Math 647 – HW 5 Soln.

§3.2

#5) Due to the homogeneous Dirichlet B.C., we begin by taking the odd extension of the I.C., which is

$$\varphi_{odd}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

By d'Alemberts formula, the soln. to the IVP

$$(*) \quad \begin{cases} w_{tt} = 4 w_{xx}, & -\infty < x < \infty, \quad t \in \mathbb{R} \\ w(x,0) = \varphi_{odd}(x), & w_t(x,0) = 0 \end{cases}$$

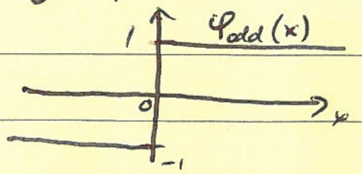
is

$$w(x,t) = \frac{1}{2}\left[\varphi_{odd}(x-2t) + \varphi_{odd}(x+2t)\right]$$

We don't worry about writing the soln. more explicitly here. Follows the soln. to the given IVBVP is given by

$$u(x,t) = w(x,t), \quad x \geq 0, \ t \in \mathbb{R}.$$

Note that since the I.C. $w(x,0)$ has a jump discontinuity, the soln. $w(x,t)$ to $(*)$



above will have TWO jump discontinuities, mainly when $x - 2t = 0$ and when $x + 2t = 0$. For $t \geq 0$, it follows that the soln. u of given IVBVP will have a singularity (jump discontinuity) whenever $x = 2t$.

§4.1

#2) Here, the appropriate IVBVP to be solved is

$$\begin{cases} u_t = k\, u_{xx}, & 0<x<L,\ t>0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = 1, & 0 \leq x \leq L \end{cases}$$

From class (or equation (17) in §4.1 of Strauss) we know any ftn. of form

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 \cdot kt} \sin\left(\frac{n\pi x}{L}\right)$$

solves the PDE + B.C.'s, where the constants A_n are arbitrary. To satisfy the I.C., we need to choose the values of A_n, $n=1,2,3,\ldots$ so that

$$1 = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \qquad \left[= u(x,0) \text{ by above}\right]$$

Well, we are given that

$$1 = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$$

so that we should choose

$$A_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

So, soln. to given IVBVP is

$$u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi} e^{-\left(\frac{n\pi}{L}\right)^2 \cdot kt} \sin\left(\frac{n\pi x}{L}\right)$$

or, more precisely,

$$u(x,t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} e^{-\left((2k+1)\pi/L\right)^2 \cdot kt} \sin\left(\frac{(2k+1)\pi x}{L}\right).$$

§4.2

#1) We seek separated solns. of form

$$u(x,t) = X(x)\,T(t)$$

Plugging this guess into the PDE, we find that X and T must satisfy

$$X(x)T'(t) = k X''(x) T(t)$$

which can be rewritten as

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

Follows there exists some constant $\lambda \in \mathbb{R}$ such that

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

which is equivalent to saying that X and T must satisfy the ODE's

$$(*)\quad \begin{cases} X'' + \lambda X = 0, & 0 < x < L \\ T' + k\lambda T = 0, & t > 0 \end{cases}$$

for some constant $\lambda \in \mathbb{R}$.

To continue, notice the boundary conditions imply

$$u(0,t) = X(0)T(t) = 0 \Longrightarrow X(0) = 0 \quad \left(\text{To avoid trivial soln!}\right)$$

$$u_x(L,t) = X'(0)T(t) = 0 \Longrightarrow X'(0) = 0 \quad \left(\text{To avoid triv. soln!}\right)$$

Thus, the given B.C.'s on u translate into "mixed" B.C.'s for the X eqn. In particular, X must satisfy the ODE-BVP

$$(**)\begin{cases} X'' + \lambda X = 0, & 0 \leq x < L \\ X(0) = X'(L) = 0 \end{cases}$$

To find all nontrivial solns. of (**), we note the gen. soln. of the ODE for X changes depending on whether $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. We consider those three cases separately:

<u>$\lambda < 0$:</u> If $\lambda < 0$, then $\lambda = -\mu^2 < 0$ for some $\mu > 0$, and hence the gen. soln. of the ODE in (**) is (when $\lambda = -\mu^2 < 0$)

$$X(x) = A \cosh(\mu x) + B \sinh(\mu x)$$

for some constants A and B. To satisfy B.C.'s, we require

$$X(0) = A \stackrel{!}{=} 0 \Longrightarrow \text{take } A = 0$$

$$X'(L) = \mu \cdot B \cosh(\mu L) \stackrel{!}{=} 0 \Longrightarrow \text{take } B = 0.$$

Thus, when $\lambda < 0$ the BVP (**) only has the trivial soln.

