), it follows

$$\begin{aligned}
a(\mu) &= \operatorname{tr}(\mathbf{M}(\mu)) = 3 + \mu \operatorname{tr}(\mathbf{M}_\mu(0)) + \frac{\mu^2}{2}\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) + \frac{\mu^3}{6}\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0)) + \mathcal{O}(|\mu|^4) \\
b(\mu) &= \frac{1}{2}\left(\operatorname{tr}((\mathbf{M}(\mu) - \mathbf{I})^2) - \operatorname{tr}(\mathbf{M}(\mu) - \mathbf{I})^2\right) \\
&= -\mu M_{3,2}^{(1)}\sigma - \mu^2\left(\frac{1}{2}\operatorname{tr}(\mathbf{M}_\mu)^2 - \frac{1}{2}\operatorname{tr}(\mathbf{M}_\mu^2) - \frac{\sigma}{2}\tilde{M}_{3,2}\right) + \mathcal{O}(|\mu|^3) \\
D(\mu, 1) &= -\sigma(M_{1,1}^{(1)}M_{3,2}^{(1)} - M_{3,1}^{(1)}M_{1,2}^{(1)})\mu^2 + (\det(\mathbf{M}_\mu(0)) + \sigma S)\,\mu^3 + \mathcal{O}(|\mu|^4)
\end{aligned}$$

From the Hamiltonian symmetry of (5.6), we have that

$$\begin{aligned}
D(\mu, 1) &= \det(\mathbf{M}(\mu) - \mathbf{I}) \\
&= \det\left(\mathbf{M}(-\mu)^{-1} - \mathbf{I}\right) \\
&= -e^{-\mu T/c} D(-\mu, 1).
\end{aligned}$$

It immediately follows that $D_{\mu\mu}(0,1) = -2\sigma(M_{1,1}^{(1)}M_{3,2}^{(1)} - M_{3,1}^{(1)}M_{1,2}^{(1)}) = 0$ and hence

$D(\mu, 1) = \mathcal{O}(|\mu|^3)$.

Similarly, we have

$$\begin{aligned} e^{-\mu T/c} D(\mu, \lambda) &= -\lambda^3 \det\left(M(-\mu) - \frac{1}{\lambda}\mathbf{I}\right) \\ &= -\left(\lambda - 1\right)^3 - \left(a(-\mu) - 3\right)\lambda\left(\lambda - 1\right)^2 \\ &\quad + b(-\mu)\lambda^2\left(\lambda - 1\right) - \lambda^3 D(-\mu, 1) \end{aligned}$$

By comparing the $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^3)$ terms above, we have the relations

$$\begin{cases} e^{-\mu T/c} a(\mu) = 2a(-\mu) - b(-\mu) - 3, & \text{and} \\ -e^{-\mu T/c} = -a(-\mu) + b(-\mu) - D(-\mu, 1) + 2. \end{cases}$$

Differentiating with respect to μ and evaluating at $\mu = 0$ immediately implies $b'(0) = 0$ and $a'(0) = \frac{T}{c}$. Similarly, it follows that $b''(0) = a''(0) - \left(\frac{T}{c}\right)^2$ which completes the proof. □

It follows that the structure of spec $(\mathcal{A})$ in a neighborhood of the origin is, to leading order, determined by the above homogeneous polynomial in κ and μ. Due to the triple root of $D(\cdot, 1)$ at $\mu = 0$ the implicit function theorem fails, but in a seemingly trivial way which can be easily corrected. This leads us to the following theorem giving a "modulational stability index" for traveling wave solutions of (5.1).

Theorem 17. *With the above notation, define*

$$\begin{aligned} \Delta\left(f; u\right) = \frac{1}{4}&\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \left(\frac{T}{c}\right)^2\right)^2 \left(2\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \left(\frac{T}{c}\right)^2\right) \\ &- 27\left(\frac{D_{\mu\mu\mu}(0,1)}{6}\right)^2 - D_{\mu\mu\mu}(0,1)\left(\frac{3\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))}{2} - \frac{5\left(\frac{T}{c}\right)^2}{3}\right)\left(\frac{T}{c}\right) \end{aligned}$$

where f denotes the dependence on the nonlinearity in (5.1)*, and suppose that $D_{\mu\mu\mu}(0,1)$ is non-zero. If $\Delta > 0$, then the spectrum of the linearized operator $\mathcal{A}$ in a neighborhood of the origin consists of the imaginary axis with a triple covering. If $\Delta < 0$, then* spec$(\mathcal{A})$ *in a neighborhood of the origin consists of the imaginary axis with multiplicity* *one together with two curves which are tangent to lines through the origin.*

Proof. Since the leading order piece of the Evans function is homogeneous by Lemma 19, it seems natural to work with the projective coordinate $y = \frac{i\mu}{\kappa}$. Making such a change of variables, Lemma 19 implies the equation $D(\mu, e^{i\kappa}) = 0$ can be written as

$$1 + \frac{yT}{c} - \frac{y^2}{2}\left(\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) - \left(\frac{T}{c}\right)^2\right) + \frac{y^3}{6}D_{\mu\mu\mu}(0,1) + \kappa E(\kappa, y) = 0 \qquad (5.17)$$

where $E(\kappa, y)$ is continuous in a neighborhood of the origin. Let $y_{1,2,3}$ denote the three roots of the above cubic in y corresponding to $E(\kappa, y) = 0$. Assuming $\Delta \neq 0$ it follows that $y_{1,2,3}$ are distinct and hence the implicit function theorem applies giving three distinct solutions of (5.17) in a neighborhood of each of the $y_{1,2,3}$. In terms of the original variable μ, this gives three solution branches

$$\mu_{1,2,3} = -iy_{1,2,3}\kappa + \mathcal{O}(\kappa^2).$$

If $\Delta > 0$, then $y_{1,2,3} \in \mathbb{R}$, giving three branches of spectrum emerging from the origin tangent to the imaginary axis. From the Hamiltonian symmetry of (5.6), the spectrum is symmetric with respect to reflections across the imaginary axis and hence $\Delta > 0$ implies these thee branches of spectrum must in fact lie on the imaginary axis, proving the existence of an interval of spectrum of multiplicity three on the imaginary axis. In the case $\Delta < 0$, it follows that one of the roots, y_1 say, is real while the other two $y_{2,3}$ occur in a complex conjugate pairs, giving one branch along the imaginary axis and two branches emerging from the origin tangent to lines through the origin with angel $\arg(-iy_{2,3})$: see Figure 5.2. □

Remark 18. *The modulational instability index Δ derived above is considerably more complicated than the one derived in chapter 2 for generalized Korteweg-de Vries equation*

$$u_t = u_{xxx} - cu_x + \left(f(u)\right)_x. \qquad (5.18)$$

When considering the spectral stability of periodic traveling wave solutions of (5.18)*, it*

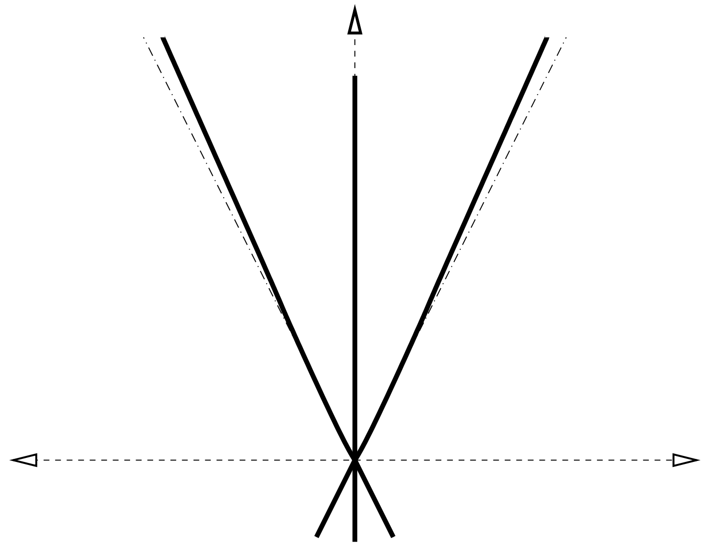

Figure 5.2: When $\Delta(f;u)<0$, the local normal form of spec($\mathcal{A}$) consists of a segment of the imaginary axis union with two straight lines making equal angles with the imaginary axis. Notice that these lines intersect at the origin, corresponding to the fact that 1 is an eigenvalue of $\mathbf{M}(0)$ with algebraic multiplicity three. In this picture, the horizontal dashed line represents the real axis, and the dashed lines which are tangent to he straight dark lines represent the true spectrum of the linearized operator.

