ON THE STABILITY OF PERIODIC SOLUTIONS OF THE GENERALIZED BENJAMIN-BONA-MAHONY EQUATION

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Abstract. We study the stability of a four parameter family of spatially periodic traveling wave solutions of the generalized Benjamin-Bona-Mahony equation to two classes of perturbations: periodic perturbations with the same periodic structure as the underlying wave, and long-wavelength localized perturbations. In particular, we derive necessary conditions for spectral instability to perturbations to both classes of perturbations by deriving appropriate asymptotic expansions of the periodic Evans function, and we outline a nonlinear stability theory to periodic perturbations based on variational methods which effectively extends our periodic spectral stability results.

1. Introduction

In the area of nonlinear dispersive waves, the question of stability is of fundamental importance as it determines those solutions which are most likely to be observed in physical applications. In this paper, we consider the stability of traveling wave solutions of the generalized Benjamin-Bona-Mahony (gBBM) equation

$$u_t - u_{xxt} + u_x + f(u)_x = 0, \tag{1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $f(\cdot) \in C^2(\mathbb{R})$ is a prescribed nonlinearity. In particular, we will be most interested in the case of a power-law nonlinearity $f(u) = u^{p+1}$ as this is when our results are most explicit. Notice that when $f(u) = u^2$, (1) is precisely the Benjamin-Bona-Mahony equation (BBM), or the regularized long-wave equation, which arises as an alternative model to the well known Korteweg-de Vries equation (KdV)

$$u_t - u_{xxx} + uu_x = 0$$

as a description of gravity water waves in the long-wave regime (see [7, 27]). In applications, various other nonlinearities arise which facilitates our consideration of the generalized BBM equation. For suitable nonlinearities, equation (1) admits traveling wave solutions of the form $u(x,t) = u(x-ct)$ with wave speed $c > 1$ which are are either periodic or asymptotically constant. Solutions which are asymptotically constant are known as the solitary waves, and they correspond to either homoclinic or heteroclinic orbits of the traveling wave ODE

$$cu_{xxx} - (c-1)u_x + (f(u))_x = 0 \tag{2}$$

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obtained from substituting the traveling wave ansatz into (1). The stability of such solutions to localized, i.e. $L^2(\mathbb{R})$, perturbations is well known [8, 26, 29]: the solitary waves form a one parameter family of traveling wave solutions of the gBBM which can be indexed by the wave speed $c > 1$. A given solitary wave solution $u_{c_0}(x - c_0 t)$ of (1) is nonlinearly (orbitally) stable if the so called momentum functional

$$\mathcal{N}(c) := \frac{1}{2}\int_{\mathbb{R}} \left(u_c(x)^2 + u_c'(x)^2\right) dx$$

is an increasing function at c_0, i.e. if $\frac{d}{dc}\mathcal{N}(c)|_{c=c_0} > 0$, and is exponentially unstable if $\frac{d}{dc}\mathcal{N}(c)|_{c=c_0}$ is negative. In the special case of a power-nonlinearity $f(u) = u^{p+1}$, it follows that such waves are nonlinearly stable if $1 \leq p \leq 4$, while for $p > 4$ there exists a critical wavespeed $c(p)$ such that the waves are nonlinearly unstable for $1 < c < c(p)$ and stable for $c > c(p)$.

In [26], Pego and Weinstein found the mechanism for the instability of the solitary waves to be as follows: linearizing the traveling gBBM equation

$$u_t - u_{xxt} + cu_{xxx} - (c-1)u_x + f(u)_x = 0 \tag{3}$$

about a given solitary wave solution u_c of (2) and taking the Laplace transform in time yields a spectral problem of the form $\partial_x \mathcal{L}[u_c]v = \mu v$ considered on the real Hilbert space $L^2(\mathbb{R})$. The authors then make a detailed analysis of the Evans function $D(\mu)$, which plays the role of a transmission coefficient familiar from quantum scattering theory: in particular, $D(\mu)$ measures intersections of the unstable manifold as $x \to -\infty$ and the stable manifold as $x \to +\infty$ of the traveling wave ODE. As a result, if $\mathrm{Re}(\mu) > 0$ and $D(\mu) = 0$, it follows that the spectral problem $\partial_x \mathcal{L}[u_c]v = \mu v$ has a non-trivial $L^2(\mathbb{R})$ solution, and hence μ belongs to the point spectrum of the linearized operator[1]. Using this machinery, Pego and Weinstein were able to prove that $\mathrm{sign}(D(\mu)) = +1$ as $\mu \to +\infty$ and

$$D(\mu) = C\left(\frac{d}{dc}\mathcal{N}(\omega)\big|_{\omega=c}\right)\mu^2 + \mathcal{O}(|\mu|^3)$$

for $|\mu| \ll 1$ and some constant $C > 0$. Thus, if $\mathcal{N}(c)$ is decreasing at c, then by continuity there must exist a real $\mu^* > 0$ such that $D(\mu^*) = 0$, which proves exponential instability of the underlying traveling wave.

The focus of the present work concerns the stability properties of spatially periodic traveling wave solutions of (1) and in contrast to its solitary wave counterpart, relatively little is known in this context. Most results in the periodic case falls into one of the following two categories: spectral stability with respect to localized or bounded perturbations [9, 11, 21, 20], and nonlinear (orbital) stability with respect to periodic perturbations[2, 4, 12, 16, 22]. The spectral stability results rely on a detailed analysis of the spectrum of the linearized operator in a given Hilbert space representing the class of admissible perturbations: in our case, we will consider both $L^2(\mathbb{R})$, corresponding to localized perturbations, and $L^2_{\mathrm{per}}([0,T])$, corresponding to T-periodic perturbations where T is the period of the underlying wave.

[1] By a standard argument, the essential spectrum can be shown to lie on the imaginary axis, and hence any spectral instability must come from the discrete spectrum.

Notice that unstable spectrum in $L^2_{\text{per}}([0,T])$ represents a high-frequency instability and hence manifests itself in the short time instability (local well-posedness) of the underlying solution, while unstable spectrum near the origin corresponds to instability to long wavelength (low-frequency) perturbations, or slow modulations, and hence manifests itself in the long time instability (global well-posedness) of the underlying solution. Notice that on a mathematical level, the origin in the spectral plane is distinguished by the fact that the traveling wave ordinary differential equation (2) is completely integrable. Thus, the tangent space to the manifold of periodic traveling wave solutions can be explicitly computed, and the null space of the linearized operator can be built up out of a basis for this tangent space.

Our analysis of the linearized spectral problem on $L^2_{\text{per}}([0,T])$ parallels that of the solitary wave theory described above, in that we compare the large real μ behavior of the corresponding periodic Evans function to the local behavior near the origin, thus deriving a sufficient condition for instability. The stability analysis to arbitrary localized perturbations is more delicate and follows the general modulational theory techniques of Bronski and Johnson [11]: unlike the solitary wave case, the spectrum of the linearized operator about a periodic wave has purely continuous L^2 spectrum and hence any spectral instability must come from the essential spectrum. As a result, there are relatively few results in this case. In the well known work of Gardner [17], it is shown that periodic traveling wave solutions of (1) of sufficiently long wavelength are exponentially unstable whenever the limiting homoclinic orbit (solitary wave) is unstable. The mechanism behind this instability is the existence of a "loop" of spectrum in the neighborhood of any unstable eigenvalue of the limiting solitary wave. More recently, Hărăguş [20] carried out a detailed spectral stability analysis in the case of power-nonlinearity $f(u) = u^{p+1}/(p+1)$ for waves sufficiently close to the constant state $u = ((p+1)(c-1))^{1/p}$ determining when the L^2 spectrum of the linearized operator is confined to the imaginary axis. We, on the other hand, consider arbitrary periodic traveling waves with essentially arbitrary nonlinearity: as a result of this level of generality we are unable to make conclusive spectral stability statements but instead can only determine L^2-spectral stability near the origin: this is determined by an asymptotic analysis of the periodic Evans function near the origin resulting in a modulational instability index which determines the local normal form of the spectrum. However, we find that this analysis yields quite a bit of information about the spectrum of the underlying wave.

The orbital stability results rely on the now familiar energy functional techniques or Grillakis, Shatah, and Strauss [18] used in the study of solitary type solutions. This program has recently been carried out by Johnson [22] in the case of periodic traveling waves of the generalized Korteweg-de Vries equation. The corresponding theory for the gBBM equation is nearly identical to the analysis in [22], and hence will only be outlined in this work.

Finally, we study the behavior of our modulational and orbital stability indices in a long-wavelength limit in the case of a power-nonlinearity $f(u) = u^{p+1}/(p+1)$. For such nonlinearities, we gain an additional scaling in the wave speed which allows explicit calculations of the leading order terms in this stability index. A naive guess would be that the value of the orientation index would converge to the solitary wave stability index. This is not the case, however, since the convergence of long-wavelength periodic waves to solitary waves is non-uniform, implying such a limit is highly singular. What is true is that the *sign* of the finite-wavelength instability index converges to the *sign* of the solitary wave stability index. Since this index was derived from an orientation index calculation, only its sign matters and thus we show that this stability index is in some sense the correct generalization of the stability index used in the study of solitary waves of (1). Moreover, the sign of the modulational stability index converges to the same sign as the solitary wave index, implying that periodic waves of (1) in a neighborhood of a solitary wave are modulationally unstable if and only if the nearby solitary wave unstable. Notice this does not follow directly from the work of Gardner described above: there it is proved that there exists a "loop" of spectrum in the neighborhood of the origin which is the it continuous image of a circle, but to our knowledge it has never been proved that this map is injective.

The outline for this paper is as follows. In section 2 we review the basic properties of the periodic traveling wave solutions of (1), and in section 3 we review the basic properties of the periodic Evans function utilized throughout this work. In section 4, we begin our analysis by considering stability of the T-periodic traveling wave to T-periodic perturbations. In particular, we first determine the orientation index, which provides sufficient information for spectral instability to such perturbations, and then discuss how this index plays into the nonlinear stability theory. In section 5, we conduct our modulational instability analysis. In section 6 we analyze the results of sections 4 and 5 in a solitary wave limit, thus extending the well known results of Garnder. Finally, in section 7 we consider analyze our results in the well-studied cases of the BBM and modified BBM equations and we close in section 8 with a brief discussion and closing remarks.

2. Properties of the Periodic Traveling Waves

In this section, we describe the basic properties of the periodic traveling waves of the gBBM equation (1). These results are similar to those for the generalized Korteweg-de Vries equation derived in [11]: the main difference being the role of the wave speed c and the structure of the conserved quantities (see (5) below). For each $c > 1$, a traveling wave is a solution of the traveling wave ODE (2), i.e. they are stationary solutions of (1) in a moving coordinate frame defined by $x - ct$. Clearly, after one integration, (2) defines a Hamiltonian ODE and can be reduced to quadrature: in particular, the traveling waves satisfy the relations

$$\begin{aligned} cu_{xx} - (c-1)u + f(u) &= a \\ \frac{c}{2}u_x^2 - \left(\frac{c-1}{2}\right)u^2 + F(u) &= au + E \end{aligned} \tag{4}$$

where a and E are real constants of integration and $F' = f$ with $F(0) = 0$. Notice that the solitary waves correspond to $a = 0$ and E fixed by the asymptotic values of the solution. In the periodic context, however, the parameters a and E are free parameters: by elementary phase plane analysis we need only require that the effective potential

$$V(u; a, c) := F(u) - \frac{c-1}{2}u^2 - au$$

have a non-degenerate local minimum (see Figure 1). Note that this places restrictions on the allowable parameter regime for our problem. Indeed, there exists an open set $\Omega \subset \mathbb{R}^3$ such that (2) has at least one periodic solution if and only if that $(a, E, c) \in \Omega$. Throughout our analysis, we will assume we are in the interior of this region, and that the roots $u_\pm$ of the equation $V(x; a, c) = E$ are simple and such that $V(x; a, c) < E$ for $x \in (u_-, u_+)$. In particular, this guarantees the classical "turning points" $u_\pm$ are C^1 functions of the parameters a, E, and c. Thus, the periodic solutions of (2) form a four parameter family of solutions $u(x + x_0; a, E, c)$ while the solitary waves form a codimension two subset. Notice however that the translation invariance, corresponding to the parameter x_0 is not essential to our theory and can be quotiented out. Hence, we consider the periodic traveling wave solutions of (1) as a three parameter family of the form $u(x; a, E, c)$. We summarize these existence results in the following lemma.

Lemma 1. *Let $\Omega \subset \mathbb{R}^3$ be the open set of all triples (a, E, c) such that the traveling wave ODE (2) admits at least one periodic solution. Then for each $(a_0, E_0, c_0) \in \Omega$ there exists a one parameter family*[2] *of periodic traveling wave solutions of the gBBM (1) of the form*

$$u_\xi(x, t) = u(x - c_0 t + \xi; a_0, E_0, c_0)$$

where $\xi \in \mathbb{R}$. In particular, for each $(a_0, E_0, c_0) \in \Omega$ the function $u(x; a_0, E_0, c_0)$ is a spatially periodic traveling wave solution of the PDE (3) with wave speed $c = c_0$.

Remark 1. *Throughout this paper, it will always be assumed that $(a, E, c) \in \Omega$ when referring to functions of (a, E, c). In particular, the function $u(x; a, E, c)$ will always denote a periodic solution of the traveling wave ODE (2).*

The partial differential equation (1) has, in general, the three conserved quantities[3]

$$\begin{aligned} M &= \int_0^T u \, dx \\ P &= \frac{1}{2}\int_0^T \left(u^2 + u_x^2\right) dx \\ H &= \int_0^T \left(\frac{1}{2}u^2 + F(u)\right) dx \end{aligned} \tag{5}$$

which correspond to the mass, momentum, and Hamiltonian (energy) of the solution, respectively. These three quantities are considered as functions of the traveling wave parameters

[2] In the case where more than one periodic solution exists for our choice of (a_0, E_0, c_0), we can distinguish them by their initial values. This does not pose an issue from the standpoint of the stability theory developed here: for example, for solutions of the mBBM with level sets of type E_2 (shown in Figure 1), although there exists two distinct families of periodic solutions of (2) the conserved quantities restricted to each of these families differs only by a constant, which will not in the end effect the stability indices introduced in this work.

[3] That is, these quantities are conserved in the space of T-periodic solutions of (1). Technically, M should be defined as $M = \int_0^T (u - u_{xx})\, dx$. However, this definition clearly agrees with the one given below on the space of T-periodic functions.

