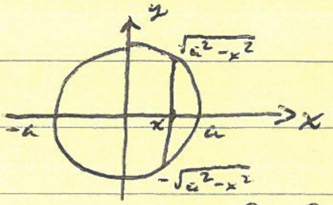
- Next, consider Laplace's equation on domain with rotational symmetry.

- <u>Ex</u>: Let $D = \{(x,y) : x^2+y^2 < a^2\} \subset \mathbb{R}^2$ and consider BVP



$$(*) \begin{cases} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

As usual, lets seek separated solns. of form $u(x,y) = X(x)Y(y)$

and note, in xy-coordinates,

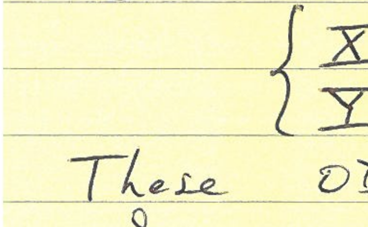
$$D = \{(x,y) : -a < x < a,\ -\sqrt{a^2-x^2} < y < \sqrt{a^2-x^2}\}.$$

Plugging into the PDE, we find, for some constant $\lambda \in \mathbb{R}$, ftns. X and Y must satisfy ODE's

$$\begin{cases} X''(x) + \lambda X(x) = 0, & -a < x < a \\ Y''(y) - \lambda Y(y) = 0, & -\sqrt{a^2-x^2} < y < \sqrt{a^2-x^2}. \end{cases}$$

These ODE's are <u>NOT</u> separated... separation of variables (in above form) fails here.

To recover, try using polar coordinates:



set

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

and note, in $r\theta$-coordinates,

$$D = \{(r,\theta) : 0 \le r < a,\ 0 \le \theta < 2\pi\}$$

FACT: Using chain rule, can show (See §6.1 in Strauss!)

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Thus, in polar coordinates, BVP $(*)$ can be written as

$$(**)\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 \le r < a,\ 0 \le \theta < 2\pi \\ u(a,\theta) = h(\theta), & 0 \le \theta < 2\pi \end{cases}$$

Now, lets seek separated solns. of form

$$u(r,\theta) = R(r)\,\Theta(\theta).$$

Plugging into PDE in $(*)$, find R and Θ must satisfy

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\,\Theta'' = 0$$

$$\Longrightarrow \frac{\Theta''}{\Theta} = -\left(\frac{r^2R''}{R} + \frac{rR'}{R}\right) = -\lambda$$

for some constant $\lambda \in \mathbb{R}$. Thus, R and Θ must satisfy the ODE;

$$(***)\begin{cases} \Theta'' + \lambda\Theta = 0, & 0 \le \theta < 2\pi \\ r^2R'' + rR' - \lambda R = 0, & 0 \le r < a \end{cases}$$

These ODE's are completely separated!!

Q: What B.C.'s do we apply to $(***)$?

Natural to require Θ is 2π-periodic, i.e.

$$\cancel{\Theta(0) = \Theta(2\pi)}.$$

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \forall\ \theta \in \mathbb{R}.$$

From (***), follows Θ eqn. has non-trivial soln. iff

$$\lambda = \lambda_n = n^2, \quad n = 0, 1, 2, 3, \ldots$$

w/ corresponding e-fcns. of form

$$\begin{cases} \Theta_0(\theta) = A_0, & n = 0 \\ \Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), & n = 1, 2, 3, \ldots \end{cases}$$

where constants A_j, B_j are arbitrary.

Now, for each $n = 0, 1, 2, \ldots$ we need to solve

$$r^2 R_n'' + r R_n' - \lambda_n R_n = 0, \quad 0 \leq r < a.$$

This is an Euler equation and so has solns. of form

$$R_n(r) = r^{\alpha_n}$$

for some $\alpha_n \in \mathbb{C}$. To determine solns. here, substitute into ODE

$$\Longrightarrow r^2\left(\alpha_n(\alpha_n - 1) r^{\alpha_n - 2}\right) + r\left(\alpha_n r^{\alpha_n - 1}\right) - \lambda_n r^{\alpha_n} = 0,$$

so (since $\lambda_n = n^2$) must choose α_n so that

$$\alpha_n(\alpha_n - 1) + \alpha_n - n^2 = 0$$

$$\Longleftrightarrow \quad \alpha_n^2 = n^2.$$

Thus, when $n = 1, 2, 3, \ldots$ this gives

$$R_n(r) = C_n r^n + D_n r^{-n}, \quad n = 1, 2, 3, \ldots$$

for some constants C_n, D_n.