<u>$\lambda = 0$:</u> If $\lambda = 0$, the gen. soln. of the ODE in (**) is

$$X(x) = A + Bx$$

for some constants A and B. To satisfy the B.C.'s, we require

$$X(0) = A \stackrel{!}{=} 0 \Longrightarrow \text{Take } A = 0$$

$$X'(L) = B \stackrel{!}{=} 0 \Longrightarrow \text{Take } B = 0$$

Thus, the only soln. to (**) when $\lambda = 0$ is the trivial soln.

$\lambda > 0$: If $\lambda > 0$, then $\lambda = \mu^2$ for some $\mu > 0$, and hence the gen. soln. of the ODE in (**) when $\lambda = \mu^2 > 0$ is

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

for some constants $A, B \in \mathbb{R}$. To satisfy B.C.'s we require

$$X(0) = A \stackrel{!}{=} 0 \Longrightarrow A = 0$$

$$X'(L) = \mu \cdot B \cdot \cos(\mu L) \stackrel{!}{=} 0.$$

The latter condition holds if $B = 0$, which leads to the trivial soln., or if $\cos(\mu L) = 0$.
Since we are seeking nontrivial solns. of (**), we take

$$\cos(\mu L) = 0, \text{ i.e. } \mu = \mu_n = \frac{(2n+1)\pi}{2L} = (n+\tfrac{1}{2})\tfrac{\pi}{L}, \; n = 0,1,2,\ldots$$

Thus, the eigenvalues of (**) are given by

$$\lambda_n = (\mu_n)^2 = \left((n+\tfrac{1}{2})\frac{\pi}{L}\right)^2, \; n = 0,1,2,\ldots$$

w/ corresponding eigenftn. (i.e. non-trivial soln.)

$$X_n(x) = \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right), \; n = 0,1,2,\ldots$$

Now, for each $\lambda = \lambda_n$ above, we solve the T eqn. in (*):

$$T_n' + k\lambda_n T_n = 0$$

$$\Longrightarrow T_n(t) = A_n e^{-((n+\frac{1}{2})\pi/L)^2 \cdot kt}, \; n = 0,1,2,3,\ldots$$

where the constants A_n are arbitrary.

This produces ∞-many solns. to the given PDE+B.C. of the form

$$u_n(x,t) = X_n(x)\,T_n(t)$$

$$= A_n e^{-((n+\frac{1}{2})\pi/L)^2 \cdot kt} \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right), \quad n = 0,1,2,\dots$$

By linearity, follows any fn. of form

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-((n+\frac{1}{2})\pi/L)^2 \cdot kt} \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)$$

where the A_n are arbitrary constants, will solve the given PDE and B.C's.

#4) (a) As in #1 above, seeking separated solns. of form

$$u(x,t) = X(x)\,T(t)$$

leads to the ODE's

$$(*)\begin{cases} X'' + \lambda X = 0, & -L < x < L \\ T' + k\lambda T = 0, & t > 0, \end{cases}$$

where here $\lambda \in \mathbb{R}$ is an arbitrary constant (for now). To determine the effect of the given periodic B.C's, note

$$u(-L,t) = u(L,t) \implies u(-L,t) - u(L,t) = 0$$

$$\implies \left(X(-L) - X(L)\right)T(t) = 0.$$

To avoid the trivial soln., we require that X satisfy $X(-L) = X(L)$. Similarly, the B.C.

$$u_x(-L) = u_x(L) \implies \text{require } X'(-L) = X'(L),$$

Thus, given B.C.'s for u provide B.C.'s for the X eqn. in (*): namely, X must satisfy BVP

$$(**) \quad \begin{cases} X'' + \lambda X = 0, & -L < x < L \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases}$$

Since gen. soln. depends on sign. of λ, we consider 3 cases, seeking non-trivial solns. in each case.

<u>$\lambda < 0$</u>: If $\lambda < 0$, then $\lambda = -\mu^2$ for some $\mu > 0$ and gen. soln. of ODE in (**) is

$$X(x) = A\cosh(\mu x) + B\sinh(\mu x), \quad A, B \in \mathbb{R}.$$

Now, since $\sinh(z) = -\sinh(-z)$ and $\cosh(z) = \cosh(-z)$ for all $z \in \mathbb{R}$,

$$X(-L) = X(L) \Longrightarrow B = 0$$
$$X'(-L) = X'(L) \Longrightarrow A = 0.$$

So, only soln. of (**) when $\lambda < 0$ is the trivial soln.