was shown that there exists a modulational instability index Δ_{gKdV} *such that* $\Delta_{gKdV}<0$ *implies modulational instability, and* $\Delta_{gKdV}>0$ *implies modulational stability. In this case, the dominant balance is somewhat simpler due to the fact that the trace of the operator* $\mathbf{M}_\mu(0)$ *vanishes, and hence the* $\kappa^2\mu$ *term in the corresponding Newton diagram vanishes. As a result, the modulational instability index took the form*

$$\Delta_{gKdV}=\frac{1}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))\right)^3-\frac{1}{3}\left(D_{\mu\mu\mu}(0,1)\right)^2,$$

where here D *represents the Evans function for equation* (2.1)*. It follows that the sign of* $D_{\mu\mu\mu}(0,1)$ *does not effect the modulational instability of such periodic solutions of equation* (5.18)*. However, in the case of the gBBM equation, the fact that* $a'(0)\neq 0$ *seems to suggest that the sign of the sign of this orientation index (see next section) has an impact on the modulational stability of the periodic traveling wave. In fact, in section 4 we prove exactly this fact in the case of a power-nonlinearity* $f(u)=u^{p+1}/(p+1)$*: we prove the long-wavelength periodic traveling wave solutions of* (5.1) *are modulationally* *unstable if and only if the limiting homoclinic orbit (solitary wave) is exponentially unstable.*

Moreover, notice that if one (formally) sets the quantity $a'(0) = \frac{T}{c}$ *to zero in Theorem 17, the modulational instability index* Δ *reduces to that derived for the KdV in chapter 2, i.e. it reduces to* Δ_{gKdV}.

Our next goal is to use the integrable nature of (5.6) to express $D_{\mu\mu\mu}(0,1)$ and $\mathrm{tr}(\mathbf{M}_{\mu\mu}(0))$ in terms of the underlying periodic traveling wave u. This can be done very explicitly by using the Hamiltonian symmetry $\mathbf{M}(\mu) \sim \mathbf{M}(-\mu)^{-1}$ for equation (5.6). Notice that while we have chosen to express the coefficients of Δ in terms of $\mathrm{tr}(\mathbf{M}_{\mu\mu}(0))$ and $D_{\mu\mu\mu}(0,1)$, which suggests they arise at second and third order in a perturbation calculation for small μ, these quantities can be expressed in terms of quantities which arise at first and second order in μ due to the invariance of the problem under the map $\mu \to -\mu$, $x \to -x$. Moreover, it is interesting to note that while all first order terms contribute, only a few terms arising at second order actually contribute: these are terms associated with the minors of the off-diagonal piece of the unperturbed Jordan form. These second order terms are computable via a single quadrature, and can be found explicitly by a first order perturbation argument.

Theorem 18. *The following identities hold:*

$$D_{\mu\mu\mu}(0,1) = -3\{T,M,P\}_{a,E,c},$$
$$c\,\mathrm{tr}\,(\mathbf{M}_{\mu\mu}(0)) = c\left(\{T,P\}_{E,c} + 2\{M,P\}_{a,E} + \frac{2}{c}\,(M_aT - T_aM)\right) - 2V'(u_-)\{T,M\}_{a,E}.$$

where $\{g,h\}_{\alpha,\beta}$ *represents the Jacobian of the transformation* $(g,h) \mapsto (\alpha,\beta)$, *and* $\{g,h,r\}_{\alpha,\beta,\gamma}$ *is defined similarly, and* $M = M(a,E,c)$ *is the functional* $\int_0^T u\,dx$, *which represents the mass of the underlying periodic traveling wave.*

Remark 19. *The above formula for* $\mathrm{tr}(\mathbf{M}(0))$ *differs from that for the gKdV by the addition of the last two terms on the right hand side. The* $(M_aT - T_aM)$ *appears from the explicit dependence on c in the formula* $P = 2cK_c$, *while the* $V'(u_-)\{T,M\}_{a,E}$ *terms* *enters via the $V'(u_-)$ in the formula for w_1^1 below. Neither of these terms are required in the parallel modulational theory for the gKdV equation.*

Proof. Let $w_i(x;\mu)$, $i = 1, 2, 3$, be three linearly independent solutions of (5.7), and let $\mathbf{W}(x, \mu)$ be the solution matrix with columns w_i. Expanding the above solutions in powers of μ as

$$w_i(x, \mu) = w_i^0(x) + \mu w_i^1(x) + \mu^2 w_i^2(x) + \mathcal{O}(|\mu|^3)$$

and substituting them into (5.7), the leading order equation becomes

$$\frac{d}{dx} w_i^0(x) = \mathbf{H}(x, 0) w_i^0(x).$$

Using Proposition 10, we choose $w_i^0(x) = Y_i(x)$.The higher order terms in the above expansion yield

$$\frac{d}{dx} w_i^j(x) = \mathbf{H}(x, 0) w_i^j(x) + V_i^{j-1}(x), \quad j \geq 1, \tag{5.19}$$

where $V_i^{j-1} = \left(0, 0, -c^{-1}\mathcal{D}(w_i^{j-1})_1\right)^t$ and $(v)_1$ denotes the first component of the vector v. Notice that for each of the higher order terms $j \geq 1$, we require $w_j^i(0) = 0$. Notice this implies that $\mathbf{W}(0, \mu) = \mathbf{U}(0, 0)$ in a neighborhood of $\mu = 0$, where $\mathbf{U}(0, 0)$ is defined in (5.12). The solution of the inhomogeneous problem is given by the variation of parameters formula

$$w_i^j(x) = \mathbf{W}(x, 0) \int_0^x \mathbf{W}(s, 0)^{-1} V_i^{j-1}(s) ds \tag{5.20}$$

$$= \begin{pmatrix} cu_x \int_0^x \mathcal{D}(w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_a \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_E \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \\ cu_{xx} \int_0^x \mathcal{D}(w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_{ax} \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_{Ex} \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \\ cu_{xxx} \int_0^x \mathcal{D}(w_i^{j-1})_1 \{u, u_x\}_{a,E}\, dz - u_{axx} \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_{Exx} \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \end{pmatrix}$$

for $j \geq 1$. Notice we have used the identities $c\{u, u_x\}_{E,x} = -1$ and $c\{u, u_x\}_{x,a} = u$ extensively in the above formula, which can be easily derived via equation (5.4). Indeed, differentiating (5.4) with respect to E and subtracting $u_E u_{xx}$ immediately yields the first identity.

Now, notice that it would be a daunting task to use (5.20) to the specified order needed. However, the integrable structure of (5.6) allows for an alternative, yet equivalent, expression in the case $i = j = 1$ which makes a seemingly second order calculation come in at first order. Indeed, in this case equation (5.19) is equivalent to $L_0 w_1^1 = u_x$ and hence it follows from Proposition 10 that we can choose w_1^1 to be the function

$$\begin{pmatrix} -cu_c \\ -cu_c' \\ -cu_c'' \end{pmatrix} + \left(u_- + \frac{1}{c}V'(u_-;a,c)\right)\begin{pmatrix} cu_a \\ cu_a' \\ cu_a'' \end{pmatrix} - \left(\frac{u_-^2}{2} + \frac{1}{c}V'(u_-;a,c)\right)\begin{pmatrix} cu_E \\ cu_E' \\ cu_E'' \end{pmatrix}$$

where the above constants in front of Y_2 and Y_3 are determined by the requirement $w_1^1(0) = 0$. Thus, one can determine the second order variation of w_1 in μ by using (5.20) to compute the *first* order variation of the function w_1^1 defined above. Defining $\delta\,\mathbf{W}(\mu) := \mathbf{W}(x,\mu)\big|_{x=0}^{T}$, we see that $\delta\,\mathbf{W}(\mu)$ satisfies the asymptotic expansion

$$\begin{pmatrix} \mathcal{O}(\mu^2) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \\ \mu V'(u_-)P(u_-) + \mathcal{O}(\mu^2) & V'(u_-)T_a & V'(u_-)T_E \\ \mathcal{O}(\mu^2) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \end{pmatrix}$$

in a neighborhood of $\mu = 0$, where $P(x) = -T_c + \left(x + \frac{1}{c}V'(x)\right)T_a - \left(\frac{x^2}{2} + \frac{1}{c}V'(x)\right)T_E$ and the higher order terms are determined by (5.20), it follows that

$$D(\mu,1) = \det\left(\delta\,\mathbf{W}(\mu)\,\mathbf{W}(0,0)^{-1}\right) = \mathcal{O}(\mu^3).$$

Now, by a straightforward yet tedious calculation, we have that

$$\det\left(\delta\,\mathbf{W}(\mu)\right) = \frac{c}{2}\{T,M,P\}_{a,E,c}\,\mu^3 + \mathcal{O}(\mu^4).$$

Since $\det(\mathbf{W}(0,0)) = -c$, it follows that $D_{\mu\mu\mu}(0,1) = -3\{T,M,P\}_{a,E,c}$ as claimed.

