Math 647 – HW8 Solutions!

§6.1

#6) In polar coordinates, we are being asked to solve the BVP

$$(*)\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 1, & a<r<b,\ 0<\theta<2\pi \\ u(a,\theta) = u(b,\theta) = 0, & 0 \le \theta < 2\pi \end{cases}$$

Notice since the boundary conditions are independent of θ, it makes sense to seek solutions of $(*)$ that are independent of θ, i.e. we may solve the ODE/BVP

$$(**)\begin{cases} u_{rr} + \frac{1}{r}u_r = 1, & a<r<b \\ u(a) = u(b) = 0. \end{cases}$$

To solve $(**)$, note we can rewrite the ODE as

$$r u_{rr} + u_r = r$$

$$\Longrightarrow (r u_r)_r = r,$$

which can be integrated to give

$$u_r(r) = \frac{1}{2}r + \frac{A}{r}$$

for some constant $A \in \mathbb{R}$. Integrating again gives

$$u(r) = \frac{1}{4}r^2 + A\ln(r) + B$$

for some constants $A, B \in \mathbb{R}$. This is the gen. soln. of the ODE in $(**)$.

Now, to satisfy the B.C.'s we must choose A and B such that

$$u(a) = \tfrac{1}{4} a^2 + A \ln(a) + B \overset{!}{=} 0$$

and

$$u(b) = \tfrac{1}{4} b^2 + A \ln(b) + B \overset{!}{=} 0.$$

Subtracting the second equation from the first gives

$$\tfrac{1}{4}(a^2 - b^2) + A \ln(a/b) = 0$$

$$\Longrightarrow \quad A = \frac{b^2 - a^2}{4 \ln(a/b)},$$

which gives

$$B = -\tfrac{1}{4} a^2 - \frac{b^2 - a^2}{4\ln(a/b)} \ln(a).$$

Thus, a soln. of $(**)$, and hence a θ-independent soln. of $(*)$, is given by

$$u(r) = \tfrac{1}{4} r^2 + \frac{b^2 - a^2}{4\ln(a/b)} \ln(r) - \tfrac{1}{4} a^2 - \frac{b^2 - a^2}{\ln(a/b)} \ln(a),$$

which, if desired, can be rewritten as

$$u(r) = \tfrac{1}{4}(r^2 - a^2) - \frac{b^2 - a^2}{4} \cdot \frac{\ln(r/a)}{\ln(b/a)}.$$

§6.2

#3) We seek separated solns. of form

$$u(x, y) = X(x)\, Y(y).$$

The requirement that u is harmonic (i.e. $\Delta u = 0$) in D means X and Y must satisfy

$$X''(x)\, Y(y) + X(x)\, Y''(y) = 0 \quad \text{in } D$$

$$\Longrightarrow \quad \frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. Thus, X and Y must satisfy the ODEs

$$\begin{cases} X'' - \lambda X = 0, & 0 < x < \pi \\ Y'' + \lambda Y = 0, & 0 < y < \pi \end{cases}$$

Now, from the homogeneous B.C.'s

$$u_y(x, 0) = u_y(x, \pi) = 0, \quad 0 < x < \pi$$

we find that Y should satisfy (to avoid the trivial soln.)

$$\begin{cases} Y'' + \lambda Y = 0, & 0 < y < \pi \\ Y'(0) = Y'(\pi) = 0 \end{cases}$$

From previous work, this BVP has nontrivial solutions only if

$$\lambda = \lambda_n = n^2, \quad n = 0, 1, 2, \ldots$$

with corresponding non-trivial solns.

$$Y_n(y) = \cos(ny), \quad n = 0, 1, 2, \ldots$$

Now, for each $n = 0, 1, 2, \ldots$ we must solve the ODE

$$X_n'' - n^2 X_n = 0.$$

When $n = 0$, the gen. soln. is

$$X_0(x) = A_0 + B_0 x, \quad A_0, B_0 \in \mathbb{R}$$

while for $n = 1, 2, 3, \ldots$ the gen. soln. is

$$X_n(x) = A_n \cosh(nx) + B_n \sinh(nx).$$

Note to satisfy the homogeneous B.C.