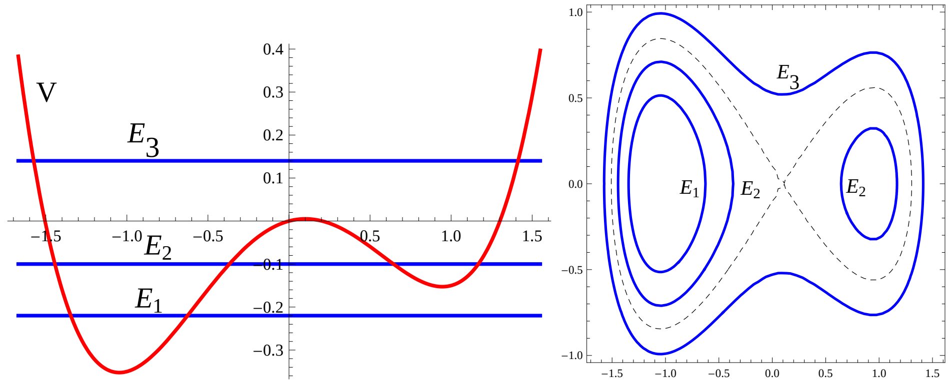

FIGURE 1. (Left) A plot of the effective potential energy $V(x; 0.1, 2)$ for the modified BBM equation ($f(u) = u^3$), as well as three energy levels $E_1 = -0.22$, $E_2 = -0.1$, and $E_3 = 0.14$. (Right) Plots in phase space of the solutions $u(x; 0.1, E_j, 2)$ corresponding to the three energy levels on the left. Notice that those solutions corresponding to energy levels E_1 and E_2 are bounded by a homoclinic orbit, i.e. they lie inside the homoclinic, in phase space (given by the thin dashed line). Moreover, notice that E_2 corresponds to two distinct periodic traveling wave solutions: however, these can be clearly distinguished by their initial values which we have chosen to mod out in our theory.

a, E, and c and their gradients with respect to these parameters will play an important role in the foregoing analysis. It is important to notice that when restricted to the four-parameter family of periodic traveling wave solutions of (1), the mass can be represented as $M = \int_0^T u\, dx$. Since all our results concern this four-parameter family, we will always work with this simplified expression for the mass functional.

As is standard, one can use equation (4) to express the period of the periodic wave u as

$$T(a, E, c) = 2\sqrt{c} \int_{u_-}^{u_+} \frac{du}{\sqrt{2\left(E - V(u; a, c)\right)}}.$$

The above interval can be regularized at the square root branch points $u_\pm$ by a standard procedure (see [11] for example) and hence represents a C^1 function of (a, E, c). Similarly, the mass and momentum can be expressed as

$$\begin{aligned} M(a, E, c) &= 2\sqrt{c} \int_{u_-}^{u_+} \frac{u\, du}{\sqrt{2\left(E - V(u; a, c)\right)}} \\ P(a, E, c) &= \int_{u_-}^{u_+} \left(\frac{\sqrt{c}\, u^2}{\sqrt{2\left(E - V(u; a, c)\right)}} + \sqrt{\frac{2}{c}} \sqrt{\left(E - V(u; a, c)\right)} \right) du \end{aligned}$$

and can be regularized as above. In particular, it follows that one can differentiate these functionals restricted to the periodic wave $u(x; a, E, c)$ with respect to the parameters

(a, E, c). The gradients of these quantities will play an important role in the subsequent theory.

It is useful to notice the following connection to the classical mechanics corresponding to the traveling wave ODE. The classical action in the sense of action angle variables is given by

$$K(a,E,c) = \oint u_x du = 2\sqrt{\frac{2}{c}} \int_{u_-}^{u_+} \sqrt{(E - V(u;a,c))} du. \tag{6}$$

While K is not itself conserved, it does provide a useful generating function for the conserved quantities of (1). Indeed, it is clear the classical action satisfies the relation

$$\nabla_{a,E,c} K(a,E,c) = \left\langle \frac{1}{c} M(a,E,c), \frac{1}{c} T(a,E,c), \frac{1}{c} \left(P(a,E,c) - K(a,E,c)\right) \right\rangle,$$

where $\nabla_{a,E,c} := \langle \partial_a, \partial_E, \partial_c \rangle$, which immediately establishes several useful identities between the gradients of T, M, and P. For example, it follows that $T_a = M_E$ and $T_c = P_E$.

Finally, we make a few notes on notation. Throughout the forthcoming analysis, various Jacobians of maps from the traveling wave parameters to the period and conserved quantities of the gBBM flow will become important. We adopt the following Possion bracket style notation

$$\{g, h\}_{x,y} := \det \left(\frac{\partial(g,h)}{\partial(x,y)} \right)$$

for the Jacobian determinants with the analogous notation for larger determinants

$$\{g, h, j\}_{x,y,z} := \det \left(\frac{\partial(g,h,j)}{\partial(x,y,z)} \right).$$

Notice that the nonvanishing of such quantities encodes geometric information about the underlying manifold of periodic traveling wave solutions of (1): more will be said on this in the coming sections.

3. The Periodic Evans Function

We now begin our stability analysis of a $T = T(a,E,c)$-periodic traveling wave $u(x; a, E, c)$ by considering a solution to the partial differential equation (1), i.e. a stationary solution of (3), of the form

$$\psi(x,t) = u(x; a, E, c) + \varepsilon v(x,t) + \mathcal{O}(\varepsilon^2)$$

where $|\varepsilon| \ll 1$ is considered as a small perturbation parameter. Substituting this into (1) and collecting terms at $\mathcal{O}(\varepsilon)$ yields the linearized equation $J\mathcal{L}[u]v = \mathcal{D}v_t$ where $J = \partial_x$ and

$$\mathcal{L}[u] := -c\partial_x^2 + (c-1) - f'(u),$$
$$\mathcal{D} := 1 - \partial_x^2.$$

Since this linearized equation is autonomous in time, we may seek separated solutions of the form $v(x,t) = e^{\mu t} v(x)$, which yields the spectral problem

$$J\mathcal{L}v = \mu \mathcal{D} v \tag{7}$$

Throughout this paper, we consider the above operators as acting on $L^2(\mathbb{R})$ corresponding to spatially localized perturbations, or on $L^2_{\mathrm{per}}([0,T])$ corresponding to T-periodic perturbations. In both cases, the operator $\mathcal{D}$ is a positive operator and is hence invertible and hence (7) can be written as a spectral problem for a linear operator:

$$\mathcal{A}v = \mu v, \quad \mathcal{A} := \mathcal{D}^{-1}J\mathcal{L}.$$

As $\mathcal{L}$ has is a differential operator with T-periodic coefficients, the natural setting to study the $L^2(\mathbb{R})$ spectrum of the operator $\mathcal{A}$ is that of Floquet theory. Indeed, by standard results in Floquet theory μ is in the L^2 spectrum of $\mathcal{A}$ if and only if (7) admits a bounded eigenfunction of the form

$$v(x) = e^{i\kappa x/T}w(x)$$

where w is T-periodic. If we write (7) as the first order system

$$\Phi(x;\mu)_x = \mathbf{H}(x,\mu)\Phi(x;\mu) \tag{8}$$

where $\Phi(x;\mu)$ is matrix valued and

$$\mathbf{H}(x,\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{c}\left(\mu + u_x f''(u)\right) & \frac{1}{c}\left(c-1-f'(u)\right) & \frac{\mu}{c} \end{pmatrix}$$

it follows that if we assume $\Phi(0;\mu) = \mathbf{I}$ for all $\mu \in \mathbb{C}$, then μ is in the L^2 spectrum of $\mathcal{A}$ if and only if the matrix

$$\mathbf{M}(\mu) := \Phi(T;\mu)$$

has an eigenvalue $e^{i\kappa}$ for some $\kappa \in \mathbb{R}$. The matrix $\mathbf{M}(\mu)$ is the monodromy matrix corresponding to the first order system (8) whose action is to spatially translate solutions of (8) by one period T. Notice, however, that the eigenvalues of $\mathbf{M}(\mu)$ do not depend in a straight forward way on the coefficient matrix $\mathbf{H}(x,\mu)$, but rather on the fundamental matrix solution of the induced system.

It now follows from an easy calculation that (7) has no point spectrum in $L^2(\mathbb{R})$. Indeed, suppose Ψ is a vector solution of (8) corresponding to a non-trivial $L^2(\mathbb{R})$ eigenfunction of $\mathcal{A}$ with eigenvalue μ. From the definition of the monodromy operator, we have

$$\Psi(NT) = \mathbf{M}(\mu)^N\Psi(0)$$

for any $N \in \mathbb{Z}$. It follows that Ψ can be at most bounded on $\mathbb{R}$ and must not decay as $N \to \pm\infty$. Summarizing, we make the following definition.

Definition 1. *We say $\mu \in \mathrm{spec}(\mathcal{A})$ if there exists a non-trivial bounded function ψ such that $\mathcal{A}\psi = \mu\psi$ or, equivalently, if there exists a $\lambda \in S^1$ such that*

$$\det(\mathbf{M}(\mu) - \lambda\,\mathbf{I}) = 0.$$

Following Gardner [17] *we define the periodic Evans function $D : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ to be*

$$D(\mu,\lambda) = \det(\mathbf{M}(\mu) - \lambda\,\mathbf{I}).$$

Finally, we say the periodic solution $u(x;a,E,c)$ is spectrally stable if spec$(\mathcal{A})$ *does not intersect the open right half plane.*

Remark 2. *First, notice by the Hamiltonian nature of* (7) *and the fact that the linearized operator is real,* $\mathrm{spec}(\mathcal{A})$ *is symmetric with respect to reflections about the real and imaginary axis. Thus, spectral stability occurs if and only if* $\mathrm{spec}(\mathcal{A}) \subset i\mathbb{R}$.

Secondly, since we are interested primarily with the roots of $D(\mu,\lambda)$ *for* λ *on the unit circle, we will frequently work with the function* $D(\mu, e^{i\kappa})$ *for* $\kappa \in \mathbb{R}$*, which is actually the function considered by Gardner. The parameter* κ *is referred to as the Floquet exponent (defined uniquely mod* 2π*) with corresponding Floquet multiplier* $e^{i\kappa}$*, which we denoted above as* λ.

It follows that we can parameterize the continuous $L^2(\mathbb{R})$ spectrum of the operator $\mathcal{A}$ by the Floquet parameter κ:

$$\mathrm{spec}(\mathcal{A}) = \bigcup_{\kappa\in[-\pi,\pi)} \{\mu \in \mathbb{C} : D(\mu, e^{i\kappa}) = 0\}.$$

In particular, the zero's of the function $D(\mu, e^{i\kappa})$ for a fixed $\kappa \in \mathbb{R}$ correspond to the $L^2_{\mathrm{per}}([0,T])$-eigenvalues of the operator resulting from the map $\partial_x \mapsto \partial_x + i\kappa$ applied to (7), and hence the continuous spectrum of A can be parameterized by a one-parameter family of eigenvalue problems. By a standard result of Gardner, if $D(\mu_0, e^{i\kappa_0}) = 0$, then the multiplicity of μ_0 as a periodic eigenvalue of the corresponding linear operator is precisely the multiplicity of μ_0 as a root of the Evans function. As we will see below, the integrable structure of (2) implies that the function $D(\mu, 1)$ has a zero of multiplicity (generically) three at $\mu = 0$. For small κ then, there will be in general three branches $\mu_j(\kappa)$ of roots of $D(\mu_j(\kappa), e^{i\kappa})$ which bifurcate from the origin. Assuming these branches are analytic[4] in κ, it follows that a necessary condition for spectral stability is thus

$$\frac{\partial}{\partial\kappa}\mu_j(\kappa)\big|_{\kappa=0} \in i\mathbb{R}. \tag{9}$$

This naturally leads to the use of perturbation methods in the study of the spectrum of $\mathcal{A}$ near the origin, i.e. modulational instability analysis of the underlying traveling wave. As we will see, the first order terms of a Taylor series expansion of the three branches $\mu_j(\kappa)$ can be encoded as roots of a cubic polynomial, and hence spectral stability is determined by the sign of the associated discriminant. Moreover, it follows by the Hamiltonian structure of (7) that in fact $\sigma(\mathcal{A}) \subset i\mathbb{R}$ if (9) holds and the roots of the cubic polynomial are distinct.

We conclude this section by reviewing some basic global features of the spectrum of the linearized operator $\mathcal{A}$ which are useful in a local analysis near $\mu = 0$. We also state some important properties of the Evans function $D(\mu,\lambda)$ which are vital to the foregoing analysis.

Proposition 1. *The* $L^2(\mathbb{R})$*-spectrum of the operator* $\mathcal{A}$ *has the following properties:*

(i) *There are no isolated points of the spectrum.*

(ii) *The entire imaginary axis is contained in the spectrum, i.e.* $i\mathbb{R} \subset \mathrm{spec}(\mathcal{A})$.

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[4]In general, for each j, the theory of branching solutions of non-linear equations guarantees the existence of a natural number m_j such that $\mu_j(\cdot)$ is an analytic function of κ^{1/m_j}. As we will see in our case, the Hamiltonian nature of the linearized operator $\mathcal{A}$ assures that $m_j = 1$, and hence the roots are in fact analytic functions of the Floquet parameter (see Lemma 5 below).

Moreover, the Evans function $D(\mu,\lambda)$ satisfies the following:

(iii) $D(\mu,0)=e^{\mu T/c}$.

(iv) $D(\mu,\lambda)=\det(\mathbf{M}(\mu)-\lambda\mathbf{I})=-\lambda^3+a(\mu)\lambda^2-a(-\mu)e^{\mu T/c}\lambda+e^{\mu T/c}$ *with* $a(\mu)=\operatorname{tr}(\mathbf{M}(\mu))$.

Proof. The first claim, that the spectrum is never discrete, follows from a basic lemma in the theory of several complex variables: namely that, if for fixed λ^* the function $D(\mu,\lambda^*)$ has a zero of order k at μ^* and is holomorphic in a polydisc about (μ^*,λ^*) then there is some smaller polydisc about (μ^*,λ^*) so that for every λ in a disc about λ^* the function $D(\mu,\lambda)$ (with λ fixed) has k roots in the disc $|\mu-\mu^*|<\delta$. For details see the text of Gunning[19].

Claim (iii) follows from Abel's formula and the fact that $c\operatorname{tr}(\mathbf{H}(x,\mu))=\mu$, where $\mathbf{H}(x,\mu)$ is as in (8). Indeed, it follows that

$$\det\left(\Phi(x;\mu)\right)=e^{\int_0^x\operatorname{tr}(\mathbf{H}(s,\mu))ds}=e^{\mu x/c}\det\left(\Phi(0;\mu)\right)=e^{\mu x/c}$$

for all $x\in\mathbb{R}$ and hence taking $x=T$ yields

$$D(\mu,0)=\det(\mathbf{M}(\mu))=e^{\mu T/c}$$

by definition.