Another straightforward yet tedious calculation yields

$$\begin{aligned}\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) &= -\frac{2}{\mu^2}\operatorname{tr}\left(\operatorname{cof}\left(\delta\,\mathbf{W}(\mu)\,\mathbf{W}(0,0)^{-1}\right)\right)\Big|_{\mu=0} \\ &= \{T,P\}_{E,c} + \{M,P\}_{a,E} + 2\{M,T\}_{a,c} - \frac{1}{c}\{T,M\}_{a,E}V'(u_-).\end{aligned}$$

By recalling that $T_c = \frac{1}{2}P_E + \frac{1}{c}T$ and $M_c = \frac{1}{2}P_a + \frac{1}{c}M$ gives

$$\{M,P\}_{a,E} + 2\{M,T\}_{a,c} = 2\{M,P\}_{a,E} + \frac{2}{c}\left(M_aT - T_aM\right)$$

which completes the proof. □

5.4.3 Analysis of spec $(\mathcal{A})\cap\mathbb{R}$: Orientation Index Calculation and Connection to Whitham Modulation Theory

In this section, we derive a finite wavelength instability index which counts mod 2 the number of unstable periodic eigenvalues in the open right half plane. Notice since the operator $\mathcal{D}^{-1}\partial_x$ is skew-adjoint and $\mathcal{L}$ is self-adjoint on $L^2(\mathbb{R})$, the number of unstable eigenvalues of $\mathcal{L}$ on the space $L^2_{\text{per}}([0,T])$ bounds above the number of unstable eigenvalues of $\mathcal{A}$ with positive real part (counting multiplicities). Since $\mathcal{L}$ is a Hill operator as in the case of the gKdV, it follows that

$$n(\mathcal{L}) = \begin{cases} 1, & \text{if } T_E > 0; \\ 2, & \text{if } T_E < 0. \end{cases}$$

Here, $n(\mathcal{L})$ represents the number of negative eigenvalues of $\mathcal{L}$ on the space $L^2_{\text{per}}([0,T])$. Thus, if $T_E > 0$ any unstable periodic eigenvalue of $\mathcal{A}$ must be real: in particular, if the nonlinearity f is such that $f(u)$ and u are both periodic with minimal period T, it follows from Lemma 10 from chapter 3 that $T_E > 0$. As a result, throughout this section we restrict ourselves to counting the number of *real* unstable periodic eigenvalues of the operator $\mathcal{A}$. Notice that since the set $\sigma(\mathcal{A})$ is continuous by Proposition (9), this immediately implies the existence of unstable spectrum supported outside a neighborhood of the origin. This is achieved by what amounts to an orientation index calculation, commonly used to study the stability properties of solitary waves. We begin with the following lemma.

Lemma 20. *The function* $D(\cdot,1)\big|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ *satisfies the following asymptotic relation:*

$$\lim_{\mathbb{R}\ni\mu\to\infty} D(\mu,1) = -\infty.$$

Proof. This follows from a simple calculation. By standard asymptotic arguments the monodromy operator satisfies

$$M(\mu) \sim \exp\left(A(\mu)T/c\right), \quad |\mu| \gg 1, \tag{5.21}$$

where

$$A(\mu) = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ -\mu & 0 & \mu \end{pmatrix}.$$

The eigenvalues λ_j of $A(\mu)$ are rather complicated, but for μ large and real they satisfy $\mathrm{Re}(\lambda_j) \sim a_j\mu$, where

$$\begin{aligned} a_1 &= 1 \\ a_2 = a_3 &= \frac{1}{3} - \frac{1}{3 2^{2/3} 2^{1/3}} - \frac{2^{1/3}}{6 2^{1/3}}. \end{aligned}$$

It follows that

$$a(\mu) \sim e^{a_1\mu T/c} + e^{a_2\mu T/2c} + e^{a_3\mu T/2c}$$

and thus, since $a_2 = a_3 < 0 < a_1 < c$, the leading order term of

$$D(\mu,1) = -1 + a(\mu) - a(-\mu)e^{\mu T} + e^{\mu T}$$

as $\mu \to +\infty$ comes from $-a(-\mu)e^{\mu T}$, which completes the proof. □

From this simple asymptotic relation, we have the following theorem guaranteeing the existence of unstable spectrum supported away from $\mu = 0$.

Theorem 19. *If $\{T, M, P\}_{a,E,c} < 0$, then the number of roots of $D(\mu, 1)$ (i.e. the number of periodic eigenvalues of $\mathcal{A}$) on the positive real axis is odd. In particular,* $\mathrm{spec}\,(\mathcal{A}) \cap \mathbb{R}^* \neq \emptyset$ *and the periodic traveling wave $u(x; a, E, c)$ is spectrally unstable.*

Proof. By our work in the proof of Lemma 19, we have that $D(0, 1) = D_\mu(0, 1) = D_{\mu\mu}(0, 1) = 0$ and hence $D(\mu, 1) = \mathcal{O}(|\mu|^3)$. Thus, if $D_{\mu\mu\mu}(0, 1) < 0$ for small positive μ, the number $D(\mu, 1)$ is negative for small positive μ. Since $D(\mu, 1)$ is positive for real μ sufficiently large, we know that $D(\pm\mu^*, 1) = 0$ for some $\mu^* \in \mathbb{R}^*$. The proof is now complete by Theorem 18. □

Corollary 9. *Suppose the nonlinearity is such that u and $f(u)$ are co-periodic of period T and suppose that $\{T, M, P\}_{a,E,c} \neq 0$ at this solution. Then the periodic traveling wave solution u of* (5.1) *is spectrally unstable to T periodic perturbations if and only if $\{T, M, P\}_{a,E,c} \neq 0$.*

We now wish to give insight into the meaning of $\{T, M, P\}_{a,E,c} = 0$ at the level of the linearized operator $\mathcal{A}$. To this end, we consider the linearized operator $\mathcal{A}$ as acting on $\mathcal{H} = L^2_{per}(0, T)$, the space of T-periodic L^2 functions on $\mathbb{R}$. To begin, we make the assumption that $\{T, M\}_{a,E}$ and $\{T, P\}_{a,E}$ do not simultaneously vanish. This assumption will be shown equivalent with the periodic null-space reflecting the structure of the monodromy at the origin (see Proposition 11). Throughout this brief discussion, we assume that $\{T, M\}_{a,E} \neq 0$: trivial modifications are needed if $\{T, M\}_{a,E}$ vanishes but $\{T, P\}_{a,E}$ does not. First, define the functions

$$\begin{aligned} \phi_0 &= \{T, u\}_{a,E}, & \psi_0 &= 1, \\ \phi_1 &= \{T, M\}_{a,E}\, u_x, & \psi_1 &= \int_0^x \mathcal{D}\phi_2(s)\,ds, \\ \phi_2 &= \{u, T, M\}_{a,E,c} & \psi_2 &= -\{T, M\}_{E,c} + \{T, M\}_{a,E}\mathcal{D}u, \end{aligned}$$

Clearly, each of these functions belong to $\mathcal{H}$ and

$$\begin{aligned} \mathcal{A}\phi_0 &= 0 & \mathcal{A}^\dagger\psi_0 &= 0 \\ \mathcal{A}\phi_1 &= 0 & \mathcal{A}^\dagger\psi_1 &= \psi_2 \\ \mathcal{A}\phi_2 &= -\phi_1 & \mathcal{A}^\dagger\psi_2 &= 0. \end{aligned}$$

In particular, we have used the fact that $\mathcal{D}^{-1}(1) = 1$ on $\mathcal{H}$. Thus, it follows that the periodic null space of $\mathcal{A}$ is generated spanned by the functions ϕ_0 and ϕ_1. Moreover, since

$$\langle \phi_0, \psi_0 \rangle = \{T, M\}_{a,E}$$
$$\langle \phi_0, \mathcal{D}u \rangle = \{T, P\}_{a,E}$$

the assumption that $\{T, M\}_{a,E}$ and $\{T, P\}_{a,E}$ do not simultaneously vanish implies that $N_{per}\left(\mathcal{A}^2\right) - N_{per}\left(\mathcal{A}\right) = \text{span}(\phi_2)$, thus reflecting the normal form of the period map (see Proposition 11).