$$u(0, y) = 0, \quad 0 < y < \pi$$

we should choose constants A_n, B_n, $n = 0, 1, 2, \ldots$ above to ensure that $X_n(0) = 0$, $n = 0, 1, 2, \ldots$

Since $X_n(0) = A_n$ for each $n = 0, 1, 2, \dots$ it follows that we should take $A_n = 0$ for all $n = 0, 1, 2, \dots$ so that

$$X_n(x) = \begin{cases} B_0 x, & n = 0 \\ B_n \sinh(nx), & n = 1, 2, 3, \dots \end{cases}$$

Together, we have found ∞-many solns. of the given PDE of form

$$u_n(x,y) = X_n(x)\, Y_n(y)$$

which also solve the 3 given homogeneous boundary conditions. By linearity, it follows any ftn. of form

$$u(x,y) = B_0 x + \sum_{n=1}^{\infty} B_n \sinh(nx)\cos(ny)$$

will satisfy the given PDE and the 3 given homogeneous B.C.'s. To satisfy the inhomogeneous B.C.

$$u(\pi, y) = \tfrac{1}{2}(1 + \cos(2y))$$

we must choose the constants B_n above so that

$$\frac{1}{2} + \frac{1}{2}\cos(2y) = B_0 \pi + \sum_{n=1}^{\infty} B_n \sinh(n\pi)\cos(n\pi)$$

$$\Longrightarrow B_0 = \frac{1}{2\pi}, \quad B_2 = \frac{1}{2\sinh(2\pi)}, \quad B_n = 0, \; n = 1, 3, 4, 5, \dots \;\; (n \neq 0, 2)$$

Thus, the soln. to the given BVP is

$$u(x,y) = \frac{x}{2\pi} + \frac{\sinh(2x)}{2\sinh(2\pi)}\cos(2y).$$

§6.4

#10) In polar coordinates, we are being asked to solve the BVP

$$(*)\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0<r<a,\ 0<\theta<\frac{\pi}{2} \\ u(r,0) = u(r,\frac{\pi}{2}) = 0, & 0<r<a \\ u_r(a,\theta) = 1, & 0<\theta<\frac{\pi}{2}. \end{cases}$$

We seek separated soln. of form

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

Plugging into PDE in $(*)$, find R and Θ must satisfy

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\Longrightarrow \frac{\Theta''(\theta)}{\Theta(\theta)} = -\left(\frac{r^2R''}{R} + \frac{rR'}{R}\right) = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. Thus, R and Θ must satisfy the ODEs

$$(**)\begin{cases} \Theta'' + \lambda\Theta = 0, & 0<\theta<\frac{\pi}{2} \\ r^2R'' + rR' - \lambda R = 0, & 0<r<a \end{cases}$$

From the homogeneous B.C.s

$$u(r,0) = u(r,\tfrac{\pi}{2}) = 0, \quad 0<r<a$$

we find, to avoid the trivial soln., that Θ must satisfy the BVP

$$(***)\begin{cases} \Theta'' + \lambda\Theta = 0, & 0<\theta<\frac{\pi}{2} \\ \Theta(0) = \Theta(\frac{\pi}{2}) = 0 \end{cases}$$

From prev. work, know $(***)$ has a non-trivial soln. only if

$$\lambda = \lambda_n = (2n)^2, \quad n = 1,2,3,\ldots$$

with corresponding e-ftns. $\Theta_n(\theta) = \sin(2n\theta)$, $n=1,2,3,\ldots$

From (**), for each $n=1,2,3,\ldots$ we must solve the ODE

$$r^2 R_n'' + r R_n' - (2n)^2 R_n = 0.$$

This is an Euler equation, so has solns. of form

$$R_n(r) = r^{\alpha_n}$$

where the $\alpha_n \in \mathbb{C}$ are roots of the polynomial

$$\alpha_n(\alpha_n - 1) + \alpha_n - (2n)^2 = 0$$

$$\Longrightarrow \quad \alpha_n = \pm 2n, \quad n=1,2,3,\ldots$$

Thus, for each $n=1,2,3,\ldots$ the general soln. of the above 'radial' ODE is

$$R_n(r) = A_n r^{2n} + B_n r^{-2n}, \quad n=1,2,3,\ldots$$

where the $A_n, B_n \in \mathbb{R}$ are arbitrary constants. Notice that to ensure $R_n(r)$ remains bounded on $0<r<a$, we take $B_n = 0$ for all $n=1,2,3,\ldots$