Next, we prove claim (iv). Since $\mathcal{A}v=\mu v$ is invariant under the transformation $x\mapsto -x$ and $\mu\mapsto-\mu$, the matrices $M(\mu)$ and $M(-\mu)^{-1}$ are similar. If we define $a(\mu)$ as above and $b(\mu)$ such that

$$\det[\mathbf{M}(\mu)-\lambda\mathbf{I}]=-\lambda^3+a(\mu)\lambda^2+b(\mu)\lambda+e^{\mu T/c},$$

it follows that

$$\begin{aligned}\det[\mathbf{M}(\mu)-\lambda\mathbf{I}] &= \det[\mathbf{M}^{-1}(-\mu)-\lambda\mathbf{I}]\\ &= -\lambda^3\det[\mathbf{M}^{-1}(-\mu)]\det[\mathbf{M}(-\mu)-\lambda^{-1}]\\ &= -e^{\mu T/c}\lambda^3\left(-\lambda^{-3}+a(-\mu)\lambda^{-2}+b(-\mu)\lambda^{-1}+e^{-\mu T/c}\right)\\ &= -\lambda^3-e^{\mu T/c}b(-\mu)\lambda^2-e^{\mu T/c}a(-\mu)\lambda+e^{\mu T/c}.\end{aligned}$$

Therefore, $b(\mu)=-e^{\mu T/c}a(-\mu)$ as claimed.

Claim (ii) now follows from a symmetry argument. Since $a(\mu)$ is real on the real axis, it follows by Schwarz reflection that for $\mu\in i\mathbb{R}$ we have $a(\overline{\mu})=\overline{a(\mu)}$. For $\mu\in i\mathbb{R}$ then the Evans function takes the form

$$D(\mu,\lambda)=-\lambda^3+a(\mu)\lambda^2-e^{\mu T}\overline{a(\mu)}\lambda+e^{\mu T/c}.$$

It follows that

$$D(\mu,\lambda)=-\lambda^3e^{\mu T/c}\overline{D(\mu,\overline{\lambda}^{-1})}$$

so that the roots of $D(\mu^*,\lambda)$ for a fixed $\mu^*\in i\mathbb{R}$ are symmetric about the unit circle, i.e. if λ_0 is a root of $D(\mu^*,\lambda)=0$, then $(\overline{\lambda})^{-1}$ is also a root. Since the equation $D(\mu^*,\lambda)=0$ clearly has three roots for each fixed $\mu^*\in\mathbb{C}$, it follows that one must lie on the unit circle and hence $\mu\in\operatorname{spec}(\mathcal{A})$ as claimed. ☐

4. Periodic Stability: Spectral and Nonlinear Stability Results

In this section, we make a detailed analysis of the stability of a given periodic traveling wave solution of (1) to perturbations with the same periodic structure, i.e. to co-periodic perturbations: such perturbations correspond to $\lambda = 1$ in the above theory. We begin by a spectral stability analysis, and then conclude with a brief discussion of a nonlinear (orbital) stability result.

4.1. **Periodic Spectral Stability.** Let $u(x; a, E, c)$ be a T-periodic traveling wave solution of (1). Considering the spectral stability of such a solution of perturbations which are T-periodic is equivalent to studying the spectrum of the linear operator $\mathcal{A}$ on the real Hilbert space $L^2_{\text{per}}([0, T])$. We begin with the following lemma.

Lemma 2. *Let $u(x; a, E, c)$ be the solution of the traveling wave equation (2) satisfying $u(0; a, E, c) = u_-(a, E, c)$ and $u_x(0; a, E, c) = 0$. A basis of solutions to the first order system*

$$Y_x = \mathbf{H}(x, 0)Y$$

is given by

$$Y_1(x) = \begin{pmatrix} cu_x(x; a, E, c) \\ cu_{xx}(x; a, E, c) \\ cu_{xxx}(x; a, E, c) \end{pmatrix}, \quad Y_2(x) = \begin{pmatrix} cu_a(x; a, E, c) \\ cu_{ax}(x; a, E, c) \\ cu_{axx}(x; a, E, c) \end{pmatrix}, \quad Y_3(x) = \begin{pmatrix} cu_E(x; a, E, c) \\ cu_{Ex}(x; a, E, c) \\ cu_{Exx}(x; a, E, c) \end{pmatrix}.$$

Moreover, a particular solution to the inhomogeneous problem

$$Y_x = \mathbf{H}(x, 0)Y + W$$

where $W = (0, 0, c\mathcal{D}u_x)$ is given by

$$Y_4(x) = \begin{pmatrix} -cu_c(x; a, E, c) \\ -cu_{cx}(x; a, E, c) \\ -cu_{cxx}(x; a, E, c) \end{pmatrix}.$$

Proof. This is easily verified by differentiating (2) with respect x and the parameters a, E, and c. Indeed, it is clear that the functions u_x, u_a, and u_E solve the differential equation

$$\partial_x \mathcal{L} v = 0$$

By fixing $u_x(0) = 0$, which modds out the translation invariance of (1), it follows the solution u satisfies

$$\begin{aligned}
&(10) & u(0; a, E, c) &= u_- = u(T; a, E, c) \\
&(11) & u_x(0; a, E, c) &= 0 = u_x(T, a, E, c) \\
&(12) & u_{xx}(0; a, E, c) &= -\frac{1}{c} V'(u_-; a, c) = u_{xx}(T; a, E, c)
\end{aligned}$$

for any a, E, c, and that $u_{xxx}(0; a, E, c) = 0$ by (2). Defining $\mathbf{U}(x, 0) = [Y_1(x), Y_2(x), Y_3(x)]$ where Y_1, Y_2, Y_3 are vector functions corresponding to the solutions cu_x, cu_a, and cu_E, respectively, it follows by differentiating (10)-(12) that

$$\mathbf{U}(0,0)=\begin{pmatrix} 0 & c\frac{\partial u_-}{\partial a} & c\frac{\partial u_-}{\partial E} \\ -V'(u_-) & 0 & 0 \\ 0 & 1+(c-1-f'(u_-))\,\frac{\partial u_-}{\partial a} & (c-1-f'(u_-))\,\frac{\partial u_-}{\partial E} \end{pmatrix}. \tag{13}$$

Differentiating the relation $E=V'(u_-;a,c)$ with respect to E gives

$$\det(\mathbf{U}(0,0))=-cV'(u_-)\frac{\partial u_-}{\partial E}=-c,$$

and hence these solutions are linearly independent at $x=0$, and hence for all x as claimed. □

Our immediate goal is to relate this spectral information to the structure of the periodic Evans function $D(\mu,1)$. As motivation, notice that by taking appropriate linear combinations Lemma 2 implies the generalized null-space of $\partial_x\mathcal{L}$ is at least three dimensional with a height one Jordan chain. In particular, $\mu=0$ is a root of $D(\mu,1)=0$ of multiplicity at least three. To determine the leading order asymptotics of the function $D(\mu,1)$ for $|\mu|\ll 1$, we utilize standard perturbation analysis. To assist in this analysis, notice using the chain rule to differentiate (10)-(12) yields

$$\mathbf{U}(T,0)=\mathbf{U}(0,0)+\begin{pmatrix} 0 & 0 & 0 \\ 0 & V'(u_-)T_a & V'(u_-)T_E \\ 0 & 0 & 0 \end{pmatrix},$$

where $T_a=\partial_a T(a,E,c)$ and T_E is defined similarly, and hence $\mathbf{U}(T,0)-\mathbf{U}(0,0)$ is a rank one matrix. As we will see, this fact allows for significant simplifications in the perturbation calculations. Without this fact, one would have to compute variations in the vector solutions at $\mu=0$ to an extra degree, which would force the use of multiple applications of variation of parameters along the same null direction.

We now present our first technical result of this section.

Lemma 3. *For $|\mu|\ll 1$, the periodic Evans function satisfies*

$$D(\mu,1)=-\{T,M,P\}_{a,E,c}\mu^3+\mathcal{O}(|\mu|^3).$$

Proof. Let $w_i(x;\mu)$, $i=1,2,3$, be three linearly independent solutions of (8), and let $\mathbf{W}(x,\mu)$ be the solution matrix with columns w_i. Expanding the above solutions in powers of μ as

$$w_i(x,\mu)=w_i^0(x)+\mu w_i^1(x)+\mu^2 w_i^2(x)+\mathcal{O}(|\mu|^3)$$

and substituting them into (8), the leading order equation becomes

$$\frac{d}{dx}w_i^0(x)=\mathbf{H}(x,0)w_i^0(x).$$

In particular, we may choose $w_i^0(x)=Y_i(x)$ where Y_i are defined as in Lemma 2. The higher order terms in the above expansion yield

$$\frac{d}{dx}w_i^j(x)=\mathbf{H}(x,0)w_i^j(x)+V_i^{j-1}(x),\quad j\geq 1, \tag{14}$$

where $V_i^{j-1} = \left(0, 0, -c^{-1}\mathcal{D}(w_i^{j-1})_1\right)^t$ and $(v)_1$ denotes the first component of the vector v. We require for each of the higher order terms $w_j^i(0) = 0$, $j \geq 1$. This implies that $\mathbf{W}(0,\mu) = \mathbf{U}(0,0)$ in a neighborhood of $\mu = 0$, where $\mathbf{U}(0,0)$ is defined in (13). The solution of the inhomogeneous problem is given by the variation of parameters formula

$$w_i^j(x) = \mathbf{W}(x,0)\int_0^x \mathbf{W}(s,0)^{-1}V_i^{j-1}(s)ds$$

(15)

$$= \begin{pmatrix} cu_x \int_0^x \mathcal{D}(w_i^{j-1})_1\{u,u_x\}_{a,E}\, dz - u_a \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_E \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \\ cu_{xx} \int_0^x \mathcal{D}(w_i^{j-1})_1\{u,u_x\}_{a,E}\, dz - u_{ax} \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_{Ex} \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \\ cu_{xxx} \int_0^x \mathcal{D}(w_i^{j-1})_1\{u,u_x\}_{a,E}\, dz - u_{axx} \int_0^x \mathcal{D}(w_i^{j-1})_1 dz + u_{Exx} \int_0^x \mathcal{D}(w_i^{j-1})_1 u\, dz \end{pmatrix}$$

for $j \geq 1$. Notice we have used the identities $c\{u,u_x\}_{E,x} = -1$ and $c\{u,u_x\}_{x,a} = u$ extensively in the above formula, which can be easily derived via equation (4). Indeed, differentiating (4) with respect to E and subtracting $u_E u_{xx}$ immediately yields the first identity.

Now, notice that it would be a daunting task to use (15) to the specified order needed. However, the integrable structure of (7) allows for an alternative, yet equivalent, expression in the case $i = j = 1$ which makes a seemingly second order calculation come in at first order. Indeed, in this case equation (14) is equivalent to $L_0 w_1^1 = u_x$ and hence it follows from Proposition 2 that we can choose

$$w_1^1(x) = \begin{pmatrix} -cu_c \\ -cu_c' \\ -cu_c'' \end{pmatrix} + \left(u_- + \frac{1}{c}V'(u_-;a,c)\right)\begin{pmatrix} cu_a \\ cu_a' \\ cu_a'' \end{pmatrix} - \left(\frac{u_-^2}{2} + \frac{1}{c}V'(u_-;a,c)\right)\begin{pmatrix} cu_E \\ cu_E' \\ cu_E'' \end{pmatrix}$$

where the above constants in front of Y_2 and Y_3 are determined by the requirement $w_1^1(0) = 0$. Thus, one can determine the second order variation of w_1 in μ by using (15) to compute the *first* order variation of the function w_1^1 defined above. Defining $\delta\,\mathbf{W}(\mu) := \mathbf{W}(x,\mu)\big|_{x=0}^{T}$, it follows that $\mathbf{W}(\mu)$ can be expanded as

$$\begin{pmatrix} \mathcal{O}(\mu^2) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \\ \mu V'(u_-)P(u_-) + \mathcal{O}(\mu^2) & V'(u_-)T_a & V'(u_-)T_E \\ \mathcal{O}(\mu^2) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \end{pmatrix}$$

where $P(x) = -T_c + \left(x + \frac{1}{c}V'(x)\right)T_a - \left(\frac{x^2}{2} + \frac{1}{c}V'(x)\right)T_E$ and the higher order terms are determined by (15). Thus,

$$D(\mu,1) = \det\left(\delta\,\mathbf{W}(\mu)\,\mathbf{W}(0,0)^{-1}\right) = \mathcal{O}(\mu^3)$$

and in particular a straightforward calculation yields

$$\det\left(\delta\,\mathbf{W}(\mu)\right) = c\{T,M,P\}_{a,E,c}\,\mu^3 + \mathcal{O}(\mu^4).$$

The proof is complete by recalling that $\det(\mathbf{W}(0,0)) = -c$. $\square$

Remark 3. *Notice that the formula for $D_{\mu\mu\mu}(0,1)$ differs from that derived by Bronski and Johnson* [11] *in the case of the generalized KdV by a factor of one-half, which comes from that fact the differences in the definitions of the momentum functionals in each work: in particular, the factor of $\frac{1}{2}$ in* (5) *is not present in* [11].

Assuming the Jacobian $\{T,M,P\}_{a,E,c}$ is non-zero, it follows that zero is a T-periodic eigenvalue of $\mathcal{A}$ of multiplicity three. By computing the orientation index

$$\operatorname{sign}\left(\{T,M,P\}_{a,E,c}\right)\operatorname{sign}\left(D(\infty,1)\right)$$

it is clear that we will have exponential periodic instability of the underlying wave if this index is positive. Indeed, positivity of this product clearly implies the existence of a real root of the function $D(\mu,1)$, i.e. there exists a real positive eigenvalue of the linearized operator $\mathcal{A}$. With this in mind, we now compute the large $\mu \gg 1$ behavior of the periodic Evans function in the next lemma.

Lemma 4. *The function $D(\cdot,1)\big|_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}$ satisfies the following asymptotic relation:*

$$\lim_{\mathbb{R}\ni\mu\to\infty}\operatorname{sign}\left(D(\mu,1)\right)=-1.$$

Proof. This follows from a simple calculation. Indeed, rescaling (7) by the change of variables $y=\mu^{-1/3}x$ for $\mu\gg 1$ yields the first order system

$$\mathbf{W}'=\left(\begin{pmatrix}0&c&0\\0&0&c\\-1&0&1\end{pmatrix}+\mathbf{B}(x,\mu)\right)\mathbf{W}$$

where $\mathbf{B}(x,\mu)=\mathcal{O}(|\mu|^{-1/3})$ is a decaying residual term. Moreover, a direct calculation shows the eigenvalues of the principle part of the above system satisfy the polynomial equation

$$z^3-z^2+c^2=0.$$

In particular, the principle part either has one negative eigenvalue with two positive roots or else one negative eigenvalue with complex conjugate roots with positive real part. It follows that for $|\mu|\gg 1$ we have

$$D(\mu,1)=\det\left(\exp\begin{pmatrix}0&c&0\\0&0&c\\-1&0&1\end{pmatrix}-\mathbf{I}+\mathcal{O}(|\mu|^{1/3})\right)$$

which is clearly negative by the above considerations. □

We are now able to state our first main theorem for this section.