Finally, we study the structure of the generalized periodic null space, and seek conditions for which there is no non-trivial Jordan chain of length two. By the Fredholm alternative, such a chain exists if and only if

$$\langle \mathcal{D}u, \phi_2 \rangle = \{T, M, P\}_{a,E,c} = 0.$$

Thus, the vanishing of $\{T, M, P\}_{a,E,c}$ is equivalent with a change in the generalized periodic-null space of the linearized operator $\mathcal{A}$. This insight has a nice relationship with formal Whitham modulation theory. One of the big ideas in Whitham theory is to locally parameterize the periodic traveling wave solution by the constants of motion for the PDE evolution. The non-vanishing of certain Jacobians is precisely what allows one to do this. In fact, the non-vanishing of $\{T, M, P\}_{a,E,c}$ is equivalent to demanding that, locally, the map $(a, E, c) \mapsto (T, M, P)$ have a unique C^1 inverse: In other words, the constants of motion for the gBBM flow are good local coordinates for the three-dimensional manifold of periodic traveling wave solutions (up to translation). Similarly, non-vanishing of $\{T,M\}_{a,E}$ and $\{T,P\}_{a,E}$ is equivalent to demanding that the matrix

$$\begin{pmatrix} T_a & M_a & P_a \\ T_E & M_E & P_E \end{pmatrix}$$

have full rank, which is equivalent to demanding that the map $(a,E)\mapsto(T,M,P)$ (for fixed c) have a unique C^1 inverse, i.e. two of the conserved quantities provide a smooth parametrization of the family of periodic traveling waves of fixed wave-speed.

To summarize, the vanishing of $\{T,M,P\}_{a,E,c}$, is connected with a change in the Jordan structure of the linearized operator $\mathcal{A}$ considered on $\mathcal{H}$. In particular, $\{T,M,P\}_{a,E,c}$ ensures the existence of a non-vanishing Jordan piece in the generalized periodic null-space of dimension *exactly* one. Moreover, and perhaps, more importantly, it guarantees that infinitesimal variations in the constants arising from reducing the family of periodic traveling waves to quadrature are enough to generate the entire generalized periodic null-space of the linearized operator $\mathcal{A}$: Such a condition is obviously necessary in our calculations. This suggests trying to develop a rigorous connection between Whitham modulation theory and the results of this paper. Notice that the above analysis of the action of $\mathcal{A}$ on $\mathcal{H}$ is the beginning of the development of a perturbation theory based on the Floquet-parameter - in essence, based on the Floquet-Bloch decomposition. While we have not yet carried out such an analysis, we believe it would be a useful and interesting calculation.

5.5 Analysis of Stability Indices in the Solitary Wave Limit

The goal of this section is to study the long-wavelength asymptotics of the stability indices derived in section 3. In particular, we restrict ourselves to the case of a power-nonlinearity $f(u)=u^{p+1}/(p+1)$. This restriction is vital to our calculation: in this case we gain an additional scaling symmetry. In particular, if $v(x;a,E)$ satisfies the differential equation

$$\frac{1}{2}v_x^2 - v^2 + \frac{1}{(p+1)(p+2)}v^{p+2} = au + E, \tag{5.22}$$

then a straight forward calculation shows that we can express the periodic solution $u(x;a,E,c)$ as

$$u(x;a,E,c) = (c-1)^{1/p} v\left(\left(\frac{c-1}{c}\right)^{1/2} x; \frac{a}{c^{1+1/p}}, \frac{E}{c^{1+2/p}}\right). \tag{5.23}$$

This additional scaling allows explicit calculations of P_c, which ends up determining the stability of periodic traveling wave solutions of (5.1) of sufficiently long wavelength.

A reasonable guess would be that long-wavelength periodic traveling wave solutions of (5.1) have the same stability properties as the limiting homoclinic orbit (solitary wave). However, as noted in the introduction, this is a highly singular limit and so it is not immediately clear whether such results are true. It is well known that the solitary wave is spectrally unstable if and only if $p > 4$ and $1 < c < c_0(p)$ for some critical wave speed $c_0(p)$. It follows from the work of Gardner into the stability of long-wavelength solutions of (5.1) that periodic waves sufficiently close to the homoclinic orbit are unstable if the solitary wave is unstable. In particular, it is proved that the linearized operator $\mathcal{A}$ for the periodic traveling wave u with sufficiently long wavelength has a “loop” of spectrum in the neighborhood of any unstable eigenvalues of the limiting solitary wave.

In terms of the finite-wavelength instability index, it seems reasonable by Theorem 19 to expect that for periodic traveling wave solutions of sufficiently long wavelength, $\{T,M,P\}_{a,E,c} < 0$ for if and only if $p > 4$ and $1 < c < c_0(p)$. What is unclear is whether such a result should be true for the modulational instability index Δ. Indeed, although Gardner’s results prove that the spectrum of the linearization about a periodic traveling wave of sufficiently long wavelength in the neighborhood of the origin contains the image of a continuous map of the unit circle, to our knowledge it has never been proved that this map is injective. Thus, it is not clear from Gardner’s results whether a modulational instability will arise from this eigenvalue since it is possible this “loop” is confined to the imaginary axis.

The following theorem proves a modulational instability does occur when ever the limiting solitary wave is unstable, and answers the corresponding question for the finite wavelength instability index. This theorem is based on asymptotic estimates of the instability indices derived in section 3.In particular, we prove the sign of both instability indices in the solitary wave limit is determined by the sign of $\frac{\partial}{\partial c}P$, where $P = P(a, E, c)$ is the momentum of the periodic wave $u(x; a, E, c)$. The proof is based on a more technical lemma, which shows that $\frac{\partial}{\partial a}M(a, E, c) < 0$ for waves of sufficiently long wavelength, i.e. for a, E sufficiently close to zero. The proofs of this lemma is given after our main theorem for this section, which is the following.

Theorem 20. *In the case of power non-linearity* $f(u) = u^{p+1}/(p+1)$*, there always exist unstable periodic traveling waves in a neighborhood of the solitary wave* $(a, E = 0)$ *if* $p > 4$ *and* $1 < c < c_0(p)$*, where* $c_0(p)$ *is the critical wavespeed determined by the non-linearity. Moreover, periodic traveling wave solutions to* (5.1) *of sufficiently long wavelength exhibit a modulational instability if and only if* $p > 4$ *and* $1 < c < c_0(p)$*.*

Proof. When a and E are sufficiently small, there exist two solutions $r_1 < r_2$ of $E = V(x; a, c)$ in a neighborhood of the origin, and a third solution r_3 bounded away from the origin. In the solitary wave limit $a, E \to 0$ a straight forward calculation gives that $r_2 - r_1 = \mathcal{O}\left(\sqrt{a^2 - 2(c-1)E}\right)$. As in chapter 2, we thus have the following asymptotics for small a and E:

$$\begin{aligned}
P(a,E,c) &= \mathcal{O}(1) \\
M(a,E,c) &= \mathcal{O}\left(a \ln\left(a^2 - 2(c-1)E\right)\right) \\
M_a(a,E,c) &= \mathcal{O}\left(\frac{a^2}{a^2 - 2(c-1)E}\right) \\
T(a,E,c) &= \mathcal{O}\left(\ln\left(a^2 - 2(c-1)E\right)\right) \\
T_a(a,E,c) &= \mathcal{O}\left(\frac{a}{a^2 - 2(c-1)E}\right) = M_E(a,E,c) \\
T_E(a,E,c) &= \mathcal{O}\left(\frac{1}{a^2 - 2(c-1)E}\right) \\
T_c(a,E,c) &= \mathcal{O}\left(\frac{E}{a^2 - 2(c-1)E}\right) = 2P_E + \frac{1}{c}T.
\end{aligned}$$