Together, we have found ∞-many solns. of the PDE in (*) of form

$$u_n(r,\theta) = A_n r^{2n} \sin(2n\theta), \quad n=1,2,3,\ldots$$

that are bounded in D and satisfy the two given homogeneous B.C's on ∂D. By linearity, follows any fn. of form

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin(2n\theta)$$

~~that satisfy~~ satisfies the PDE + 2 hom. B.C's

To satisfy the inhomogeneous (Neumann) B.C.

$$u_r(a,\theta)=1, \quad 0<\theta<\pi/2,$$

we must choose the constants A_n above so that

$$1=\sum_{n=1}^{\infty} 2n A_n a^{2n-1}\sin(2n\theta).$$

Recalling that on $0<\theta<\pi/2$ the fn. 1 has Fourier sine series

$$1=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2(2k+1)\theta)}{2k+1}, \quad 0<\theta<\pi/2$$

we must take

$$2nA_n a^{2n-1}=\begin{cases}\frac{4}{(2k+1)\pi}, & \text{if } n=2k+1 \text{ for some } k=0,1,2,\dots\\ 0, & \text{else.}\end{cases}$$

Thus, the soln. of our given BVP is

$$u(r,\theta)=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2 a^{4k+1}}\, r^{2(2k+1)}\sin(2(2k+1)\theta).$$

<u>Extra Problem</u>:

(a) In polar coordinates, the given BVP can be written as

$$(*)\ \begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0, & 0<r<1,\ 0<\theta<2\pi\\ u_r(1,\theta)=h(\theta), & 0\le\theta<2\pi\end{cases}$$

(b) Now, assume $\int_0^{2\pi} h(\theta)\,d\theta = 0$. From class, we know the PDE in $(*)$ has ∞-many bounded solns. of form

$$u_n(r,\theta) = \begin{cases} \frac{1}{2} A_0, & n=0 \\ r^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right), & n=1,2,3,\ldots \end{cases}$$

and so, by linearity, any ftn. of form

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right) \quad (**)$$

will solve the PDE in $(*)$.

To satisfy the inhomogeneous B.C. $u_r(1,\theta) = h(\theta)$ we must choose the A_n, B_n above so that

$$\left.\begin{aligned} h(\theta) &= \frac{\partial}{\partial r}\Big|_{r=1} u(r,\theta) \\ &= \sum_{n=1}^{\infty} n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right) \end{aligned}\right\}$$

Note: A_0 is NOT determined here!

$$\Longrightarrow \begin{cases} A_n = \dfrac{1}{n\pi}\displaystyle\int_0^{2\pi} h(\varphi)\cos(n\varphi)\,d\varphi, & n = 1,3\ldots \\ B_n = \dfrac{1}{n\pi}\displaystyle\int_0^{2\pi} h(\varphi)\sin(n\varphi)\,d\varphi, & n=1,2,3,\ldots \end{cases}$$

Note since $\int_0^{2\pi} h(\varphi)\,d\varphi = 0$, the "zeroth" Fourier cosine coeff. of h vanishes!

Thus, with A_n, B_n chosen above for $n=1,2,3,\ldots$ the ftn. $(**)$ is a soln. of BVP $(*)$ for ANY value of $A_0 \in \mathbb{R}$. Thus, when $\int_0^{2\pi} h(\varphi)\,d\varphi = 0$, solns. of the given BVP exist but they are not unique.

(c) Above, we needed the boundary data h to admit a Fourier series expansion of form

$$h(\theta) = \sum_{n=1}^{\infty} n\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

which is impossible unless $\int_0^{2\pi} h(\theta)d\theta = 0$. Thus, if $\int_0^{2\pi} h(\theta)d\theta \neq 0$, no soln. of the given Neumann BVP exists...