Theorem 1. *Let $u(x;a_0,E_0,c_0)$ be a periodic solution of* (2)*. If $\{T,M,P\}_{a,E,c}$ is negative at (a_0,E_0,c_0), then the number of roots of $D(\mu,1)$ (i.e. the number of periodic eigenvalues of $\mathcal{A}$) on the positive real axis is odd. Moreover, if the period is an increasing function of energy at (a_0,E_0,c_0), i.e. $T_E(a_0,E_0,c_0)>0$, then the periodic traveling wave $u(x;a_0,E_0,c_0)$ is spectrally unstable to $T(a_0,E_0,c_0)$-periodic perturbations if and only if $\{T,M,P\}_{a,E,c}$ is negative at (a_0,E_0,c_0).*

Proof. By our work in the proof of Lemma 3, we know that $D(\mu,1)=\mathcal{O}(|\mu|^3)$. Moreover, if $D_{\mu\mu\mu}(0,1)=-6\{T,M,P\}_{a,E,c}>0$ the number $D(\mu,1)$ is positive for small positive μ. Since $D(\mu,1)$ is negative for real μ sufficiently large, we know that $D(\pm\mu^*,1)=0$ for some $\mu^*\in\mathbb{R}^*$ from which instability follows.

Next, we show that if $T_E>0$, then the positivity of $\{T,M,P\}_{a,E,c}$ is sufficient for spectral stability to co-periodic perturbations. To this end, notice that since $\mathcal{L}u_x=0$ it follows by Sturm-Liouville theory that $\mathcal{L}$ has either one or two negative T-periodic eigenvalues. Also, Theorem 3.1 of [26] implies the number of unstable T-periodic eigenvalues of $\mathcal{A}$ is at most the number of negative eigenvalues of $\mathcal{L}$. Moreover, it was shown in Lemma 4.1 of [22] that $\mathcal{L}$ has precisely one negative T-periodic eigenvalue in the case that $T_E>0$. Combining these results, it follows that $\mathcal{A}$ can have at most one unstable T-periodic eigenvalue in the open right half plane. As $\mathrm{spec}(\mathcal{A})$ is symmetric about the real axis, it follows that any unstable eigenvalues with positive real part must in fact be real and hence it is enough to determine the parity of the number of roots of the function $D(\mu,1)$ on $\mathbb{R}^+$. By noting that if $\{T,M,P\}_{a,E,c}>0$ the number of positive roots of $D(\mu,1)$ is even (and hence zero), the result now follows. $\square$

We now wish to give insight into the meaning of $\{T,M,P\}_{a,E,c}=0$ at the level of the linearized operator $\mathcal{A}$. To this end, we consider the linearized operator $\mathcal{A}$ as acting on $L^2_{\text{per}}(0,T)$, the space of T-periodic L^2 functions on $\mathbb{R}$. To begin, we make the assumption that $\{T,M\}_{a,E}$ and $\{T,P\}_{a,E}$ do not simultaneously vanish. This assumption will be shown equivalent with the periodic null-space reflecting the Jordan structure of the monodromy at the origin. Throughout this brief discussion, we assume that $\{T,M\}_{a,E}\neq 0$: trivial modifications are needed if $\{T,M\}_{a,E}$ vanishes but $\{T,P\}_{a,E}$ does not. First, define the functions

$$
\begin{aligned}
&\phi_0=\{T,u\}_{a,E}, && \psi_0=1,\\
&\phi_1=\{T,M\}_{a,E}\,u_x, && \psi_1=\int_0^x\mathcal{D}\phi_2(s)ds,\\
&\phi_2=\{u,T,M\}_{a,E,c} && \psi_2=-\{T,M\}_{E,c}+\{T,M\}_{a,E}\mathcal{D}u,
\end{aligned}
\tag{16}
$$

Clearly, each of these functions belong to $L^2_{\text{per}}([0,T])$ and

$$
\begin{aligned}
&\mathcal{A}\phi_0=0,\ \mathcal{A}^\dagger\psi_0=0,\\
&\mathcal{A}\phi_1=0,\ \mathcal{A}^\dagger\psi_1=\psi_2,\\
&\mathcal{A}\phi_2=-\phi_1,\ \mathcal{A}^\dagger\psi_2=0
\end{aligned}
$$

where $\mathcal{A}^\dagger$ denotes the adjoint of $\mathcal{A}$ defined on $L^2_{\text{per}}([0,T])$. Notice we have used the fact that $\mathcal{D}^{-1}(1)=1$ on $L^2_{\text{per}}([0,T])$. Thus, it follows that the periodic null space of $\mathcal{A}$ is generated spanned by the functions ϕ_0 and ϕ_1. Moreover, since

$$
\begin{aligned}
\langle\psi_0,\phi_0\rangle&=\{T,M\}_{a,E}\\
\langle\mathcal{D}u,\phi_0\rangle&=\{T,P\}_{a,E},
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2_{\text{per}}([0,T])$, the assumption that $\{T,M\}_{a,E}$ and $\{T,P\}_{a,E}$ do not simultaneously vanish implies that $N_{\text{per}}\left(\mathcal{A}^2\right)-N_{\text{per}}\left(\mathcal{A}\right)=\text{span}(\phi_2)$, where $N_{\text{per}}(G)$ denotes the null-space of the linear operator G acting on $L^2_{\text{per}}([0,T])$, thus reflecting the Jordan normal form of the period map at $\mu=0$.

Finally, we study the structure of the generalized periodic null space, and seek conditions for which there is no non-trivial Jordan chain of height at least two. By the Fredholm alternative, such a chain exists if and only if

$$\langle\mathcal{D}u,\phi_2\rangle=\{T,M,P\}_{a,E,c}=0.$$

Thus, the vanishing of $\{T,M,P\}_{a,E,c}$ is equivalent with a change in the generalized periodic-null space of the linearized operator $\mathcal{A}$. This insight has a nice relationship with formal Whitham modulation theory. One of the big ideas in Whitham theory is to locally parameterize the periodic traveling wave solution by the constants of motion for the PDE evolution. The non-vanishing of certain Jacobians is precisely what allows one to do this. In fact, the non-vanishing of $\{T,M,P\}_{a,E,c}$ is equivalent to demanding that, locally, the map $(a,E,c)\mapsto(T,M,P)$ have a unique C^1 inverse: In other words, the constants of motion for the gBBM flow are good local coordinates for the three-dimensional manifold of periodic traveling wave solutions (up to translation). Similarly, non-vanishing of $\{T,M\}_{a,E}$ and $\{T,P\}_{a,E}$ is equivalent to demanding that the matrix

$$\begin{pmatrix} T_a & M_a & P_a \\ T_E & M_E & P_E \end{pmatrix}$$

have full rank, which is equivalent to demanding that the map $(a,E)\mapsto(T,M,P)$ (for fixed c) have a unique C^1 inverse, i.e. two of the conserved quantities provide a smooth parametrization of the family of periodic traveling waves of fixed wave-speed. For more information concerning these connections in the context of the generalized Korteweg-de Vries equation, see the recent work [13].

To summarize, the vanishing of $\{T,M,P\}_{a,E,c}$, is connected with a change in the Jordan structure of the linearized operator $\mathcal{A}$ considered on $L^2_{\text{per}}([0,T])$. In particular, $\{T,M,P\}_{a,E,c}$ ensures the existence of a non-vanishing Jordan piece in the generalized periodic null-space of dimension *exactly* one, i.e. that $\mu=0$ is a T-periodic eigenvalue of the linearized operator with algebraic multiplicity three and geometric multiplicity two. Moreover, and perhaps more importantly for our analysis, it guarantees that infinitesimal variations in the constants arising from reducing the family of periodic traveling waves to quadrature are enough to generate the entire generalized periodic null-space of the linearized operator $\mathcal{A}$: such a condition is obviously not necessary in our calculations, but provides significant simplifications in the theory.

4.2. **Nonlinear Periodic Stability.** We now compliment Theorem 1 by considering in what sense the Jacobian $\{T,M,P\}_{a,E,c}$ affects the nonlinear stability of a periodic traveling wave solution of (1) to T-periodic perturbations. Clearly, the positivity of this index is necessary for such stability but it is not clear if this is also necessary. Indeed, the analysis presented below allows for the possibility that a periodic traveling wave which is spectrally stable to T-periodic perturbations could be nonlinearly unstable to such perturbations: such a result would stand in stark contrast to the solitary wave theory where these two notions of stability are equivalent (assuming the nondegeneracy condition $\frac{d}{dc}\mathcal{N}(c) \neq 0$). More will be said on this at the end of this section.

Such analysis has recently been carried out in the context of the generalized Korteweg-de Vries equation in [22]. There, sufficient conditions for nonlinear stability to T-periodic perturbations were derived in terms of Jacobians of various maps between the parameter space (a, E, c) and the period, mass, and momentum. As the theory for the gBBM equation is nearly identical to this work, we only review the main points of the analysis here and refer the reader to [22] for details.

To begin, we note that any complete stability study must start with analysis of the Cauchy problem for the corresponding partial differential equation. In the case of the gBBM equation (1), several global well-posedness results are available in the periodic context. For example, in the case of the BBM equation, corresponding to $f(u) = u^2$, global well-posedness in $H^1_{\mathrm{per}}([0,T])$ has been established in [5][5] and [15]. For more general power-law nonlinearities $f(u) = u^{p+1}$, $p \in \mathbb{N}$, one can use the ideas of [7] to obtain the global well-posedness in the Sobolev space $H^s_{\mathrm{per}}([0,T])$ with $s \geq 1$. For general smooth nonlinearities f, global well-posedness in $H^s_{\mathrm{per}}([0,T])$, $s \geq 1$ can be obtained from the work of Albert in [1] and again from the methods of [7]. Using these results as motivation, throughout our analysis we assume the nonlinearity f is such that the Cauchy problem for (1) is globally well posed on a real Hilbert space X of T-periodic functions defined on $\mathbb{R}$, which we equip with the standard $L^2([0,T])$ inner product, denoted here as $\langle \cdot, \cdot \rangle$. In particular, we require X to be a subspace of $L^2_{\mathrm{per}}([0,T])$. As seen from the well-posedness results cited above, we can in general take $X = H^1_{\mathrm{per}}([0,T])$. Also, we identify the dual space X^* through the usual pairing.

Now, we fix a periodic traveling wave solution of (1) $u_0(x; a_0, E_0, c_0)$ and notice that we can write the linearized spectral problem (7) about u_0 in the form

$$\mathcal{D}^{-1}\partial_x \mathcal{E}_0''(\phi) = \mu\phi$$

where $\mathcal{D} = 1 - \partial_x^2$ and $\mathcal{E}_0(\phi)$ is an augmented energy functional defined by

$$\mathcal{E}_0(\phi) = -\mathcal{E}(\phi) + c_0\mathcal{P}(\phi) + a_0\mathcal{M}(\phi)$$

[5]In [5], the existence and stability of cnoidal type waves for the BBM equation with arbitrary minimal period was proved. In the present analysis, however, our goal is to derive general conditions for stability which applies to a general class of nonlinearities. As an application of our theory, in section 7 we establish the stability of the cnoidal waves of the BBM under the assumption of spectral stability.

where $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{M}$ are the energy, momentum, and energy functionals defined on X defined by

$$\begin{aligned}\mathcal{E}(\phi) &= \int_0^T \left(\frac{1}{2}\phi(x)^2 + F(\phi(x))\right) dx\\ \mathcal{M}(\phi) &= \int_0^T \phi(x)dx\\ \mathcal{P}(\phi) &= \frac{1}{2}\int_0^T \left(\phi(x)^2 + \phi_x(x)^2\right) dx,\end{aligned}$$

and the corresponding variations are evaluated at the underlying wave u_0. In particular, notice that each of these functionals are left invariant under spatial translations and hence it is appropriate to study the stability of periodic solutions of (2) up to translation, so called orbital stability. To this end, we introduce a semi-distance $\rho : X \times X \to \mathbb{R}$ defined via

$$\rho(\phi, \psi) := \inf_{\xi\in\mathbb{R}} \|\phi(\cdot) - \psi(\cdot + \xi)\|_X$$

and we seek conditions for which the following statement is true: If $\phi_0 \in X$ is near u_0 as measured by the semidistance ρ, then the solution $\phi(x,t)$ of (1) with initial data ϕ_0 stays close to a translate of u_0 for all time. Specifically, the main result of this section is the following theorem.

Theorem 2. *Let $u(x) = u_0(x; a_0, E_0, c_0)$ solve (2) with $(a_0, E_0, c_0) \in \Omega$, as defined in Lemma 1. Further, suppose the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are all positive at (a_0, E_0, c_0). Then there exist positive constants C_0 and ε_0 such that if $\phi_0 \in X$ satisfies $\rho(\phi_0, u) < \varepsilon$ for some $\varepsilon < \varepsilon_0$, then the solution $\phi(x,t)$ of (1) with initial data ϕ_0 satisfies $\rho(\phi(\cdot, t), u) \le C_0\varepsilon$ for all $t > 0$.*

Remark 4. *Throughout the text, nonlinear orbital stability (or just orbital stability) will always mean stability in the sense described in Theorem 2.*

Remark 5. *While the condition that $\{T, M, P\}_{a,E,c}$ is positive at (a_0, E_0, c_0) is necessary for stability by Theorem 1, it is not clear whether the positivity of the Jacobians T_E and $\{T, M\}_{a,E}$ at this point are necessary or just artifacts of our proof. Indeed, by a more delicate analysis along the lines of that found in* [12]*, it may be possible to show that these conditions can relaxed somewhat. However, we do not attempt such a generalization here.*

As a first step in proving Theorem 2, we make the crucial observation that $u_0(x; a_0, E_0, c_0)$ is a critical point of the augmented energy functional $\mathcal{E}_0$. In order to determine the nature of this critical point, it is necessary to analyze the second derivative of $\mathcal{E}_0$. If $\mathcal{E}_0''(u_0) \in \mathcal{L}(X, X^*)$ is positive definite, then nonlinear stability follows by standard arguments. However, one sees after an easy calculation that

$$\mathcal{E}_0''(u_0) = \mathcal{L}$$

which is clearly not positive definite by the translation invariance of (1). Indeed, since $\mathcal{L}u_{0,x} = 0$ and u_0 is not monotone it follows by standard Sturm-Liouville arguments that zero is either the second or third eigenvalue of $\mathcal{L}$ with respect to the natural ordering on $\mathbb{R}$.

