Math 647 – HW 3 Solns!

§1.5

#5) (a) Here, the char. curves $(x, y(x))$ satisfy

$$\frac{dy}{dx} = y,$$

and hence they are of the form

$$y(x) = Ce^{x}, \quad C \in \mathbb{R}.$$

Along the char. curves, we have

$$\frac{d}{dx} u(x, y(x)) = 0$$

when u solves the given PDE

$\Longrightarrow$ Solns. of the PDE are constant along the characteristics.

Thus, $\forall x \in \mathbb{R}$ we have

$$u(x, y(x)) = u(0, y(0)) = u(0, C) =: f(C).$$

Thus, gen. soln. of the PDE is

$$u(x,y) = f(ye^{-x})$$

where f is an arb. C^1 ftn.

To satisfy given condition, must have

$$u(x, 0) = f(0) = \varphi(x)$$

i.e. f must be chosen s.t. $\varphi(x) = f(0)$ for all $x \in \mathbb{R}$. This can only be done if $\varphi(x)$ is constant in x, hence there is no soln. when $\varphi(x) = x$.

(b) By above, the gen. soln. of the PDE is
$$u(x,y) = f(ye^{-x})$$
where f is C^1. To satisfy the given condition, we need
$$u(x,0) = f(0) = 1.$$
Thus, given ANY C^1 fn. f with $f(0) = 1$,
$$u(x,y) = f(ye^{-x})$$
provides a soln. of the given problem.

§2.1

#2) Using D'Alemberts formula, we find the soln. is given by
$$u(x,t) = \frac{1}{2}\left[\log\left(1+(x+ct)^2\right) + \log\left(1+(x-ct)^2\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)\,ds.$$
Noting that
$$\int_{x-ct}^{x+ct}(4+s)\,ds = 8ct + 2xct,$$
the above formula can be simplified to
$$u(x,0) = \frac{1}{2}\left[\log\left(1+(x+ct)^2\right) + \log\left(1-(x-ct)^2\right)\right] + 4t + xt.$$

#7) Given $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$, the soln. of the wave eqn. is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[\varphi(x+ct) - \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

If φ and ψ are odd functions, then for all t we have

$$\begin{aligned} u(-x,t) &= \frac{1}{2}\left[\varphi(-x+ct) + \varphi(-x-ct)\right] \\ &\qquad + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds \\ &= \frac{1}{2}\left[-\varphi(x-ct) - \varphi(x+ct)\right] \\ &\qquad - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-s)\,ds \\ &= -\frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] \\ &\qquad - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \\ &= -u(x,t) \end{aligned}$$

Change variables $s \mapsto -s$

Use $\psi(-s) = -\psi(s)$ and $\int_{x+ct}^{x-ct} = -\int_{x-ct}^{x+ct}$

so that u is an odd ftn. of x, as claimed.

#9) The given PDE can be rewritten as

$$\left(\partial/\partial x + \partial/\partial t\right)\left(\partial/\partial x - 4\,\partial/\partial t\right) u = 0.$$

Setting $v = u_x - 4u_t$, we find v must satisfy

$$v_x + v_t = 0.$$

Using characteristics, we find v must be of form

$$v(x,t) = f(x-t)$$

for some C^1 ftn. f. The soln. u to original PDE can now be found by solving

$$u_x - 4u_t = f(x-t).$$

Here, char. curves are given by $(x, t(x))$ where

$$t(x) = -4x + C, \quad C \in \mathbb{R}.$$

Along these char. curves, soln. u satisfies

$$\frac{d}{dx} u(x, t(x)) = f(5x - C).$$

Integrating, it follows that

$$u(x, t(x)) = G(C) + F(5x - C),$$

where G is an arb. C^1 ftn. and F is such that

$$F'(z) = \frac{1}{5} f(z);$$

regardless, can just treat F as an arb. ftn, since f was. Thus, the gen. soln. of given PDE is

$$u(x,t) = F(x-t) + G(\overset{t+4x}{\cancel{x+4t}})$$

where F and G are arb. ftns. of class C^2 (so PDE makes sense).

To satisfy given condition, we need

$$\begin{cases} u(x,0) = F(x) + G(4x) = x^2 & (1) \\ u_t(x,0) = -F'(x) + G'(4x) = e^x. & (2) \end{cases}$$

Differentiating (1) w.r.t. x gives

$$F'(x) + 4G'(4x) = 2x. \qquad (3)$$

Together, (2) and (3) form a closed 2×2 linear system for unknowns F' and G'. Solving we find

$$\begin{cases} 5F'(x) = 2x - 4e^x \\ 5G'(4x) = 2x + e^x, \end{cases}$$

which can be integrated to give

$$F(x) = \frac{1}{5}\left(x^2 - 4e^x\right) + A$$

and

$$G(4x) = \frac{4}{5}\left(x^2 + e^x\right) + B$$

for some constants A and B. Since then

$$G(x) = \frac{4}{5}\left(\frac{x^2}{16} + e^{x/4}\right) + B,$$

and since $A + B = 0$ by (1), it follows the soln. of the given problem is

$$u(x,t) = \frac{1}{5}\left((x-t)^2 - 4e^{x-t}\right) + \frac{4}{5}\left(\frac{(t+4x)^2}{16} + e^{x+t/4}\right),$$

which can be simplified to

$$u(x,t) = \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + x^2 + \frac{1}{4}t^2.$$

§2.2

#3) Throughout this exercise, let $u(x,t)$ denote a soln. of the wave eqn.

$$(*) \quad u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \; t \in \mathbb{R}.$$

(a) Given any fixed $y \in \mathbb{R}$,

$$\partial_x^2 u(x-y,t) = u_{xx}(x-y,t)$$

and

$$\partial_t^2 u(x-y,t) = u_{tt}(x-y,t)$$

so that

$$\partial_t^2 \big(u(x-y,t)\big) - c^2 \partial_x^2 \big(u(x-y,t)\big)$$

$$= u_{tt}(x-y,t) - c^2 u_{xx}(x-y,t)$$

$$= 0,$$

by $(*)$. Thus, $u(x-y,t)$ solves $(*)$ for any fixed $y \in \mathbb{R}$.

(b) If D represents any constant coefficient linear differential operator, then, since derivatives commute, applying D to both sides of $(*)$ gives

$$D\left(u_{tt} - c^2 u_{xx}\right) = D(0)$$

$$\Longleftrightarrow \quad (Du)_{tt} - c^2 (Du)_{xx} = 0$$

Thus, if D is as defined above and u solves $(*)$, then Du solves $(*)$ as well.

(c) Given any fixed $a \in \mathbb{R}$, the chain rule gives

$$\partial_x^2 u(ax, at) = a\,\partial_x u_x(ax, at)$$
$$= a^2 u_{xx}(ax, at)$$

and
$$\partial_t^2 u(ax, at) = a\,\partial_t u_t(ax, at)$$
$$= a^2 u_{tt}(ax, at).$$

Thus,
$$\partial_t^2\big(u(ax,at)\big) - c^2 \partial_x^2\big(u(ax,at)\big)$$
$$= a^2\Big[u_{tt}(ax, at) - c^2 u_{xx}(ax, at)\Big]$$
$$= 0$$

by (*). Thus, $u(ax, at)$ solves (*) for any fixed $a \in \mathbb{R}$.