When $n=0$, get only one linearly independent soln.

$$R(r) = C = \text{const.}$$

To find 2nd linearly independent soln., note when $n=0$ we must solve

$$\underbrace{r^2 R_0'' + r R_0'}_{r(rR_0')'} = 0\,, \quad 0<r<a$$

$$\Longrightarrow R_0(r) = C_0 + D_0 \ln(r)\,, \quad n=0.$$

Q: What B.C.'s do we impose on R?

Well, notice as $r \to 0^+$ we have

$$\lim_{r\to 0^+} |R_n(r)| = +\infty \quad \text{for each } n=0,1,2,3,\ldots$$

unless we require $D_j = 0$ for all $j=0,1,2,\ldots$, in which case fcns.

$$R_n(r) = C_n r^n$$

are bounded on $0<r<a$.

$\Longrightarrow$ Boundedness is the B.C. at $r=0$!!

Thus, have produced ∞-many solns. of PDE in (**) of form

$$U_n(r,\theta) = R_n(r)\Theta_n(\theta) = \begin{cases} \frac{1}{2}A_0\,, & n=0 \\ r^n\left(A_n\cos(n\theta)+B_n\sin(n\theta)\right), & n=1,2,3,\ldots \end{cases}$$

that are bounded on disk D.

By linearity, follows any ftn. of form

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right) \quad (\square)$$

will solve the PDE in (**).

Finally, to satisfy the inhomogeneous B.C. $u(a,\theta) = h(\theta)$, we require that

$$h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n \left(A_n \cos(n\theta) + B_n \sin(n\theta) \right)$$

$$\Longrightarrow \begin{cases} A_n = \dfrac{1}{\pi a^n} \displaystyle\int_0^{2\pi} h(\varphi) \cos(n\varphi)\, d\varphi, & n = 0, 1, 2, \dots \\ B_n = \dfrac{1}{\pi a^n} \displaystyle\int_0^{2\pi} h(\varphi) \sin(n\varphi)\, d\varphi, & n = 1, 2, 3, \dots \end{cases}$$

With these choices, ftn. in $(\square)$ solves the BVP posed in (**). ✓

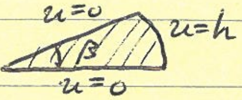
• <u>Rmk</u>: Using above ideas, can also solve

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \text{B.C.} & \text{on } \partial D \end{cases}$$

when D is...

(i) $D = \{ (r,\theta) : 0 \le r < a, \; 0 < \theta < \beta < 2\pi \}$ (A Wedge!)

<u>Ex</u>:



Now, Θ will solve BVP

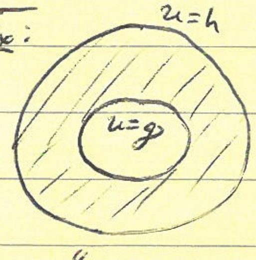
$$\begin{cases} \Theta'' + \lambda \Theta = 0, & 0 < \theta < \beta \\ \Theta(0) = \Theta(\beta) = 0 \end{cases}$$

$\Longrightarrow$ E.V.'s are $\lambda = \lambda_n = \left(\frac{n\pi}{\beta}\right)^2$ w/ e-ftn.
$\Theta_n(\theta) = \sin\left(\frac{n\pi\theta}{\beta}\right)$, $n = 1, 2, 3, \dots$ Get "gen. soln."

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\left(\frac{n\pi\theta}{\beta}\right). \quad (\text{Check!})$$

(ii) $D = \{(r,\theta) : 0 < a < r < b,\ 0 \le \theta < 2\pi\}$ (An Annulus!)