Thanks to the above scaling we know $M_c = 2P_a + \frac{1}{c}T$ can be expressed as a linear combination of M, M_a, and $M_E = T_a$. Similarly, P_c can be expressed as a linear combination of P, $2P_a = M_c - \frac{1}{c}T$ and P_E. It follows that the asymptotically largest minor of $\{T,M,P\}_{a,E,c}$ is $-T_E M_a P_c$ and, moreover,

$$\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) \sim T_E P_c.$$

Since $M_a < 0$ and $T_E > 0$ by Lemmas 21 and 10 for a and E sufficiently small, it follows that both stability indices are determined by the sign of $P_c(a,E,c)$ in the solitary wave limit. The theorem now follows by Lemma 22. □

Remark 20. *Notice that the fact that the finite wavelength instability index is determined by the sign of P_c in the solitary wave limit is not surprising, since this is exactly what detects the stability of solitary waves. What is surprising is that the same quantity controls the modulational stability index in the same limit. As mentioned above, it has not been know if the instability of the limiting solitary wave forces a modulational instability: the answer is shown to be affirmative by Theorem 20.*

In order to complete the proof Theorem 20, we must prove a few more technical lemmas. The first two are used in showing that the sign of the modulational and finite-wavelength instability indices are determined completely by the sign of $P_c(a,E,c)$ in the limit as a, E tend to zero.

Lemma 21. *For a, E sufficiently small, $M_a(a,E,c) \leq 0$ for all $c > 1$.*

Proof. Notice it is sufficient to prove $\frac{\partial}{\partial a} M(a,0,c) \leq 0$ for all $c > 1$ and a sufficiently small. Now, $M(a,0,c)$ can be written as

$$M(a,0,c) = \sqrt{2c} \int_0^{r(a,c)} \frac{\sqrt{u}\, du}{\sqrt{a + \frac{c-1}{2} u - \frac{1}{(p+1)(p+2)} u^{p+1}}}$$

where $r(a,c)$ is the smallest positive root of the polynomial equation $a + \frac{c-1}{2} r - \frac{1}{(p+1)(p+2)} r^{p+1} = 0$. Setting $a = ((p+1)(p+2))^{1/p} \alpha$, we have

$$M\left(\alpha, E, c\right) = ((p+1)(p+2))^{1/p} \sqrt{2c} \int_0^{\widetilde{r}(\alpha,c)} \frac{\sqrt{u}\, du}{\sqrt{\alpha + \frac{c-1}{2} u - u^{p+1}}}$$

where $\widetilde{r}(\alpha,c)$ is the smallest positive root of the polynomial $\alpha + \frac{c-1}{2} r - r^{p+1} = 0$. Notice for a fixed wave speed $c > 1$, $\widetilde{r}(\alpha,c)$ is a smooth function of α for α sufficiently small and satisfies

$$\widetilde{r}(\alpha,c) = \left(\frac{c-1}{2}\right)^{1/p} + \frac{2a}{p(c-1)} + \mathcal{O}(a^2).$$

The goal is to now rewrite the above integral over a fixed domain and show that the integrand is a decreasing function of α for a fixed $c > 1$.

Making the substitution $u \to \widetilde{r}(\alpha,c) u$ yields the expression

$$\frac{M(\alpha,0,c)}{((p+1)(p+2))^{1/p}\sqrt{2c}} = \int_0^1 \frac{\sqrt{u}\, du}{\sqrt{\alpha\, \widetilde{r}(\alpha,c)^{-3} + \left(\frac{c-1}{2}\right) u\, \widetilde{r}(\alpha,c)^{-2} - \widetilde{r}(\alpha,c)^{p-2}\, u^{p+1}}}$$

Now, as in chapter 2 we can use the above expansion of $r(a,c)$ to conclude that

$$\frac{\partial}{\partial \alpha}\Big|_{\alpha=0} \left(\alpha\, \widetilde{r}(\alpha,c)^{-3} + \left(\frac{c-1}{2}\right) u\, \widetilde{r}(\alpha,c)^{-2} - \widetilde{r}(\alpha,c)^{p-2}\, u^{p+1} \right)$$

is positive on the open interval $(0,1)$ for all $p \geq 1$ and $c > 1$, which completes the proof. $\square$

To complete the proof of Theorem 20, we must understand the asymptotic behavior for a fixed wavespeed $c > 1$ of the quantity $P_c(a, E, c)$ in the solitary wave limit. This is the content of the following lemma.

Lemma 22. *In the case of power non-linearity $f(u) = u^{p+1}/(p+1)$, we the momentum $P = P(a, E, c)$ satisfies*

$$\frac{\partial}{\partial c} P\left(a, E, c\right) = \frac{(c-1)^{2/p-1/2} c^{1/2} I\left(\frac{4}{p}\right)}{2pc(c-1)} \left(4c - p + \frac{(4c+p)(c-1)p}{(4+p)c}\right) + \mathcal{O}(|a| + |E|)$$

in the solitary wave limit $(a, E) \to (0, 0)$, where $I(r) = \int_{-\infty}^{\infty} \operatorname{sech}^r(x)dx$. In particular, for a and E sufficiently small, if $p < 4$ then $\frac{\partial}{\partial c} P\left(a, E, c\right) > 0$ for all $c > 1$ while if $p > 4$ then $\frac{\partial}{\partial c} P\left(a, E, c\right) < 0$ for $1 < c < c_0(p)$ and $\frac{\partial}{\partial c} P\left(u; a, E, c\right) > 0$ for $c > c_0(p)$, where

$$c_0(p) = \frac{p\left(1 + \sqrt{2 + \frac{1}{2}p}\right)}{4 + 2p}.$$

Proof. The proof is based on scaling and a limiting argument. To begin, let $v = v(x; a, E)$ satisfy the differential equation (5.22) so that $u(x; a, E, c)$ can be expressed via scaling as in (5.23), and assume with out loss of generality that $x = 0$ be an absolute max of $v(x; a, E, c)$. Clearly, the solitary wave limit corresponds to taking $(a, E) \to (0, 0)$ with fixed wave speed $c > 1$. Notice that on any compact subset Γ of $\mathbb{R}$, we have

$$v(x; a, E) \to \left(\frac{(p+2)(p+1)}{2}\right)^{1/p} \operatorname{sech}^{2/p}\left(\frac{p}{2}x + x_0\right)$$

uniformly as $(a, E) \to (0, 0)$ on Γ for some $x_0 \in \mathbb{R}$. Using (5.22), it follows that

$$\int_0^{T((c-1)/c)^{1/2}} v^2(x)dx = \sqrt{2} \int_{\tilde{u}_-}^{\tilde{u}_+} \frac{u^2\, du}{\sqrt{E + v^2 - \frac{1}{(p+1)(p+2)} v^{p+2} + av}}$$

where $\tilde{u}_\pm$ are the roots of $E + v^2 - \frac{1}{(p+1)(p+2)} v^{p+2} + av = 0$ satisfying the original hypothesis of the roots $u_\pm$ of $E - V(u; a, c) = 0$. Since $\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v^2(x)dx = \mathcal{O}(1)$ as $(a, E) \to (0, 0)$, the dominated convergence theorem along with the fact that

$\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v^2(x)dx$ is a C^1 function of a and E implies that

$$\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v(x;a,E)^2 dx = \left(\frac{(p+2)(p+1)}{2}\right)^{2/p} \frac{1}{p} I\left(\frac{4}{p}\right) + \mathcal{O}(|a|+|E|).$$

Similarly, it follows that

$$\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v_x(x;a,E)^2 dx = \left(\frac{(p+2)(p+1)}{2}\right)^{2/p} \frac{1}{4+p}\, I\left(\frac{4}{p}\right) + \mathcal{O}(|a|+|E|).$$