By Lemma 4.1 of [22], it follows that $\mathcal{L}$ considered on $L^2_{\rm per}([0,T])$ has precisely one negative eigenvalue, a simple eigenvalue at zero, and the rest of the spectrum is positive and bounded away from zero if $T_E > 0$. If $T_E \leq 0$, then either the null space or the number or negative eigenvalues jumps by one: a situation which can seemingly not be handled by the present variational analysis[6].

Thus, assuming $T_E > 0$ it follows that u_0 is a degenerate critical point of $\mathcal{E}_0$ with one unstable direction and one neutral direction. In order to get rid of the unstable direction, simply notice that the evolution of (1) does not occur on the entire space X, but on the codimension two subset

$$\Sigma_0 := \{\phi \in X : \mathcal{M}(\phi) = M(a_0, E_0, c_0),\ \ \mathcal{P}(\phi) = P(a_0, E_0, c_0)\}.$$

Clearly Σ_0 is a smooth submanifold of X containing all translates of the function u_0. Defining $\mathcal{T}_0$ to be the tangent space of Σ_0 at u_0, i.e.

$$\mathcal{T}_0 := \left\{\phi \in X : \left\langle \mathcal{P}'(u_0), \phi \right\rangle = \left\langle \mathcal{M}'(u_0), \phi \right\rangle = 0\right\}$$

it follows immediately from the identity

$$\left\langle \mathcal{P}'(u_0), \phi_2 \right\rangle = \langle \mathcal{D}u_0, \phi_2 \rangle = \{T, M, P\}_{a,E,c}$$

and the assumption that $\{T, M, P\}_{a,E,c} \neq 0$ that the generalized translational eigenfunction ϕ_2 defined in (16) does not belong to $\mathcal{T}_0$. Hence, following the proof of Lemma 4.4 of [22], assuming that $T_E > 0$ and that the quantity

$$\left\langle \mathcal{E}_0''(u_0)\phi_2, \phi_2 \right\rangle = -\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$$

is negative we can use the spectral resolution of the operator $\mathcal{E}_0''(u_0) = \mathcal{L}$ to prove that the quadratic form induced by $\mathcal{E}_0''(u_0)$ is positive definite on the subspace of functions in $\mathcal{T}_0$ which are orthogonal to the periodic null-space span$\{u_{0,x}\}$ of $\mathcal{E}''(u_0)$. Since the underlying periodic wave is spectrally unstable if $\{T, M, P\}_{a,E,c} < 0$ by Theorem 1, the only interesting case in which $\langle \mathcal{E}_0''(u_0)\phi_2, \phi_2 \rangle$ is negative is when $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are both positive. We can now follow the proof of Proposition 4.3 in [22] to show the augmented energy $\mathcal{E}_0$ is coercive on Σ_0 near u_0 with respect to the semi-distance ρ, i.e. one can prove the existence of positive constants C and δ, which depend on the underlying solution, such that

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq C\rho(\phi, u)^2 \tag{17}$$

for all $\phi \in \Sigma_0$ with $\rho(\phi, u) < \delta$. With this estimate in hand, we now present the proof of Theorem 2.

Proof of Theorem 2: To begin, assume that $\phi_0 \in X$ satisfies $\rho(\phi_0, u) \leq \varepsilon$ for some small $\varepsilon > 0$. Notice then that by replacing ϕ_0 with $\phi_0(\cdot + \xi)$ for some $\xi \in \mathbb{R}$ if needed, we can assume that $\|\phi_0 - u\|_X \leq \varepsilon$. Since u is a critical point of the functional $\mathcal{E}_0$, it is clear that $\mathcal{E}_0(\phi_0) - \mathcal{E}_0(u) \leq C\varepsilon^2$ for some constant $C > 0$. To conclude nonlinear stability, we consider the two cases $\phi_0 \in \Sigma_0$ and $\phi_0 \notin \Sigma_0$. In the case that $\phi_0 \in \Sigma_0$ it follows that the unique solution $\phi(\cdot, t)$ of (1) with initial data ϕ_0 satisfies $\phi(\cdot, t) \in \Sigma_0$ for all $t \geq 0$. Thus,

[6]However, see the recent work of Bronski, Johnson, and Kapitula [9].

the estimate in equation (17) holds and implies that $\rho(\phi(\cdot,t),u) \leq C\varepsilon$ for some positive constant C and all $t>0$, which proves Theorem 2 in this case.

The case when $\phi_0 \notin \Sigma_0$ requires more care as the estimate (17) does not directly apply. However, we claim that we can vary the constants (a_0,E_0,c_0) slightly in order to reduce this case to the previous one. To this end, notice that the non-vanishing of $\{T,M,P\}_{a,E,c}$ at (a_0,E_0,c_0) implies by the Implicit Function Theorem that the map

$$(a,E,c) \mapsto (T(a,E,c),M(a,E,c),P(a,E,c))$$

is a diffeomorphism from a neighborhood of (a_0,E_0,c_0) onto a neighborhood of

$$(T(a_0,E_0,c_0),M(a_0,E_0,c_0),P(a_0,E_0,c_0))\,.$$

Recalling Lemma 1 then, it follows that we can find a curve $[0,1] \ni s \to (a(s),E(s),c(s))$ in $\mathbb{R}^3$ with $(a(0),E(0),c(0)) = (0,0,0)$ such that for each $s \in [0,1]$ the function

$$\widetilde{u}(x;s) = u(x;a_0+a(s),E_0+E(s),c_0+c(s))$$

is a $T = T(a_0,E_0,c_0)$ periodic traveling wave solution of (1) in the space X and, moreover, the endpoint condition

$$\begin{aligned} M(a_0+a(1),E_0+E(1),c_0+c(1)) &= \mathcal{M}(\phi_0),\\ P(a_0+a(1),E_0+E(1),c_0+c(1)) &= \mathcal{P}(\phi_0) \end{aligned}$$

is satisfied. Defining a new augmented functional on X by

$$\widetilde{\mathcal{E}}(\phi) = \mathcal{E}_0(\phi) + (c_0+c(1))\mathcal{P}(\phi) + (a_0+a(1))\mathcal{M}(\phi) + (E_0+E(1))T,$$

it follows as before that

$$\widetilde{\mathcal{E}}(\phi(\cdot,t)) - \mathcal{E}(\widetilde{u}(\cdot;1)) \geq C\rho(\phi(\cdot,t)\widetilde{u}(\cdot;1))^2$$

for some $C>0$ so long as $\rho(\phi(\cdot,t),\widetilde{u}(\cdot;1))$ is sufficiently small. Since $\widetilde{u}(\cdot;1)$ is a critical point of $\widetilde{\mathcal{E}}$ by construction, we have that

$$C\rho(\phi(\cdot,t),\widetilde{u}(\cdot;1))^2 \leq \widetilde{\mathcal{E}}(\phi(\cdot,t)) - \mathcal{E}(\widetilde{u}(\cdot;1)) \leq C'\rho(\phi(\cdot,t)\widetilde{u}(\cdot;1))^2$$

for some constants C and C'. Moreover, it follows by the triangle inequality that

$$\|\phi_0 - \widetilde{u}(\cdot;1)\|_X \leq \|\phi_0 - u\|_X + \|u - \widetilde{u}(\cdot;1)\|_X \leq C\varepsilon$$

for some $C>0$ and hence there exists a positive constant $C>0$ such that

$$\rho(\phi(\cdot,t),u) \leq \rho(\phi(\cdot,t),\widetilde{u}(\cdot;1)) + \|\widetilde{u}(\cdot;1) - u\|_X \leq C\varepsilon$$

for all $t>0$. This completes the proof of Theorem 2 in the case $\phi_0 \notin \Sigma_0$. □

By Theorem 2, it follows that $\{T,M,P\}_{a,E,c} > 0$ may not be sufficient for nonlinear stability of a periodic traveling wave of (1). In particular, notice that in the solitary wave theory it is always true that the operator $\mathcal{L}$ has only one $L^2(\mathbb{R})$ eigenvalue, and hence it is always true that any unstable eigenvalues of the linearized operator must be real. In the periodic context, however, we see this is only true if $T_E > 0$, which is not true for all periodic traveling wave solutions of (1): for example, it is clear that the cnoidal wave solutions of the modified BBM equation corresponding to (1) with $f(u) = u^3$ of sufficiently long wave period satisfy $T_E < 0$. Thus, it may be possible in certain situations that the linearized operator $\mathcal{A}$ has unstable T-periodic eigenvalues which are not real. Moreover, even if T_E and $\{T, M, P\}_{a,E,c}$ are positive, and hence one has periodic spectral stability, it is not clear from this analysis whether one has nonlinear stability since the sign of the Jacobian $\{T, M\}_{a,E}$ still plays a seemingly large role[7]. This phenomenon, which stands in contrast to the solitary wave theory, is a reflection of how the periodic traveling wave solutions of (1) have a much richer structure than the solitary waves, allowing for possibly more interesting dynamics.

5. Modulational Instability Analysis

In this section, we begin our study of the spectral stability of periodic traveling wave solutions of the gBBM equation (1) to arbitrary localized perturbations. In particular, our methods will detect instabilities of such solutions to long wavelength perturbations, i.e. to slow modulations of the underlying wave. Such stability analysis seems to be a bit more physical than the periodic stability analysis conducted in the previous section in the following sense: in physical applications, one should probably never expect to find an exact spatially periodic wave. Instead, what one often sees is a solution which on small space-time scales seems to exhibit periodic behavior, but over larger scales is clearly seen not to be periodic due to a slow variations in the amplitude, frequency, etc, i.e. one sees slow modulations physical parameters defining the solution. Thus, it seems natural to study the stability of such solutions by idealizing them as exact spatially periodic waves and then study the stability of this idealized wave to slow modulations in the underlying parameters. This is precisely the goal of the modulational stability analysis in this paper. Moreover, in terms of the long-time stability of such solutions it is clear that a low frequency analysis of the linearized operator is vital to obtaining suitable bounds on the corresponding solution operator of the nonlinear equation.

Remark 6. *Throughout the text, modulational stability will always refer to the spectral stability of the underlying periodic wave to long-wavelength (low-wave number) perturbations. In other words, spectral stability in a neighborhood of the origin.*

To begin, notice that by Lemma 3 the linearized operator $\mathcal{A}$ has a T-periodic eigenvalue at the origin in the spectral plane of multiplicity (generically) three. Thus, as we allow small variations in the Floquet parameter, we expect that there will be three branches of continuous spectrum which bifurcate from the origin. According to Proposition 1, one of these branches must be confined to the imaginary axis, and hence will not contribute to any spectral instability. In order to determine if the other two branches bifurcate off the imaginary axis or not, we derive an asymptotic expansion of the function $D(\mu, e^{i\kappa})$ for $|\mu| + |\kappa| \ll 1$. As a result, we will see that, to leading order, the local structure of the spectrum near the origin is governed by a homogeneous polynomial of degree three in the

[7]However, one should be aware of the recent works [16] and [12] in which perturbation methods and more delicate functional analysis were used to analyze this problem for the gKdV in a way which provides more precise results than the above variational methods.

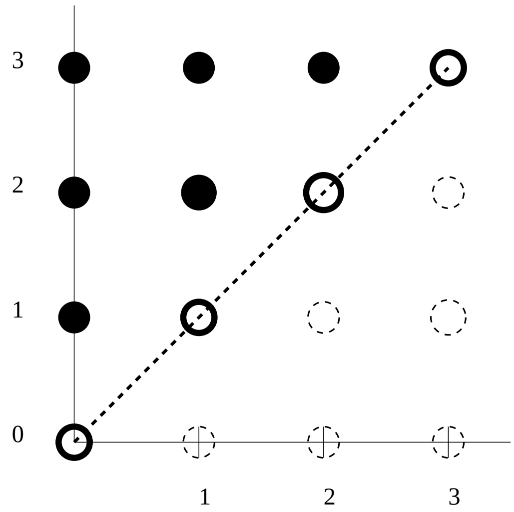

FIGURE 2. The Newton diagram corresponding to the asymptotic expansion of $D(\mu, e^{i\kappa}) = 0$ in a neighborhood of $(\mu,\kappa) = (0,0)$ is shown to $O(|\mu|^3 + \kappa^3)$. Terms associated to open circles with dashed boundary are shown to vanish due to the natural symmetries inherent in (7). The open circles with dark boundary are non-vanishing terms which are a part of the lower convex hull, and hence contribute to the dominant balance. The closed dark circles lie above the lower convex hull and thus do not contribute to the leading order asymptotics. See [6] and [11] for more details.

variables μ and κ. We then evaluate the coefficients of the resulting polynomial in terms of Jacobians of various maps from the parameters (a, E, c) to the quantities T, M, and P. To this end, we begin with the following Lemma.

Lemma 5. *If $\{T, M, P\}_{a,E,c} \neq 0$, the equation $D(\mu, e^{i\kappa}) = 0$ has the following normal form in a neighborhood of $(\mu,\kappa) = (0,0)$:*

$$-(i\kappa)^3 + \frac{(i\kappa)^2\mu T}{c} + \frac{i\kappa\mu^2}{2}\left(\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) - \frac{T^2}{c^2}\right) - \mu^3\{T, M, P\}_{a,E,c} + \mathcal{O}(|\mu|^4 + \kappa^4) = 0 \tag{18}$$

whose Newton diagram is depicted in Figure 2.