Ex:



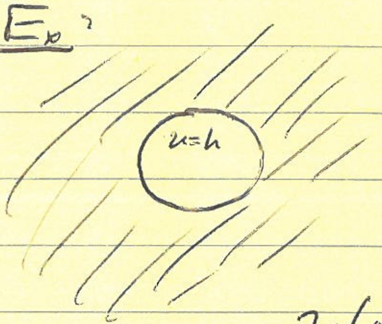
Here, Θ eqn. / e.v.'s are same, but solns. r^{-n} and $\log(r)$ are now bounded in D, so don't throw them away!

Get "gen. soln."

$$u(r,\theta) = \frac{1}{2}\left(C_0 + D_0 \log(r)\right) + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos(n\theta) + \left(A_n r^n + B_n r^{-n}\right)\sin(n\theta)\right]$$

(iii) $D = \{(r,\theta) : 0 < a < r,\ 0 \le \theta < 2\pi\}$ (Exterior of disk!)

Ex:



Here, Θ eqn. / e.v.'s are same, but solns. r^n and $\log(r)$ are unbounded in D, so discard them. Get "gen. soln."

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left(A_n \cos(n\theta) + B_n \sin(n\theta)\right).$$

For more details / ex's, see §6.4 in Strauss!

- Now, returning to prev. example on disk, have the...

AMAZING FACT: The series in ($\square$) can be summed explicitly!

Indeed, plugging formulas for A_n, B_n into ($\square$) we find

$$u(r,\theta) = \frac{1}{2}\left(\frac{1}{\pi}\int_0^{2\pi} h(\varphi)\,d\varphi\right)$$

$$+ \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n}\int_0^{2\pi} h(\varphi)\Big(\cos(n\varphi)\cos(n\theta) + \sin(n\varphi)\sin(n\theta)\Big)d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} h(\varphi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos\big(n(\theta-\varphi)\big)\right]d\varphi.$$

To proceed, use Euler's formula

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha), \quad \alpha \in \mathbb{R}$$

which gives $\cos\big(n(\theta-\varphi)\big) = \frac{1}{2}\left(e^{in(\theta-\varphi)} + e^{-in(\theta-\varphi)}\right).$

So, $1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos\big(n(\theta-\varphi)\big)$

$$= 1 + \underbrace{\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\varphi)}\right)^n + \sum_{n=1}^{\infty}\left(\frac{r}{a}e^{-i(\theta-\varphi)}\right)^n}_{\text{Geometric Series!}}$$

(Cvgs. since $\left|\frac{r}{a}e^{\pm i(\theta-\varphi)}\right| = \frac{r}{a} < 1$)

$$= 1 + \frac{\left(\frac{r}{a}e^{i(\theta-\varphi)}\right)}{1-\left(\frac{r}{a}e^{i(\theta-\varphi)}\right)} + \frac{\left(\frac{r}{a}e^{-i(\theta-\varphi)}\right)}{1-\left(\frac{r}{a}e^{-i(\theta-\varphi)}\right)}$$

= ... simplify ...

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2} \quad \text{(Check!)}$$

Thus, if $D = \{(r,\theta) : 0 \le r < a,\ 0 \le \theta < 2\pi\}$ (A disk!)
then soln. of

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

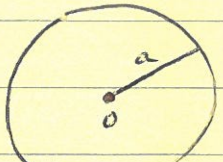
is given explicitly by

$$\boxed{u(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\varphi)}{a^2 - 2ar\cos(\theta - \varphi) + r^2}\, d\varphi}$$

This is Poisson's formula... expresses any harmonic ftn. on circle in terms of bdry values!!!
Alternatively, provides "harmonic extension" of h to interior of D... this intrepretation is useful in MANY applications ~~intrepretat~~ (ex: fluids!)

Poisson formula has many impt. consequences...