Using (5.23), we now have

$$\begin{aligned} p\left(\frac{(p+2)(p+1)}{2}\right)^{-2/p} &\int_{-T/2}^{T/2} \left(u(x;a,E,c)^2 + u_x(x;a,E,c)^2\right) dx \\ &= (c-1)^{2/p-1/2} c^{1/2} I\left(\frac{4}{p}\right) + (c-1)^{2/p+1/2} c^{-1/2} \frac{p}{4+p}\, I\left(\frac{4}{p}\right) \\ &+ \mathcal{O}(|a|+|E|) \end{aligned}$$

as $a, E \to 0$, and hence it follows by differentiation that

$$\begin{aligned} p\left(\frac{(p+2)(p+1)}{2}\right)^{-2/p} \frac{\partial}{\partial c} &\int_{-T/2}^{T/2} \left(u(x;a,E,c)^2 + u_x(x;a,E,c)^2\right) dx = \\ &\frac{(c-1)^{2/p-1/2} c^{1/2} I\left(\frac{4}{p}\right)}{2pc(c-1)} \left(4c - p + \frac{(4c+p)(c-1)p}{(4+p)c}\right) \\ &+ \mathcal{O}(|a|+|E|) \end{aligned}$$

as claimed, where we have used that $T_c u_-^2 = \mathcal{O}(|a|+|E|)$. The lemma now follows by solving the quadratic equation $(4+p)c(4c-p) + (4c+p)(c-1)p = 0$ for c and recalling the restriction that $c > 1$. $\square$

The proof of Theorem 20 is now complete by Lemmas 21, 10, and 22. As a consequence, the finite-wavelength instability index $\{T, M, P\}_{a,E,c}$ seems to be a somewhat natural generalization of the solitary wave stability index, in the sense that the well known stability properties of solitary waves are recovered in a long-wavelength limit. Moreover, this gives an extension of the results of Gardner in the case of generalized

BBM equation by proving the marginally stable eigenvalue of the solitary wave at the origin contributes to modulational instabilities of nearby periodic waves when ever the solitary wave is unstable.

5.6 Transverse Instabilities of gBBM

In this final section, we use the above Evans function techniques to derive a sufficient condition for the periodic gBBM traveling waves to be spectrally unstable to long-wavelength transverse perturbations. Suppose we have a periodic traveling wave solution $u(x; a, E, c)$ of (5.1) which is spectrally stable as a solution of the gBBM equation. We wish to examine the spectral stability of u to long-wavelength perturbations in the framework of the Zakharov-Kuznetsov-gBBM (ZK-gBBM) equation

$$u_t - (c-1)u_x + (f(u))_x + cu_{xxx} + (u_{xt} + u_{yy})_x = 0 \tag{5.24}$$

As u is a solution to (5.1), it is clearly a solution to (5.24) and hence it makes sense to discuss its spectral stability (in this section, spectral stability will refer to spectral stability in the ZK-gBBM model). Linearizing (5.24) around u yields

$$\mathcal{D}v_t = \partial_x \left(\mathcal{L} + \partial_y^2\right) v$$

where $\mathcal{D}$ and $\mathcal{L}$ are defined as before. We now seek separated solutions of the form

$$v(x, y, t) = v(x)e^{\mu t - iky}$$

where $\mu \in \mathbb{C}$ and $k \in \mathbb{R}$. This leads one to the (ordinary differential equation) spectral problem

$$\partial_x \left(\mathcal{L} + k^2\right) v = \mu \mathcal{D} v$$

on $L^2(\mathbb{R})$. Our goal is to study the spectrum of the above (non-local) operator on $L^2(\mathbb{R})$ near the origin $(\mu, k) = (0, 0)$. As before, the spectrum is purely continuous and consists of piecewise smooth arcs. In particular, for a given $k \in \mathbb{R}$, $\mu \in \text{spec}\left(\mathcal{D}^{-1}\partial_x\left(\mathcal{L}+k^2\right)\right)$ if and only if there exists a $\lambda \in S^1$ such that

$$D(\mu,k,\lambda) = \det\left(\mathbf{M}(\mu,k) - \lambda I\right) = 0$$

where $\mathbf{M}(\mu,k)$ is the monodromy map corresponding to the first order system

$$Y_x = \mathbf{H}(x,\mu,k)Y, \quad Y(0,\mu,k) = \mathbf{I}, \tag{5.25}$$

where

$$\mathbf{H}(x,\mu,k) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{c}\left(\mu + f''(u)u_x\right) & \frac{1}{c}\left(c-1+k^2-f'(u)\right) & 0 \end{pmatrix}.$$

We now state our main technical lemma for this section.

Lemma 23. *The equation $D(\mu,k,1) = 0$ has the following local normal form in a neighborhood of the origin $(\mu,k) = (0,0)$:*

$$-\frac{\mu^3}{2}\{T,M,P\}_{a,E,c} + 2\mu k^2\{T,M\}_{a,E}\int_0^T u_x^2\,dx + \mathcal{O}(4) = 0.$$

Proof. This proof is essentially the same as given in chapter 4. By Theorem 18, we need only compute the $\mathcal{O}(\mu k^2)$ term in the above expansion. Let $\mathbf{W}(x,\mu,k)$ be a matrix solution of (5.25) such that

$$\mathbf{W}(x,0,0) = \begin{pmatrix} cu_x & cu_a & cu_E \\ cu_{xx} & cu_{ax} & cu_{Ex} \\ cu_{xxx} & cu_{axx} & cu_{Exx} \end{pmatrix}.$$

The goal is to treat $\mathbf{W}(x,\mu,k)$ as a small perturbation of $W(x,0,0)$ for $|(\mu,k)|_{\mathbb{C}\times\mathbb{R}} \ll 1$. Following the proof of Theorem 18 we define $\delta\,\mathbf{W}(\mu,k) = \mathbf{W}(x,\mu,k)|_{x=0}^{T}$ and notice that by the form of $\delta\,\mathbf{W}(\mu,0)$ we need only compute the k^2 variation of the first column of $W(x,0,k)$. Using the variation of parameters formula (5.20) we have

$$\mathbf{W}(T,0,0)\int_0^T \mathbf{W}(z,0,0)^{-1}\begin{pmatrix}0\\0\\u_{xx}(z)\end{pmatrix}dz=\begin{pmatrix}c\frac{\partial u_-}{\partial E}\int_0^T u_x^2dx\\ *\\ (c-1-f'(u_-))\frac{\partial u_-}{\partial E}\int_0^T u_x^2dx\end{pmatrix},$$

where the term $*$ is can be explicitly computed, but is not necessary at this order in the perturbation argument. Therefore, a straightforward calculation gives

$$\frac{1}{\mu k^2}\det\left(\delta\,\mathbf{W}(\mu,k)\right)\big|_{(\mu,k)=(0,0)}=-c\{T,M\}_{a,E}\int_0^T u_x^2dx.$$

Since $\det(\mathbf{W}(x,0,0))=-c$, this completes the proof. □

Lemma 23 readily yields a necessary condition for the underlying (gBBM) periodic traveling wave $u(x;a,E,c)$ to exhibit a modulational transverse instability in the ZK-gBBM model.