<u>$\lambda = 0$</u>: If $\lambda = 0$, gen. soln. of ODE in (**) is

$$X(x) = A + Bx, \quad A, B \in \mathbb{R}.$$

Here, the B.C.'s imply

$$X(-L) = X(L) \Longrightarrow A - BL = A + BL \Rightarrow B = 0$$
$$X'(-L) = X'(L) \Longrightarrow B = B \text{ (No restriction on } A!)$$

Thus, when $\lambda = 0 = \lambda_0$ a non-trivial soln. of (**) is given by

$$X_0(x) = 1.$$

$\underline{\lambda > 0}$: If $\lambda > 0$, then $\lambda = \mu^2$ for some $\mu > 0$ and hence gen. soln. of ODE in (**) is

$$X(x) = A\cos(\mu x) + B\sin(\mu x), \quad A, B \in \mathbb{R}.$$

Such a ftn. will clearly satisfy the given B.C.'s if and only if it is $2L$-periodic, i.e. if and only if

$$X(x + 2L) = X(x) \quad \forall x \in \mathbb{R}.$$

Since $\cos(\mu x)$ and $\sin(\mu x)$ are $2L$-periodic if and only if (recall $\mu > 0$ by above)

$$\mu = \mu_n = \frac{n\pi}{L} \quad \text{for some } n = 1, 2, 3, \ldots$$

it follows that when $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)$, $n = 1, 2, 3, \ldots$ a non-trivial soln. of BVP (**) is given by

$$X_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

where the constants A_n, B_n are arbitrary (but not both zero).

By above analysis, follows the eigenvalues for the given PDE + B.C.'s are given by

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, 3, \ldots$$

as claimed.

(b) For each $n = 0, 1, 2, 3, \ldots$ we can solve the T eqn. in (*) when $\lambda = \lambda_n$:

$$T_n' + k\lambda_n T_n = 0, \quad n = 0, 1, 2, \ldots$$

$$\Longrightarrow T_n(t) = C_n e^{-\lambda_n k t}$$

$$= C_n e^{-(n\pi/L)^2 \cdot kt}, \quad C_n \in \mathbb{R} \text{ arbitrary}.$$

Thus, the ftns.

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$= \left(A_n \cos\left(\tfrac{n\pi x}{L}\right) + B_n \sin\left(\tfrac{n\pi x}{L}\right)\right) \cdot C_n e^{-(n\pi/L)^2 \cdot k t}$$

$$= \left(\tilde{A}_n \cos\left(\tfrac{n\pi x}{L}\right) + \tilde{B}_n \sin\left(\tfrac{n\pi x}{L}\right)\right) e^{-(n\pi/L)^2 \cdot k t},$$

where the constants $\tilde{A}_n = A_n \cdot C_n$, $\tilde{B}_n = B_n \cdot C_n$ are arbitrary, provide ∞-many solns. to given PDE + B.C. By linearity then, follows the gen. soln. of given PDE + B.C. is

$$u(x,t) = \sum_{n=0}^{\infty} \left(\tilde{A}_n \cos\left(\tfrac{n\pi x}{L}\right) + \tilde{B}_n \sin\left(\tfrac{n\pi x}{L}\right)\right) e^{-(n\pi/L)^2 \cdot k t}$$

$$= \tilde{A}_0 + \sum_{n=1}^{\infty} \left(\tilde{A}_n \cos\left(\tfrac{n\pi x}{L}\right) + \tilde{B}_n \sin\left(\tfrac{n\pi x}{L}\right)\right) e^{-(n\pi/L)^2 \cdot k t},$$

which is what is in Strauss after relabeling $\tilde{A}_0 = \frac{A_0}{2}$, $\tilde{A}_n = A_n$, $\tilde{B}_n = B_n$.