Proof. Define functions a and b on a neighborhood of $\mu = 0$ by

$$D(\mu, e^{i\kappa}) = -\eta^3 + (a(\mu) - 3)\eta^2 + b(\mu)\eta + D(\mu, 1). \tag{19}$$

where $\eta = e^{i\kappa} - 1$ is small. In particular, notice that

$$\begin{aligned} a(\mu) &= \operatorname{tr}(\mathbf{M}(\mu)) \\ b(\mu) &= \frac{1}{2}\left(\operatorname{tr}((\mathbf{M}(\mu) - I)^2) - \operatorname{tr}(\mathbf{M}(\mu) - I)^2\right). \end{aligned}$$

Now, using the fact that the spectral problem (7) is invariant under the transformation $(x,\mu) \mapsto (-x,-\mu)$, it follows that the matrices $\mathbf{M}(-\mu)$ and $\mathbf{M}(\mu)^{-1}$ are similar for all $\mu \in \mathbb{C}$. In particular, it follows that

$$\begin{aligned}
e^{-\mu T}\det(M(\mu)-\lambda) &= -\lambda^3 \det\left(M(-\mu)-\frac{1}{\lambda}\right)\\
&= -\lambda^3\left(\left(1-\frac{1}{\lambda}\right)^3 + (a(-\mu)-3)\left(\frac{1}{\lambda}-1\right)^2 + b(-\mu)\left(\frac{1}{\lambda}-1\right) + D(-\mu,1)\right)\\
&= -(\lambda-1)^3 - (a(-\mu)-3)\,\lambda\,(\lambda-1)^2 + b(-\mu)\lambda^2(\lambda-1) - \lambda^3 D(-\mu,1)
\end{aligned}$$

By comparing the $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^3)$ terms above to those in (19), we have the relations

$$\begin{cases} e^{-\mu T/c}a(\mu) = 2a(-\mu) - b(-\mu) - 3, \\ -e^{-\mu T/c} = -a(-\mu) + b(-\mu) - D(-\mu,1) + 2. \end{cases}$$

Differentiating with respect to μ and evaluating at $\mu = 0$ immediately implies $b'(0) = 0$ and $a'(0) = \frac{T}{c}$. Similarly, it follows that $b''(0) = a''(0) - \frac{T^2}{c^2} = \operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \frac{T^2}{c^2}$. The proof is now complete by Lemma 3 and the fact that $\eta = i\kappa + \mathcal{O}(\kappa^2)$. □

It follows that the structure of $\operatorname{spec}(\mathcal{A})$ in a neighborhood of the origin is, to leading order, determined by the above homogeneous polynomial in κ and μ. Due to the triple root of $D(\cdot,1)$ at $\mu = 0$ the implicit function theorem fails, but can be trivially corrected by considering the appropriate change of variables. This leads us to the following theorem giving a modulational stability index for traveling wave solutions of (1).

Theorem 3. *With the above notation, define*

$$\begin{aligned}
\Delta = \frac{1}{4}&\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \left(\frac{T}{c}\right)^2\right)^2\left(2\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \left(\frac{T}{c}\right)^2\right)\\
&- 27\{T,M,P\}_{a,E,c}^2 + 6\{T,M,P\}_{a,E,c}\left(\frac{3\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))}{2} - \frac{5\left(\frac{T}{c}\right)^2}{3}\right)\left(\frac{T}{c}\right)
\end{aligned}$$

and suppose that $\{T,M,P\}_{a,E,c} \neq 0$. If $\Delta > 0$, then the spectrum of the linearized operator $\mathcal{A}$ in a neighborhood of the origin consists of the imaginary axis with a triple covering. If $\Delta < 0$, then $\sigma(\mathcal{A})$ in a neighborhood of the origin consists of the imaginary axis with multiplicity one together with two curves which are tangent to lines through the origin.

Proof. Since, to leading order in μ and κ, the Evans function is homogeneous by Lemma 5 it seems natural to work with the projective coordinate $y = \frac{i\mu}{\kappa}$. Making such a change of variables, Lemma 5 implies the equation $D(\mu, e^{i\kappa}) = 0$ can be written as

$$1 + \frac{yT}{c} - \frac{y^2}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) - \left(\frac{T}{c}\right)^2\right) - y^3\{T,M,P\}_{a,E,c} + \kappa E(\kappa,y) = 0 \tag{20}$$

where $E(\kappa,y)$ is continuous in a neighborhood of the origin. Let $y_{1,2,3}$ denote the three roots of the above cubic in y corresponding to $E(\kappa,y) = 0$. Assuming $\Delta \neq 0$ it follows that $y_{1,2,3}$ are distinct and hence the implicit function theorem applies giving three distinct solutions of (20) in a neighborhood of each of the $y_{1,2,3}$. In terms of the original variable μ, this gives three solution branches

$$\mu_{1,2,3} = -iy_{1,2,3}\kappa + \mathcal{O}(\kappa^2).$$

If $\Delta > 0$, then $y_{1,2,3} \in \mathbb{R}$, giving three branches of spectrum emerging from the origin tangent to the imaginary axis. From the Hamiltonian symmetry of (7), the spectrum is symmetric with respect to reflections across the imaginary axis and hence $\Delta > 0$ implies these thee branches of spectrum must in fact lie on the imaginary axis, proving the existence of an interval of spectrum of multiplicity three on the imaginary axis. In the case $\Delta < 0$, it follows that one of the roots, y_1 say, is real while the other two $y_{2,3}$ occur in a complex conjugate pairs yielding one branch along the imaginary axis and two branches emerging from the origin tangent to lines through the origin with angel $\arg(-iy_{2,3})$, which clearly implies the desired instability. $\square$

Remark 7. *The modulational instability index Δ derived above is considerably more complicated than the one derived by Bronski and Johnson* [11] *for generalized Korteweg-de Vries equation*

$$u_t = u_{xxx} - cu_x + (f(u))_x \,. \tag{21}$$

When considering the spectral stability of periodic traveling wave solutions of (21)*, it was shown that there exists a modulational instability index Δ_{gKdV} such that $\Delta_{gKdV} < 0$ implies modulational instability, and $\Delta_{gKdV} > 0$ implies modulational stability. In this case, the dominant balance is somewhat simpler due to the fact that the trace of the operator $\mathbf{M}_\mu(0)$ vanishes, and hence the $\kappa^2\mu$ term in the corresponding Newton diagram vanishes. As a result, the modulational instability index took the form*

$$\Delta_{gKdV} = \frac{1}{2}\left(\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))\right)^3 - \frac{1}{3}\left(\frac{3}{2}\{T,M,P\}_{a,E,c})\right)^2.$$

It follows that the sign of $\{T,M,P\}_{a,E,c}$ does not a priori effect the modulational stability of such periodic solutions of equation (21)*. However, in the case of the gBBM equation, the fact that $\operatorname{tr}(\mathbf{M}_\mu(0)) \neq 0$ seems to suggest that the sign of $\{T,M,P\}_{a,E,c}$ does indeed affect the modulational stability of an underlying periodic wave. This may be somewhat unexpected as the instability detected by $\{T,M,P\}_{a,E,c}$ in Theorem 1 is not local to the origin in the spectral plane.*

Our next goal is to use the integrable nature of (2) to express $\operatorname{tr}(\mathbf{M}_{\mu\mu}(0))$ in terms of the underlying periodic traveling wave u. This is the content of the following lemma.

Lemma 6. *We have the following identity:*

$$\frac{1}{2}\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) = \{T,P\}_{E,c} + 2\{M,P\}_{a,E} - V'(u_-)\{T,M\}_{a,E}.$$

Proof. This proof is essentially an extension of that of Lemma 3. Using the same notation, a straightforward yet tedious calculation yields

$$\begin{aligned}\operatorname{tr}(\mathbf{M}_{\mu\mu}(0)) &= -\frac{2}{\mu^2}\operatorname{tr}\left(\operatorname{cof}\left(\delta\,\mathbf{W}(\mu)\,\mathbf{W}(0,0)^{-1}\right)\right)\big|_{\mu=0}\\ &= 2\{T,P\}_{E,c} + 2\{M,P\}_{a,E} + 2\{T,M\}_{a,c} - \frac{2}{c}V'(u_-)\{T,M\}_{a,E}\end{aligned}$$

The proof is completed by noting that $\{T,M\}_{a,c} = \{M,P\}_{E,a}$ since $T_c = P_E$ and $M_c = P_a$. □

Thus, the modulational stability of a given periodic traveling wave solution to (1) can be determined from information about the underlying solution itself. Interestingly, from Lemma 6 the modulational instability index seems to depend directly on the turning point u_-, a feature not seen in the corresponding analysis of the generalized KdV equation in [11]. However, one can trace this dependence back to the definition of w_1^1 in Lemma 3 and hence seems to be unavoidable. In the next section, we will analyze the above stability indices in neighborhoods of homoclinic orbits in phase space, complimenting the results of Gardner [17] by providing stability results in this limit.

6. Analysis of Stability Indices in the Solitary Wave Limit

The goal of this section is to study the long wavelength asymptotics of the stability indices derived in the previous sectios. Throughout, we restrict ourselves to the case of a power-nonlinearity $f(u) = u^{p+1}/(p+1)$: this restriction is vital to our calculation since in this case we gain an additional scaling symmetry. In particular, if $v(x;a,E)$ satisfies the differential equation

$$\frac{1}{2}v_x^2 - v^2 + \frac{1}{(p+1)(p+2)}v^{p+2} = au + E, \tag{22}$$

then a straight forward calculation shows that we can express the periodic solution $u(x;a,E,c)$ as

$$u(x;a,E,c) = (c-1)^{1/p}v\left(\left(\frac{c-1}{c}\right)^{1/2}x;\frac{a}{c^{1+1/p}},\frac{E}{c^{1+2/p}}\right). \tag{23}$$

This additional scaling allows explicit calculations of P_c, which ends up determining the stability of periodic traveling wave solutions of (1) of sufficiently long wavelength.

A reasonable guess would be that long-wavelength periodic traveling wave solutions of (1) have the same stability properties as the limiting homoclinic orbit (solitary wave). However, as noted in the introduction, this is a highly singular limit and so it is not immediately clear whether such results are true. It is well known that the solitary wave is spectrally unstable if and only if $p > 4$ and $1 < c < c_0(p)$ for some critical wave speed $c_0(p)$. It follows from the work of Gardner [17] that periodic orbits bounded by the homoclinic orbit in phase space which are sufficiently close to the homoclinic orbit are unstable if the solitary wave is unstable. In particular, it is proved that the linearized operator $\mathcal{A}$ for the periodic traveling wave u with sufficiently long wavelength has a "loop" of spectrum in the neighborhood of any unstable eigenvalues of the limiting solitary wave. Thus, Gardner's analysis deduces instability of long wavelength periodic waves from instability of the limiting solitary waves. The results of this section compliment this theory by also proving that the stability of the limiting wave is inherited by nearby periodic waves bounded by the homoclinic orbit in phase space.

In terms of the finite-wavelength instability index, it seems reasonable by Theorem 1 to expect that for periodic traveling wave solutions below the seperatrix of sufficiently long wavelength, $\{T, M, P\}_{a,E,c} < 0$ for if and only if $p > 4$ and $1 < c < c_0(p)$. What is unclear is whether such a result should be true for the modulational instability index Δ. Indeed, although Gardner's results prove that the spectrum of the linearization about a periodic traveling wave of sufficiently long wavelength in the neighborhood of the origin contains the image of a continuous map of the unit circle, to our knowledge it has never been proved that this map is injective. Thus, it is not clear from Gardner's results whether a modulational instability will arise from this eigenvalue since it is possible this "loop" is confined to the imaginary axis: we show that in fact one has modulational instability in this limit precisely when the limiting solitary wave is unstable.

The main result for this section is the following theorem, which is based on asymptotic estimates of the instability indices derived in section 3. In particular, we prove the sign of both instability indices in the solitary wave limit is determined by the sign of $\frac{\partial}{\partial c}P = P_c$, where $P = P(a, E, c)$ is the momentum of the periodic wave $u(x; a, E, c)$. The proof is based on a more technical lemma, which shows that $\frac{\partial}{\partial a}M(a, E, c) < 0$ for waves of sufficiently long wavelength, i.e. for a, E sufficiently close to zero. We begin by outlining the proof of the following theorem, and then fill in the necessary lemma's afterward.

Theorem 4. *Let $f(u) = u^{p+1}$ for some $p \geq 1$ and let $u(x; a, E, c)$, with a sufficiently small, be a periodic solution of the traveling wave ODE* (2) *which corresponds to an orbit inside homoclinic orbit in phase space. If $T(a, E, c)$ is sufficiently large, then u is modulationally and nonlinearly stable for all $1 \leq p < 4$. For $p > 4$, there exists a critical wave speed $c(p) > 1$ such that if $T(a, E, c)$ is sufficiently large, the solution u is orbitally and modulationally stable if $c > c(p)$, while it is modulationally unstable for $1 < c < c(p)$.*

Proof. Without loss of generality, suppose that the periodic solution $u(x; a, E, c)$ of (2) corresponds to a branch cut of the function $\sqrt{E - V(u; a, c)}$ with positive right end point. Now, when a and E are small there are two turning points r_1, r_2 in the neighborhood of the origin and a third turning point r_3 which is bounded away from the origin. In the solitary wave limit $a, E \to 0$ a straight forward calculation gives that $r_2 - r_1 = \mathcal{O}\left(\sqrt{a^2 - 2(c-1)E}\right)$ From this, it follows that the period satisfies the asymptotic relation

$$T(a, E, c) = \mathcal{O}\left(\ln\left(a^2 - 2(c-1)E\right)\right). \tag{24}$$

To see this, notice that by our assumptions on r_1, r_2, and r_3 we can write $E - V(u; a, 1) = (r - r_1)(r - r_2)(r_3 - r)Q(r)$ where $Q(r)$ is positive on the set $[r_1, r_3]$. The period can then be expressed as

$$\begin{aligned}
T(a,E,c) &= \sqrt{2c}\int_{r_2}^{r_3}\frac{dr}{\sqrt{(r-r_1)(r-r_2)(r_3-r)Q(r)}}\\
&= \sqrt{2c}\int_{r_2-r_1}^{r_2-r_1+\delta}\frac{dr}{\sqrt{r(r-(r_2-r_1))(r_3-r_1-r)Q(r+r_1)}}\\
&+ \sqrt{2c}\int_{r_2-r_1+\delta}^{r_3-r_1}\frac{dr}{\sqrt{r(r-(r_2-r_1))(r_3-r_1-r)Q(r+r_1)}}
\end{aligned}$$

The integral over the set $(r_2-r_1+\delta, r_3-r_1)$ is clearly $\mathcal{O}(1)$ as $(a,E)\to(0,0)$. For the other integral, notice that

$$(r_3-r_1-r)Q(r+r_1)>0$$

on the set $[r_2-r_1, r_2-r_1+\delta]$. Thus, in the limit as $(a,E)\to\infty$ we have

$$\begin{aligned}
T(a,E,c) &\sim \int_{r_2-r_1}^{r_2-r_1+\delta}\frac{dr}{\sqrt{r(r-(r_2-r_1))}}\\
&= -4\ln(4(r_2-r_1)) + 2\ln(2(\sqrt{r_2-r_1+\delta}+\sqrt{\delta}))\\
&\sim -\ln(r_2-r_1)
\end{aligned}$$

from which (24) follows. Similar computations yield the following asymptotic relations for a and E sufficiently small:

$$\begin{aligned}
P(a,E,c) &= \mathcal{O}(1)\\
M(a,E,c) &= \mathcal{O}\left(a\ln\left(a^2-2(c-1)E\right)\right)\\
M_a(a,E,c) &= \mathcal{O}\left(\frac{a^2}{a^2-2(c-1)E}\right)\\
T_a(a,E,c) &= \mathcal{O}\left(\frac{a}{a^2-2(c-1)E}\right) = M_E(a,E,c)\\
T_E(a,E,c) &= \mathcal{O}\left(\frac{1}{a^2-2(c-1)E}\right)\\
T_c(a,E,c) &= \mathcal{O}\left(\frac{E}{a^2-2(c-1)E}\right) = 2P_E + \frac{1}{c}T.
\end{aligned}$$