Theorem 21. *If $\{T,M,P\}_{a,E,c}\neq 0$, then the spectrally stable periodic (gBBM) traveling wave $u(x;a,E,c)$ is spectrally unstable to long-wavelength transverse perturbations in the ZK-gBBM model if $\{T,M\}_{a,E}>0$.*

Proof. By Lemma 23, there are three periodic eigenvalues in a neighborhood of the origin which are given by $\mu_0=o(k)$ and

$$\mu_\pm=\pm|k|\sqrt{\frac{4\{T,M\}_{a,E}\int_0^T u_x^2dx}{\{T,M,P\}_{a,E,c}}}+o(k)$$

Since $u(x;a,E,c)$ was assumed to be a spectrally stable solution of (5.1), we know from Theorem 19 that $\{T,M,P\}_{a,E,c}>0$ and hence there will be two (non-zero) periodic eigenvalues off the imaginary axis in the neighborhood of the origin if $\{T,M\}_{a,E}>0$. □

Notice that since $\{T,M\}_{a,E}=M_E^2-T_EM_a$, it follows that $T_EM_a<0$ is a sufficient condition for u to be spectrally unstable to such long-wavelength transverse perturbations in the ZK-gBBM model. By our analysis in Section 4 it follows that for power-law non-linearities, periodic traveling wave solutions of (5.1) sufficiently close to the homoclinic orbit are transversely unstable in the ZK-gBBM model. Moreover, we have the following stronger result in the case of the BBM equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \tag{5.26}$$

Theorem 22. *Let $u(x; a_0, E_0, c_0)$ be a spectrally stable periodic traveling wave solution of the BBM equation* (5.26)*. Then u is transversely unstable as a solution of the ZK-gBBM equation if $\{T, M, P\}_{a,E,c}$ is non-zero at (a_0, E_0, c_0).*

Indeed, this holds by noticing that equation (5.26) is Galilean invariant, and hence the proof of Lemma 17 implies[3] that $M_a < 0$ for all non-equilibrium point solutions.

5.7 Stability Analysis of the Generalized Camassa-Holm Equation

We now briefly explain how the methods of this paper can be applied to the Generalized Camassa-Holm equation (5.2). As mentioned in the introduction, this equation admits a four parameter family of traveling wave solutions of the form $u(x, t) = u(x - ct)$, which are stationary solutions of the equation

$$u_t - u_{xxt} = 2u_x u_{xx} + (u - c)u_{xxx} - (f(u)/2)_x + (c - k)u_x. \tag{5.27}$$

Searching for stationary solutions of (5.27) leads to the traveling wave ordinary differential equation

$$(2u_x - c)u_{xx} + uu_{xxx} - (f(u)/2)_x + (c - k)u_x = 0. \tag{5.28}$$

[3]The only difference is that we can't scale out the wave speed parameter c. However, since we have the requirement that $c > 1$ the proof still holds verbatim.

This equation can easily be integrated to quadrature, and is seen to satisfy the relations

$$\frac{1}{2}u_x^2 + uu_{xx} - \frac{1}{2}f(u) - (k-c)u - cu_{xx} = \frac{a}{2} \tag{5.29}$$

$$(c-u)u_x^2 = E - F(u) - (k-c)u^2 - au \tag{5.30}$$

In order to avoid technical issues, we assume we can choose (a, E, c) such that $u(x; a, E, c)$ is a periodic solution of (5.30) satisfying $\|u\|_{L^\infty(\mathbb{R})} < c$. Defining the effective potential to be

$$V(u; a, c) = F(u) + (k-c)u^2 + au$$

it follows that (5.30) can be rewritten as $(c-u)u_x^2 = E - V(u; a, c)$.

Linearizing (5.27) about the stationary periodic solution u and taking the Laplace transform in time leads to a spectral problem of the form $\partial_x \mathcal{K} v = \mu \mathcal{D} v$ considered on $L^2(\mathbb{R})$, where

$$\mathcal{K} = u_{xx} + \partial_x\left(u-c\right)\partial_x - \frac{1}{2}f'(u) - (k-c), \quad \text{and} \quad \mathcal{D} = 1 - \partial_x^2.$$

As before, the $L^2(\mathbb{R})$ spectrum of the operator $\mathcal{A} = \mathcal{D}^{-1}\partial_x\mathcal{K}$ is purely continuous and consists of piecewise smooth arcs. Moreover, $\mu \in \text{spec}(\mathcal{A})$ if and only if there exists a $\lambda \in S^1$ such that

$$D(\mu, \lambda) = \det\left(\mathbf{M}(\mu) - \lambda\mathbf{I}\right) = 0,$$

where $\mathbf{M}(\mu)$ is the corresponding monodromy operator associated to the first order system

$$Y' = \mathbf{H}(x, \mu)Y, \;\; Y(0, \mu) = \mathbf{I}$$

where the matrix $\mathbf{H}(x, \mu)$ can be determined through the usual procedure. As before, the vector valued functions

$$Y_1(x) = \begin{pmatrix} u_x \\ u_{xx} \\ u_{xxx} \end{pmatrix}, \; Y_2(x) = \begin{pmatrix} u_a \\ u_{ax} \\ u_{axx} \end{pmatrix}, \; Y_3(x) = \begin{pmatrix} u_E \\ u_{Ex} \\ u_{Exx} \end{pmatrix}$$

satisfy $Y' = \mathbf{H}(x,0)Y$, where we have suppressed the dependence on $(x;a,E,c)$, and the vector $Y_4(x) = [u_c, u_{cx}, u_{cxx}]^t$ solves the inhomogeneous problem

$$Y' = \mathbf{H}(x,0)Y + W$$

where $W = [0,0,-(c-u)^{-1}\mathcal{D}u_x]^t$. A straight forward calculation in the spirit of that in section 3.1 implies that

$$\mathbf{M}(0) \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix}$$

for some σ which vanishes if and only if $D_{a,E}T$ vanishes. Thus, the equation $D(\mu,\lambda)$ has an expansion as in (5.14) and, in particular we see that $D_\mu(0,1) = 0$. Moreover, using the fact that $\mathcal{D}$ and $\mathcal{K}$ are self adjoint, we know that $\mathbf{M}(\mu) \sim \mathbf{M}(-\mu)^{-1}$. Thus, we immediately have the following lemma which is the analogue of Lemma 19 for the generalized Camassa-Holm equation.

Lemma 24. *If $D_{\mu\mu\mu}(0,1) \neq 0$, the equation $D(\mu, e^{i\kappa}) = 0$ has the following normal form in a neighborhood of $(\mu,\kappa) = (0,0)$:*

$$-(i\kappa)^3 + (i\kappa)^2\mu\omega'(0) + \frac{i\mu^2\kappa}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \omega'(0)^2\right) + \frac{\mu^3}{6}D_{\mu\mu\mu}(0,1) + \mathcal{O}(4) = 0,$$

where $\omega(\mu) = \int_0^T \operatorname{tr}(\mathbf{H}(x,\mu))\,dx$ and $\mathcal{O}(4)$ represents terms of order four and higher in the variables κ and μ. In particular, there are three branches spectrum bifurcating from the $\mu = 0$ state which are analytic functions of κ in a neighborhood of the origin.

Proof. As in the proof of Lemma 19, we begin by defining a function b in a neighborhood of $\mu = 0$ such that

$$\det\left(\mathbf{M}(\mu) - e^{i\kappa}\mathbf{I}\right) = -\eta^3 + a(\mu)\eta^2 + b(\mu)\eta + D(\mu,1)$$

where $a(\mu) = \operatorname{tr}(\mathbf{M}(\mu))$ and $\eta = e^{i\kappa} - 1$. Since the spectral problem $\mathcal{A}v = \mu v$ is invariant under the transformation $(x,\mu) \to (-x,-\mu)$, it follows that $\mathbf{M}(\mu) \sim \mathbf{M}(-\mu)^{-1}$. As a consequence, we see that $D_{\mu\mu}(0,1) = 0$ and hence $D(\mu,1) = \mathcal{O}(|\mu|^3)$ near $\mu = 0$. Moreover, defining $\omega(\mu) = \int_0^T \operatorname{tr}(H(x,\mu))\,dx$ we see that

$$\begin{aligned} e^{-\omega(\mu)}D(\mu,\lambda) = &-(\lambda-1)^3 - (a(-\mu)-3)\,\lambda\,(\lambda-1)^2 \\ &+ b(-\mu)\lambda^2(\lambda-1) - \lambda^3 D(-\mu,1) \end{aligned}$$

This immediately implies the relations

$$\begin{cases} e^{-\omega(\mu)}a(\mu) = 2a(-\mu) - b(-\mu) - 3, & \text{and} \\ -e^{-\omega(\mu)} = -a(-\mu) + b(-\mu) - D(-\mu,1) + 2. \end{cases}$$

Since $\omega(0) = 0$, $\omega'(0) \neq 0$, and $\partial_\mu^k \omega(\mu)|_{\mu=0}$ for $k \geq 2$, it follows as before that $b'(0) = 0$, $a'(0) = \omega'(0)$, and $b''(0) = \operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \omega'(0)^2$. $\square$

Therefore, by defining the variable $y = \frac{i\mu}{\kappa}$, it follows that the equation $D(\mu, e^{i\kappa}) = 0$ can be written as

$$1 + y\omega'(0) - \frac{y^2}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \omega'(0)^2\right) + \frac{y^3}{6}D_{\mu\mu\mu}(0,1) + \kappa E(\kappa, y)$$

where $E(\kappa, y)$ is continuous in a neighborhood of the origin. It follows that the discriminant of this polynomial determines the local structure of the spectrum in a neighborhood of the origin, just as in Theorem 17. By using variation of parameters, we can determine closed form expressions for the quantities $\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))$ and $D_{\mu\mu\mu}(0,1)$ and thus have a modulational instability theory for the generalize Camassa-Holm equation in spirit of that for the generalized Benjamin-Bona-Mahony equation and the generalized Korteweg-de Vries equation from chapter 2.