Extra Problem #2) Our goal is to solve the BVP

$$(*) \begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a,\ 0 < y < b \\ u(0,y) = u(a,y) = 0 \\ u(x,0) = 0, \quad u(x,b) = g(x) \end{cases}$$

where the ftn. $g(x)$ is given. To begin, seek separated solns. of form

$$u(x,y) = X(x)\, Y(y)$$

and note plugging this into the PDE in $(*)$

implies

$$\mathrm{X}''(x)\,\mathrm{Y}(y) + \mathrm{X}(x)\,\mathrm{Y}''(y) = 0$$

$$\Longrightarrow \quad \frac{\mathrm{X}''(x)}{\mathrm{X}(x)} = \frac{\mathrm{Y}''(y)}{-\mathrm{Y}(y)} \quad \text{for all } 0<x<a,\ 0<y<b.$$

Follows there exists a constant $\lambda \in \mathbb{R}$ such that

$$\frac{\mathrm{X}''(x)}{\mathrm{X}(x)} = \frac{\mathrm{Y}''(y)}{-\mathrm{Y}(y)} = -\lambda,$$

i.e.

$$(**)\begin{cases} \mathrm{X}''(x) + \lambda\,\mathrm{X}(x) = 0, & 0<x<a \\ \mathrm{Y}''(y) - \lambda\,\mathrm{Y}(y) = 0, & 0<y<b. \end{cases}$$

Now, the B.C.'s imply

$$\left.\begin{aligned} u(0,y) &= \mathrm{X}(0)\cdot\mathrm{Y}(y) \overset{!}{=} 0 \Longrightarrow \mathrm{X}(0) = 0 \\ u(a,y) &= \mathrm{X}(a)\cdot\mathrm{Y}(y) \overset{!}{=} 0 \Longrightarrow \mathrm{X}(a) = 0 \\ u(x,0) &= \mathrm{X}(x)\,\mathrm{Y}(0) \overset{!}{=} 0 \Longrightarrow \mathrm{Y}(0) = 0, \end{aligned}\right\} \text{To avoid trivial soln!}$$

while B.C. $u(x,b) = g(x)$ will serve same purpose as an I.C. in previously studied wave and diffusion problems. Thus, the B.C.'s on $x=0,\ x=a$ yield B.C.'s on X eqn. in $(**)$

$$(***)\begin{cases} \mathrm{X}''(x) + \lambda\,\mathrm{X}(x) = 0, & 0<x<a \\ \mathrm{X}(0) = \mathrm{X}(a) = 0 \end{cases}$$

while B.C. at $y=0$ provides B.C. for Y eqn. in $(**)$:

$$(\star)\begin{cases} \mathrm{Y}''(y) - \lambda\,\mathrm{Y}(y) = 0, & 0<y<b \\ \mathrm{Y}(0) = 0 \end{cases}$$

We have already seen the eigenvalues for $(***)$ are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2,\ n=1,2,3,\ldots$$

w/ corresponding eigenfunction $\mathrm{X}_n(x) = \sin\left(\frac{n\pi x}{l}\right),\ n=1,2,3,\ldots$

For each $n=1,2,3,\dots$ we can solve $(\star)$:

$$(\star\star)\begin{cases} Y_n''(y) - \lambda_n Y_n(y) = 0, & 0<y<b \\ Y_n(0) = 0 \end{cases},$$

which has gen. soln.

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right), \quad A_n, B_n \in \mathbb{R}.$$

The B.C. $Y_n(0) = 0$

$$\Longrightarrow Y_n(0) = A_n \overset{!}{=} 0 \Longrightarrow A_n = 0, \quad n=1,2,3,\dots$$

Thus, for each $n=1,2,3,\dots$ a non-trivial soln. of $(\star\star)$ is given by

$$Y_n(y) = B_n \sinh\left(\frac{n\pi y}{a}\right), \quad B_n \in \mathbb{R}.$$

It follows that the ftns.

$$\begin{aligned} u_n(x,y) &= X_n(x) Y_n(y) \\ &= B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \end{aligned}$$

provide ∞-many solns. of the given PDE + B.C. at $x=0$, $x=a$, $y=0$. By linearity, follows any ftn. of form

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

will solve the given PDE + B.C. at $x=0, x=a, y=0$.

To satisfy the B.C. at $y=b$, we require

$$\begin{aligned} u(x,b) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) \\ &= \sum_{n=1}^{\infty} \left(B_n \sinh\left(\frac{n\pi b}{a}\right)\right) \sin\left(\frac{n\pi x}{a}\right) \\ &\overset{!}{=} g(x). \end{aligned}$$

It follows if we expand $g(x)$ in a Fourier Sine series on $[0, a]$, i.e. if we write

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)$$

w/

$$b_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx, \quad n = 1, 2, 3, \ldots$$

then the condition $u(x,b) = g(x)$ will be satisfied provided

$$B_n \sinh\left(\frac{n\pi b}{a}\right) = b_n$$

for all $n = 1, 2, 3, \ldots$, i.e. if

$$B_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx, \quad n = 1, 2, 3, \ldots$$

With B_n chosen as above, it follows that the ftn.

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

provides a soln. to the given PDE + BVP.