Thanks to the above scaling we know $M_c = 2P_a + \frac{1}{c}T$ can be expressed as a linear combination of M, M_a, and $M_E = T_a$. Similarly, P_c can be expressed as a linear combination of P, $2P_a = M_c - \frac{1}{c}T$ and P_E. It follows that the asymptotically largest minor of $\{T,M,P\}_{a,E,c}$ for a and E small is $-T_E M_a P_c$ and, moreover,

$$\operatorname{tr}\left(\mathbf{M}_{\mu\mu}(0)\right) \sim T_E P_c.$$

In Lemma 7 below, we will show that $M_a<0$ for such a and E. Moreover, it is clear that for such periodic waves with sufficiently long wavelength satisfy $T_E>0$. Therefore, it follows that both stability indices are determined by the sign of $P_c(a,E,c)$ in the solitary wave limit. The theorem now follows by Lemma 8 below. $\square$

Remark 8. *Notice that since the theorem holds for a small, i.e. for nonsymmetric potentials, one does not need to rely on positivity of the underlying wave. However, the hypothesis that the periodic orbit lie inside the homoclinic orbit in phase space is essential in Theorem 4. That is, referring back to Figure 1, Theorem 4 applies to the orbits with energy levels E_1 and E_2, but not to the orbit with energy E_3. Indeed, considering cnoidal waves of the modified BBM (corresponding to $p = 2$ with $a = 0$ and energy levels of the type E_3 in Figure 1) of sufficiently large period, it is clear that one has $T_E < 0$ in which case the above proof fails. Indeed, in the recent work* [12] *it was shown that such solutions of the modified KdV equation are indeed spectrally unstable even through the homoclinic orbit is orbitally stable to localized perturbations. Although we have not carried out the analogous computations in this case, we highly expect the same phenomenon to occur in the present context. Nevertheless, Theorem 4 does apply to the well studied dnoidal wave solutions of the modifed BBM equation.*

Remark 9. *Notice that the fact that the finite wavelength instability index is determined by the sign of P_c in the solitary wave limit is not surprising, since this is exactly what detects the stability of the limiting solitary waves. What is surprising is that the same quantity controls the modulational stability index in the same limit. As mentioned above, it has not been know if the instability of the limiting solitary wave forces a modulational instability: the answer is shown to be affirmative by Theorem 4.*

In order to complete the proof Theorem 4, we must prove a few more technical lemmas. The first is used in showing that the sign of the modulational and finite-wavelength instability indices are determined completely by the sign of $P_c(a, E, c)$ in the limit as a, E tend to zero. This is the result of the following lemma.

Lemma 7. *For a, E sufficiently small, $M_a(a, E, c) \leq 0$ for all $c > 1$.*

Proof. Notice it is sufficient to prove $\frac{\partial}{\partial a} M(a, 0, c) \leq 0$ for all $c > 1$ and a sufficiently small. Now, $M(a, 0, c)$ can be written as

$$M(a,0,c) = \sqrt{2c} \int_0^{r(a,c)} \frac{\sqrt{u}\, du}{\sqrt{a + \frac{c-1}{2} u - \frac{1}{(p+1)(p+2)} u^{p+1}}}$$

where $r(a, c)$ is the smallest positive root of the polynomial equation $a+\frac{c-1}{2} r - \frac{1}{(p+1)(p+2)} r^{p+1} = 0$. Setting $a = ((p+1)(p+2))^{1/p} \alpha$, we have

$$M\left(\alpha, E, c\right) = ((p+1)(p+2))^{1/p} \sqrt{2c} \int_0^{\widetilde{r}(\alpha,c)} \frac{\sqrt{u}\, du}{\sqrt{\alpha + \frac{c-1}{2} u - u^{p+1}}}$$

where $\widetilde{r}(\alpha, c)$ is the smallest positive root of the polynomial $\alpha + \frac{c-1}{2} r - r^{p+1} = 0$. Notice for a fixed wave speed $c > 1$, $\widetilde{r}(\alpha, c)$ is a smooth function of α for α sufficiently small and satisfies

$$\widetilde{r}(\alpha, c) = \left(\frac{c-1}{2}\right)^{1/p} + \frac{2a}{p(c-1)} + \mathcal{O}(a^2).$$

The goal is to now rewrite the above integral over a fixed domain and show that the integrand is a decreasing function of α for a fixed $c > 1$.

Making the substitution $u \to \widetilde{r}(\alpha, c)u$ yields the expression

$$\frac{M(\alpha,0,c)}{((p+1)(p+2))^{1/p}\sqrt{2c}} = \int_0^1 \frac{\sqrt{u}\,du}{\sqrt{\alpha\,\widetilde{r}(\alpha,c)^{-3} + \left(\frac{c-1}{2}\right) u\,\widetilde{r}(\alpha,c)^{-2} - \widetilde{r}(\alpha,c)^{p-2}\,u^{p+1}}}$$

Now, we can use the above expansion of $r(a, c)$ to conclude that

$$\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\left(\alpha\,\widetilde{r}(\alpha,c)^{-3} + \left(\frac{c-1}{2}\right) u\,\widetilde{r}(\alpha,c)^{-2} - \widetilde{r}(\alpha,c)^{p-2}\,u^{p+1}\right)$$

is positive on the open interval $(0, 1)$ for all $p \geq 1$ and $c > 1$, which completes the proof. $\square$

Finally, it is left to analyze the asymptotic behavior for a fixed wavespeed $c > 1$ of the quantity $P_c(a, E, c)$ in the solitary wave limit. This is the content of the following lemma.

Lemma 8. *In the case of power non-linearity $f(u) = u^{p+1}/(p + 1)$, the momentum $P = P(a, E, c)$ satisfies*

$$\frac{\partial}{\partial c}P\left(a,E,c\right) = \frac{(c-1)^{2/p-1/2}c^{1/2}I\left(\frac{4}{p}\right)}{2pc(c-1)}\left(4c - p + \frac{(4c+p)(c-1)p}{(4+p)c}\right) + \mathcal{O}(|a|+|E|)$$

in the solitary wave limit $(a, E) \to (0, 0)$, where $I(r) = \int_{-\infty}^{\infty} \operatorname{sech}^r(x)dx$. In particular, for a and E sufficiently small, if $p < 4$ then $\frac{\partial}{\partial c}P(a, E, c) > 0$ for all $c > 1$ while if $p > 4$ then $\frac{\partial}{\partial c}P(a, E, c) < 0$ for $1 < c < c_0(p)$ and $\frac{\partial}{\partial c}P(u; a, E, c) > 0$ for $c > c_0(p)$, where

$$c_0(p) = \frac{p\left(1+\sqrt{2+\frac{1}{2}p}\right)}{4+2p}.$$

Proof. The proof is based on scaling and a limiting argument, as well as a modification of the analysis in [26]. To begin, let $v = v(x; a, E)$ satisfy the differential equation (22) so that $u(x; a, E, c)$ can be expressed via scaling as in (23), and assume with out loss of generality that $x = 0$ be an absolute max of $v(x; a, E, c)$. Clearly, the solitary wave limit corresponds to taking $(a, E) \to (0, 0)$ with fixed wave speed $c > 1$. Notice that on any compact subset Γ of $\mathbb{R}$, we have

$$v(x;a,E) \to \left(\frac{(p+2)(p+1)}{2}\right)^{1/p} \operatorname{sech}^{2/p}\left(\frac{p}{2}x + x_0\right)$$

uniformly as $(a, E) \to (0, 0)$ on Γ for some $x_0 \in \mathbb{R}$. Using (22), it follows that

$$\int_0^{T((c-1)/c)^{1/2}} v^2(x)dx = \sqrt{2}\int_{\tilde{u}_-}^{\tilde{u}_+} \frac{u^2\,du}{\sqrt{E + v^2 - \frac{1}{(p+1)(p+2)}v^{p+2} + av}}$$

where $\tilde{u}_\pm$ are the roots of $E+v^2-\frac{1}{(p+1)(p+2)}v^{p+2}+av = 0$ satisfying the original hypothesis of the roots $u_\pm$ of $E - V(u; a, c) = 0$. Since $\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v^2(x)dx = \mathcal{O}(1)$ as $(a, E) \to (0, 0)$, the dominated convergence theorem along with the fact that $\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v^2(x)dx$ is a C^1 function of a and E implies that

$$\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v(x;a,E)^2dx = \left(\frac{(p+2)(p+1)}{2}\right)^{2/p}\frac{1}{p}I\left(\frac{4}{p}\right)+\mathcal{O}(|a|+|E|).$$

Similarly, it follows that

$$\int_{-T((c-1)/c)^{1/2}/2}^{T((c-1)/c)^{1/2}/2} v_x(x;a,E)^2dx = \left(\frac{(p+2)(p+1)}{2}\right)^{2/p}\frac{1}{4+p}\,I\left(\frac{4}{p}\right)+\mathcal{O}(|a|+|E|).$$

Using (23), we now have

$$\begin{aligned}p\left(\frac{(p+2)(p+1)}{2}\right)^{-2/p}&\int_{-T/2}^{T/2}\left(u(x;a,E,c)^2+u_x(x;a,E,c)^2\right)dx\\ &=(c-1)^{2/p-1/2}c^{1/2}I\left(\frac{4}{p}\right)+(c-1)^{2/p+1/2}c^{-1/2}\frac{p}{4+p}\,I\left(\frac{4}{p}\right)\\ &\quad+\mathcal{O}(|a|+|E|)\end{aligned}$$

as $a,E\to 0$, and hence it follows by differentiation that

$$\begin{aligned}p\left(\frac{(p+2)(p+1)}{2}\right)^{-2/p}&\frac{\partial}{\partial c}\int_{-T/2}^{T/2}\left(u(x;a,E,c)^2+u_x(x;a,E,c)^2\right)dx=\\ &\frac{(c-1)^{2/p-1/2}c^{1/2}I\left(\frac{4}{p}\right)}{2pc(c-1)}\left(4c-p+\frac{(4c+p)(c-1)p}{(4+p)c}\right)\\ &+\mathcal{O}(|a|+|E|)\end{aligned}$$

as claimed, where we have used that $T_cu_-^2=\mathcal{O}(|a|+|E|)$. The lemma now follows by solving the quadratic equation $(4+p)c(4c-p)+(4c+p)(c-1)p=0$ for c and recalling the restriction that $c>1$. □

The proof of Theorem 4 is now complete by Lemmas 7 and 8. As a consequence, the finite-wavelength instability index $\{T,M,P\}_{a,E,c}$ seems to be a somewhat natural generalization of the solitary wave stability index, in the sense that the well known stability properties of solitary waves are recovered in a long-wavelength limit. Moreover, this gives an extension of the results of Gardner in the case of generalized BBM equation by proving the marginally stable eigenvalue of the solitary wave at the origin contributes to modulational instabilities of nearby periodic waves when ever the solitary wave is unstable.

7. Analysis of Stability Indices for BBM and modified BBM Equations

Finally, we consider as case examples the elliptic function solutions of the BBM and modified BBM (mBBM) equations. To begin, we consider the mBBM equation with $f(u)=u^3$. In the special case in our theory of $a=0$, it turns out that all periodic traveling wave solutions $u(x;0,E,c)$ can be expressed in terms of Jacobi elliptic functions: when $E>0$ such solutions correspond to cnoidal waves, expressible in terms of the $\mathrm{cn}(x,k)$ function, while when $E < 0$ such solutions correspond to dnoidal waves, expressible in terms of the function $\mathrm{dn}(x, k)$. In particular, if $a = 0$ then solutions with $E > 0$ correspond to orbits outside the homoclinic in phase space, while solutions with $E < 0$ correspond to orbits inside the homoclinic.

These claims can be verified by using arguments parallel to those given in Chapter 3 of [3] for the modified KdV equation. For completeness, however, we given an outline of the results here. First, notice that when $f(u) = u^3$ and $a = 0$, the traveling wave ODE (2) can be integrated twice to yield the relation

$$u_x^2 = \frac{1}{2c}\left(2(c-1)u^2 - u^4 + 4E\right) \tag{25}$$

Defining $g(t) = \frac{1}{c}\left(2(c-1)t^2 - t^4 + 4E\right)$, it follows that the equation $g(t) = 0$ has either zero, two, or four real roots. In the case of zero roots, phase plane analysis implies (25) has no periodic solutions. Moreover, for $(0, E, c) \in \Omega$, using the notation as in Lemma 1, $g(t)$ has four real solutions in the case $E < 0$ and two real solutions in the case $E > 0$. Lets first consider the case $E < 0$ here. Noting that g is an even function, we may rewrite (25) as

$$u_x^2 = \frac{1}{2c}\left(u^2 - \eta_1^2\right)\left(\eta_2^2 - u^2\right). \tag{26}$$

Since we are interested in the case where g has four real solutions, we can assume the η_i are the positive roots of g which we order as $0 < \eta_1 < \eta_2$ and notice this implies we need $\eta_1 \leq u \leq \eta_2$ for all x. Defining then $\phi = \frac{u}{\eta_1}$ and $k^2 = \left(\eta_1^2 - \eta_2^2\right)/\eta_1^2$, we can write (26) as

$$\phi_x^2 = \frac{\eta_1^2}{2c}\left(\phi^2 - 1\right)\left(1 - k^2 - \phi^2\right), \tag{27}$$

which, after defining the additional variable ψ via the relation $\phi = 1 - k^2\sin^2(\psi)$ yields

$$\psi_x^2 = \frac{\eta_1^2}{2c}\left(1 - k^2\sin^2(\psi)\right).$$

This final equation can be integrated via quadrature, yielding the explicit solution

$$\int_0^{\psi(x)} \frac{dt}{1 - k^2\sin^2(t)} = \frac{\eta_1 x}{\sqrt{2c}},$$

which immediately yields the equality $\sin(\psi) = \mathrm{sn}\left(\frac{\eta_1 x}{\sqrt{2c}}; k\right)$. Therefore, we obtain the solution

$$\phi(x) = \sqrt{1 - k^2\,\mathrm{sn}^2\left(\frac{\eta_1 x}{\sqrt{2c}}, k\right)} = \mathrm{dn}\left(\frac{\eta_1 x}{\sqrt{2c}}; k\right)$$

of the equation (27). It follows that when $E < 0$ and $(0, E, c) \in \Omega$, all solutions of the traveling wave ODE (25) are expressible in terms of the Jacobi-elliptic dnoidal function dn. The constant $k \in [0, 1)$ is the elliptic modulus: notice that when $k \to 1^-$ the root η_1 tends to zero and hence the period of the corresponding solution tends to infinity, i.e. the periodic orbits approach the bounding homoclinic orbit from the inside in this limit. Similarly, when $k \to 0^+$ the two roots η_1 and η_2 coalesce, corresponding to the equilibrium solution of (25). In the alternate case, when seeking negative solutions alternating between the two negative roots, a similar computation shows this solution may also be expressed in terms of the Jacobi-elliptic dnoidal function. Notice that the symmetry of the function g, equivalently, the symmetry of the corresponding effective potential energy V, was essential in classifying the solutions as standard Jacobi elliptic functions. Without this property, no such elementary representation can be given: this explains our restriction to the special case $a = 0$.