APPENDIX

In this appendix, we review the constructions of the Newton diagrams used throughout this text. This construction is quite general, and while equivalent to the method of dominant balance introduced in most asymptotic methods courses, we believe this to be somewhat more systematic. For details see the book of Baumgartel[6] or Hilton[37].

To begin, suppose we have a function of two complex variables of the form

$$p(\lambda, z) = a_0(z)\lambda^n + a_1(z)\lambda^{n-1} + a_2(z)\lambda^{n-2} + ... + a_{n-1}(z)\lambda + a_n(z) \tag{.1}$$

where the functions $a_j(z)$ are analytic at $z = 0$. Moreover, we assume that $a_0(0) \neq 0$ and $a_j(0) = 0$ for $j = 1, 2, ..., n$ so that $\lambda = 0$ is an n-fold root of the polynomial $p(\lambda, 0)$. Our goal is to study the roots of the equation $p(\lambda, z) = 0$ in a neighborhood of $(0, 0)$ and to understand the possible bifurcations of the n-roots from the $(\lambda, z) = (0, 0)$ state[1]. By the analyticity of the a_j in a neighborhood of $z = 0$, we can find non-zero constants $b_j \in \mathbb{R}$ and $\beta_j \in \mathbb{N} \cup \{0\}$ such that the expansions

$$a_j(z) = b_j z^{\beta_j} + o\left(z^{\beta_j}\right) \tag{.2}$$

holds for $|z| \ll 1$ for each $j = 0, 1, ..., n$. In particular, notice our above assumptions force $\beta_0 = 0$ and $\beta_j \geq 1$ for $j = 1, 2, ..., n$. We now make the ansatz

$$\lambda = \epsilon z^{\alpha} + o\left(z^{\alpha}\right) \tag{.3}$$

[1]In this appendix, we restrict ourselves to determining the roots to*first order* only. Higher order expansions can be dealt with in similar means: see the above references for more information.

where ϵ is non-zero and $\alpha > 0$. Substituting (.2) and (.3) into the equation $p(\lambda, z) = 0$ and grouping terms by powers of z, it is clear that we must require that the coefficients of each power of z must vanish. In particular, the coefficient of the lowest power of z must vanish. By the form of (.1), this lowest power of z must be among one of the following terms:

$$n\alpha,\ (n-1)\alpha + \beta_1,\ (n-2)\alpha + \beta_2,\ \dots\ ,\ \beta_n. \tag{.4}$$

Clearly, in order for the lowest of these terms to vanish, we must determine a value of α such that at least one of the numbers in (.4) must occur twice. In order to determine such values of α, notice that if two of the numbers from the list (.4) are equal, there must be two distinct non-negative integers k_1 and k_2 such that

$$\beta_{k_1} + (n - k_1)\alpha = \beta_{k_2} + (n - k_2)\alpha,$$

which happens when

$$\alpha = \frac{\beta_{k_1} - \beta_{k_2}}{k_1 - k_2}.$$

To understand this condition, we consider the points $A_j = (j, \beta_j)$ on the non-negative integer lattice $\mathbb{Z}^+ \times \mathbb{Z}^+$. In the case when a particular a_j vanishes identically, we omit the corresponding point A_j. It follows that for two of the numbers in (.4) to occur twice, we should choose α to be among the slopes between all pairs of points A_{k_1} and A_{k_2} with $k_1, k_2 = 0, 1, ..., n$ distinct.

However, recall that we want the two numbers in (.4) which coincide to be as small as possible. This adds yet another constraint to α. In particular, suppose that the point A_{k_3} lies below the line containing the points A_0 and A_{k_2}. Then clearly

$$\frac{\beta_{k_3} - \beta_0}{k_3 - 0} < \frac{\beta_{k_2} - \beta_0}{k_2 - 0},$$

i.e. the line between A_0 and A_{k_3} lies *below* the line between A_0 and A_{k_2}. Thus, we see that none of the points A_k should lie below the line which corresponds to the numbers $\beta_0 + n\alpha$ and $\beta_r + (r - j)\alpha$ which coincide. By restarting the argument again at the point

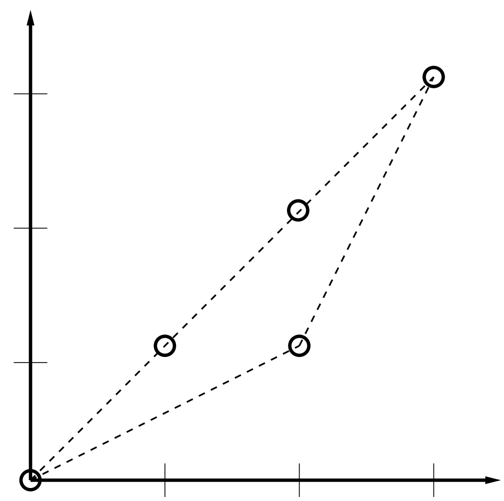

Figure .1: The Newton diagram corresponding to equation (.5). Notice that in this picture, the point $(2,2)$ should technically not be listed, but its presence does not change the lower convex hull.

A_r in place of A_0, it follows that the straight lines with admissible slopes α we should consider are those which form the lower convex hull of the points A_j for $j = 0, 1, ..., n$. For each such α, we have at least one solution of the form (.3). As a matter of fact, the number of solutions of form (.3) for a particular admissible α equals the length of the projection of the line of slope α from the Newton diagram onto the vertical axis, i.e. its rise in height. Moreover, the equation expressing the vanishing of the power of z in (.1) of smallest power yields a polynomial equation which determines the coefficients ϵ in (.3).

As an elementary example, consider the equation

$$5z^3 - (z + 2z^3)\lambda + z\lambda^2 + \lambda^3 = 0. \tag{.5}$$

Clearly, when $z = 0$ the point $\lambda = 0$ is root of algebraic multiplicity three. To determine the manner in which these roots bifurcate for small $|z|$, we use the above construction. From the corresponding Newton diagram, see Figure , it follows that there are two admissible α's for this problem: $\alpha = \frac{1}{2}$ and $\alpha = 2$. The slope $\alpha = \frac{1}{2}$ corresponds to the points $(0,0)$ and $(2,1)$ on the Newton diagram, which from the specific form of the left hand side of (.5) corresponds to the equation

$$\lambda^3 - z\lambda = 0.$$

Making the substitution $\lambda = \epsilon z^{1/2} + o(z^{1/2})$, as motivated by the above analysis, yields the equation

$$z^{3/2}\left(1 - \epsilon\right) = 0$$

and hence one solution of (.5) is of the form $\lambda = z^{1/2} + o(z^{1/2})$. Similarly, the slope $\alpha = 2$ corresponds to the equation

$$-z\lambda + 5z^3 = 0$$

Making the substitution $\lambda = \epsilon z^2 + o(z^2)$ yields the equation

$$z^3\left(-1 + 5\epsilon^2\right) = 0$$

from which we conclude there are two roots of (.5) near the origin of the form

$$\lambda_{\pm} = \pm\frac{z^2}{\sqrt{5}} + o(z^2).$$

Notice that if the term in (.5) corresponding to the point $(2, 2)$ in the Newton diagram vanished, then there would be only one admissible value of α: namely $\alpha = 1$. It follows that three roots bifurcate from the origin with expansions $\lambda_j = a_j z + o(z)$, where the coefficients a_j are determined by the roots of a cubic polynomial. This is the case which occurs in our analysis throughout this thesis.

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