① Spse. u is a harmonic ftn. (ie. $\Delta u = 0$) in a disk D that is cts. on $\overline{D} = D \cup \partial D$.
Then setting $r = 0$ above gives



$$\underbrace{u(0)}_{\substack{\text{Soln. of Laplace eqn.} \\ \text{at center of disk}}} = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} u(a,\varphi)\, d\varphi}_{\substack{\text{Avg. of } u \text{ over boundary} \\ \text{of disk...}}}$$

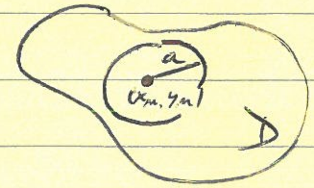
This is called the "Mean Value Property".

② Spse. u is harmonic in some connected bounded, open region $D \subset \mathbb{R}^2$, and spse. u is cts. on $\overline{D} = D \cup \partial D$. Since $\overline{D}$ is closed + bounded, and u is cts. on $\overline{D}$, follows $\exists (x_m, y_m) \in \overline{D} \ni$

$$u(x,y) \le u(x_m, y_m) \quad \forall (x,y) \in D \quad (*)$$

(i.e. its max is attained somewhere in $\overline{D}$...).

Claim: Unless u is constant, $(x_M, y_M) \in \partial D$.



Indeed, if $(x_M, y_M) \in D$ (the interior) can draw open disk

$$B_M(a) = \{(x,y) : (x-x_M)^2 + (y-y_M)^2 < a\}$$

with $0 < a << 1$ small enough so that $B_M(a) \subset D$.

By mean-value property,

$$u(x_M, y_M) = \frac{1}{2\pi}\int_0^{2\pi} u(x_M + a\cos\theta, y_M + a\sin\theta)\, d\theta$$

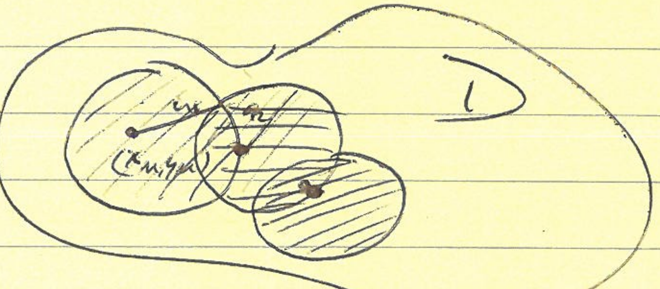
$$= \text{Avg. of } u \text{ around } \partial B_M(a).$$

But by (*), follows

$$u(x_M, y_M) = u(x_M + a\cos\theta, y_M + a\sin\theta)$$

for all $\theta \in [0, 2\pi)$, i.e. u is constant on $\partial B_M(a)$ w/ constant value $u(x_M, y_M)$.

~~So~~ By varying a then, follows that u must be constant in $B_M(a)$



Now, can repeat this argument w/ a different centers, now on $\partial B_M(a)$. We can now fill all of D up w/ disks and show that u is constant on each disk. Since these disks overlap and D is connected, follows u is constant in D. Thus, if u is <u>not</u> constant, then

$(x_M, y_M) \in \partial D$ as claimed.

• Rmk: Above property is called the maximum principle for Harmonic Ftns.

• Rmk: Both the maximum principle and the mean value property hold in ALL dimensions!

• Equipped w/ the Max. Princ., can prove uniqueness of the Dirichlet problem

$$(*)\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$

where D is a given connected, bounded, open domain in $\mathbb{R}^2$ and f, h are given ftns. If both u_1 and u_2 solve $(*)$, then $w = u_1 - u_2$ solves

$$\begin{cases} \Delta w = 0 & \text{in } D \\ w = 0 & \text{on } \partial D \end{cases}$$

Thus, by maximum principle,

$$\max_{(x,y)\in\bar{D}} w(x,y) = 0.$$

BUT, applying max princ. to ftn. "$-u$" see also

$$\min_{(x,y)\in\bar{D}} w(x,y) = 0 \qquad \text{(The "Min. Princ.")}$$

which implies

$$w = 0 \quad \text{on } \bar{D},$$

verifying uniqueness.