Moreover, we note that when $a = 0$ and $E > 0$ the function g has only two real (and hence two imaginary) solutions and similar computations as above implies that all the corresponding solutions of (25) are expressible in terms of the Jacobi-elliptic cnoidal function $\mathrm{cn}(\cdot, k)$ with the limit $k \to 1^-$ again corresponding to the homoclinic orbit: notice in this case, however, the orbits approach the homoclinic from the outside in phase space.

Now that we have characterized the traveling wave solutions of the modified BBM equation with $a = 0$, we apply our stability theories to these special cases. Notice, however, that although we have just identified the solutions as Jacobi elliptic functions, we at no point use this special structure, i.e. the following stability arguments rely on direct analysis of the Jacobians via asymptotic analysis and complex analytic methods rather than complicated elliptic integral arguments. This approach has the advantage of applying to larger classes of periodic waves than just the elliptic ones considered here.

First, note by Theorem 4 it follows that the dnoidal waves (corresponding to $a = 0$ and $E < 0$) with $1 - k$ sufficiently small (corresponding to $0 < -E \ll 1$) are always modulationally stable and nonlinearly (orbitally) stable to perturbations with the same periodic structure. What is unclear from our above analysis is whether the cnoidal waves with k sufficiently close to 1 (corresponding to $0 < E \ll 1$) are exhibit a sense of stability: such waves correspond to periodic orbits outside the seperatrix in phase space, and hence Theorem 4 does not apply to such solutions. Moreover, such waves do not approach any particular solitary wave in any uniform way, even on compact subsets, and hence it is not clear if the stability of the limiting solitary waves at all influences the cnoidal waves. However, notice that for k sufficiently close to unity one must have $T_E < 0$ and hence by the analysis in the proof of Theorem 4 we have $\Delta < 0$. Therefore, the cnoidal waves of the mBBM with sufficiently long wavelength are modulationally unstable. Similarly, in this solitary wave limit one has

$$\lim_{E\to 0^+} \mathrm{sign}\left(\{T, M, P\}_{a,E,c}(0, E, c)\right) = \lim_{E\to 0^+} \mathrm{sign}\left(M_a(0, E, c)\right)$$

and hence one has spectral instability to periodic perturbations if $M_a(0, E, c) < 0$ for $E > 0$ sufficiently small. Using the complex analytic calculations of Bronski, Johnson, and Kapitula [12], we can solve the corresponding Picard-Fuchs system and see that

$$\lim_{E\to 0^+} \frac{\mathrm{disc}(R(u; 0, E, c))M_a(0, E, c)}{(c - 1)T(0, E, c)} = 1. \tag{28}$$

See the appendix for more details of this calculation. Since $\mathrm{disc}(R(u; 0, E, c)) < 0$ for such solutions, we find that $M_a < 0$. Thus, the cnoidal wave solutions of the mBBM equation of sufficiently large wavelength are exponentially unstable to perturbations with the same periodic structure. In particular, it seems like the index $\mathrm{sign}(T_E)$ serves as Maslov index in this problem: when $T_E > 0$ it is $P_c < 0$ which signals the stability near the solitary wave, while it is $P_c > 0$ when[8] $T_E < 0$. This index arises naturally in the solitary wave setting: see [14] for details and discussion. In particular, notice that in the solitary wave setting of Pego and Weinstein and in the long wavelength analysis of Gardner, the Maslov index always has the same sign: however, as seen here this is not the case when considering the full family of periodic traveling waves of the gBBM.

Finally, the same complex analytic methods of [12] can be used to show that in the case of the BBM equation with $f(u) = u^2$, one has the expression

$$\{T, M\}_{a,E} = \frac{-T^2 V'\left(\frac{M}{T}\right)}{12\ \mathrm{disc}(E - V(\,\cdot\,; a, c))}$$

for all periodic traveling waves $u(x; a, E, c)$. Since V' is strictly convex in this case, Jensen's inequality implies that

$$V'\left(\frac{M}{T}\right) \leq \frac{1}{T}\int_0^T V'(u(x))dx = 0$$

and hence $\{T, M\}_{a,E}$ is positive for all such solutions. Moreover, a result of Schaaf [28] implies that all periodic traveling wave solutions satisfy $T_E > 0$: indeed, notice that by Galilean invariance one can always assume that $a = 0$ and hence the result of Schaaf always applies in this case. It follows by Theorem 2 that in the case of the BBM equation, spectral stability in $L^2_{\mathrm{per}}([0, T])$ implies orbital stability of the corresponding T-periodic traveling wave.

8. Concluding Remarks

In this paper, we considered the stability of periodic traveling wave solutions of the generalized Benjamin-Bona-Mahony equation with respect to periodic and localized perturbations. Two stability indices were introduced of the full four parameter family of periodic traveling waves. The first, which is given by the Jacobian of the map between the constants of integration of the traveling wave ordinary differential equation and the conserved quantities of the partial differential equation restricted to the class of periodic traveling wave solutions, serves to count (modulo 2) the number of periodic eigenvalues along the real axis. This is, in some sense, a natural generalization of the analogous calculation for the solitary wave solutions, and reduces to this is the solitary wave limit. The second index, which arises as the discriminant of a cubic which governs the normal form of the linearized operator in a neighborhood of the origin, can also be expressed in terms of the conserved quantities of the partial differential equation and their derivatives with respect to the constants of integration of the ordinary differential equation. This discriminant detects modulational instabilities of the underlying periodic wave, i.e. bands of spectrum off of the imaginary axis in an arbitrary neighborhood of the origin. Finally, using the recent complex analytic methods of [12] it is possible to calculate the stability indices for a large class of equations

[8] While we have not actually proven this here (we only showed this is the case for cnoidal solutions of mBBM), we suspect this is the case for all nonlinearities of the form $f(u) = u^{2k+1}$ for $k \in \mathbb{N}$. For examples where this phenomenon is studied in more cases for the gKdV equation, see [12].

of form (1) with power-law nonlinearity. Such a representation has the obvious advantage of being well suited for numerical investigation. For more general nonlinearities, it is still possible to use numerical techniques to evaluate the given stability indices although the results will (most likely) not be as accurate due to the need to numerically differentiate the underlying wave with respect to the traveling wave parameters.

In our calculations, we heavily used the fact that the ordinary differential equation defining the traveling wave solutions of (1) have sufficient first integrals, and as such is doubtlessly related to the multi-symplectic formalism of Bridges [10]. Many of the ideas from this paper were recently used by the author, in collaboration with Jared C. Bronski, in the analogous study of the spectral stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation [11]. Thus, the general technique of this paper has been successful in two different cases. These successes gives an indication that a rather general modulational stability theory can be developed using these techniques. It would be quite interesting to compare the rigorous results from this paper to the formal modulational stability predictions of Whitham's modulation theory [30]. Moreover, the techniques in this paper could eventually lead to a rigorous justification of the Whitham modulation equations for such dispersive equations: for example, see the recent work of Johnson and Zumbrun [25] in which the Whitham equations are rigorously justified (in the sense long-wavelength stability) for periodic traveling waves of the generalized KdV equations. Further more, the low-frequency Evans analysis and perturbation techniques utilized here apply in a straightforward way to yields results concerning the spectral instability of periodic traveling waves to long-wavelength transverse perturbations in higher dimensional models: for recent examples, see [23] and [24].

It is believed that the methods of this paper can be applied in a direct way to analyze the stability of nonlinear PDE possessing a sufficient number of first integrals to explicitly construct the periodic null-space of the corresponding linearized operator: for example, the NLS equation

$$i\psi_t + \psi_{xx} + F(|\psi|^2)\psi = 0$$

for suitable functions $F : \mathbb{R}^+ \to \mathbb{R}$. However, it is not clear if our techniques can be extended to equations which are deficient in the number of first integrals. An example would be the generalized regularized Boussinesq equation

$$u_{tt} - u_{xx} - (f(u))_{xx} - u_{xxtt} = 0.$$

A straight forward calculation shows that although this equation admits a five parameter family of traveling wave solutions with a *four* parameter submanifold of periodic solutions. Although one can still build a four dimensional basis of null-space of the corresponding linearized operator, it is not clear whether the resulting stability indices can be directly related back to geometric information concerning the original periodic traveling wave. Such a result would be very interesting, as it would allow the use of such techniques to a broader class of partial differential equations admitting traveling wave solutions.

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9. Appendix

In this appendix, we outline the general complex analytic methods used to derive the identity in equation (28). To begin, consider (1) with $f(u) = u^3$ and let $u(x; a, E, c)$ be a given periodic traveling wave. To begin, we notice that that in this case T, M, and P can be represented integrals on a Riemann surface corresponding to the classically allowed region for $\sqrt{E - V(u; a, c)}$, where $V(u; a, c) = E + au + \left(\frac{c-1}{2}\right) u^2 - u^4/4$. In particular, defining $R(u; a, E, c) := E - V(u; a, c)$ we have the following integral representations:

$$T(a, E, c) = \sqrt{\frac{c}{2}} \oint_\Gamma \frac{du}{\sqrt{R(u)}}$$
$$M(a, E, c) = \sqrt{\frac{c}{2}} \oint_\Gamma \frac{u\ du}{\sqrt{R(u)}}$$
$$P(a, E, c) = \frac{1}{2} \left(\sqrt{\frac{c}{2}} \oint_\Gamma \frac{u^2\ du}{\sqrt{R(u)}} + K \right)$$

where K is the classical action defined in (6) and the integrals are taken around the classically allowed region corresponding to the periodic orbit $u(x; a, E, c)$. By differentiating the above relations with respect to the traveling wave parameters, we have the following identities:

$$\nabla_{a,E,c} T(a, E, c) = \left\langle -\frac{1}{2}\sqrt{\frac{c}{2}} \oint_\Gamma \frac{u\ du}{R(u)^{3/2}},\ -\frac{1}{2}\sqrt{\frac{c}{2}} \oint_\Gamma \frac{du}{R(u)^{3/2}},\ -\frac{1}{4}\sqrt{\frac{c}{2}} \oint_\Gamma \frac{u^2\ du}{R(u)^{3/2}} + \frac{1}{2c} T \right\rangle$$
$$\nabla_{a,E,c} M(a, E, c) = \left\langle -\frac{1}{2}\sqrt{\frac{c}{2}} \oint_\Gamma \frac{u^2\ du}{R(u)^{3/2}},\ T_a,\ -\frac{1}{4}\sqrt{\frac{c}{2}} \oint_\Gamma \frac{u^3\ du}{R(u)^{3/2}} + \frac{1}{2c} M \right\rangle$$
$$\nabla_{a,E,c} P(a, E, c) = \left\langle M_c,\ T_c,\ -\frac{1}{8}\sqrt{\frac{c}{2}} \oint \frac{u^4\ du}{R(u)^{3/2}} + \frac{1}{c} P - \frac{3}{4} K \right\rangle$$

In particular, we see that each of the above derivatives can be expressed as linear combinations of the classical action K, the k^{th} moment of the solution u

$$\mu_k = \oint \frac{u^k\ du}{\sqrt{R(u)}}$$

and an integral of the form

$$I_k = \oint \frac{u^k\ du}{R(u)^{3/2}}.$$

We now show that in the case of the mBBM equation, we can find a closed system of seven equations for the integrals I_k in terms of the traveling wave parameters and the quantities

T, M, P and K. To this end, first notice that for any integer $j \geq 1$ one has the identity

$$\begin{aligned}\mu_j = \oint \frac{u^j\ du}{\sqrt{R(u)}} &= \oint \frac{u^j R(u) du}{R(u)^{3/2}} \\ &= E\ I_j + a\ I_{j+1} + \left(\frac{c-1}{2}\right) I_{j+2} - \frac{1}{4} I_{j+4}.\end{aligned}$$

Moreover, integration by parts yields the relation

$$\begin{aligned}2j\mu_{j-1} \oint \frac{u^{j-1}\ du}{\sqrt{R(u)}} &= \oint \frac{u^j R'(u) du}{R(u)^{3/2}} \\ &= aI_j + (c-1)I_{j+1} - I_{j+2}.\end{aligned}$$

This yields a linear system of seven equations in seven unknowns $\{I_j\}_{j=0}^6$ known as the Picard-Fuchs system:

$$\begin{pmatrix} E & a & \frac{c-1}{2} & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & E & a & \frac{c-1}{2} & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & E & a & \frac{c-1}{2} & 0 & -\frac{1}{4} \\ a & (c-1) & 0 & -1 & 0 & 0 & 0 \\ 0 & a & (c-1) & 0 & -1 & 0 & 0 \\ 0 & 0 & a & (c-1) & 0 & -1 & 0 \\ 0 & 0 & 0 & a & (c-1) & 0 & -1 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{c}}T \\ \sqrt{\frac{2}{c}}M \\ \mu_2 \\ 0 \\ 2\sqrt{\frac{2}{c}}T \\ 4\sqrt{\frac{2}{c}}M \\ 6\mu_2 \end{pmatrix}.$$

The matrix which arises in the above linear system is known as the Sylvester matrix of the polynomials $R(u)$ and $R'(u)$. In commutative algebra, a standard result is that the determinant of the Sylvester matrix of two polynomials $P(u)$ and $Q(u)$ vanishes if and only if they have a common root. If we now restrict ourselves to $a = 0$, it follows that the above matrix is invertible for all periodic traveling wave solutions $u(x; 0, E, c)$ of the corresponding mBBM equation. In particular, we can solve the above Picard-Fuchs system and use the fact that $M_a = -\frac{1}{2}\sqrt{\frac{c}{2}} I_2$ to see that

$$\lim_{E \to 0^+} \frac{\operatorname{disc}(R(u; 0, E, c)) M_a(0, E, c)}{(c-1)T(0, E, c)} = 1$$

as claimed.

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