Math 951 – Advanced PDE II
Homework 1: Due Wednesday, February 12 at 3pm

Spring 2020

Turn in solutions to all problems not marked as "suggested". Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. Let $p \in [1, \infty)$ and $U \subset \mathbb{R}^n$ be open. Prove that translation is continuous in $L^p(U)$. More precisely, prove that for all $f \in L^p(U)$ and for every open V such that the closure $\overline{V}$ is compact and $\overline{V} \subset U$, denoted $V \Subset U$, we have
$$\lim_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(V)} = 0.$$
Is this result true for $p = \infty$? Either give a proof or a counterexample.
(*Hint: For the first part, prove this for smooth functions with compact support, and then use the density of such functions in $L^p(U)$. Note: You will receive VERY little credit for this problem if you DO NOT show the details of your density argument.*)

2. Suppose that $U \subset \mathbb{R}^n$ is open and bounded with smooth boundary and that $\Omega_1, \Omega_2 \subset U$ are two non-empty disjoint open sets with smooth boundaries which decompose U in the sense that $U = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is a smooth hypersurface in $\mathbb{R}^n$. Suppose $u_i \in H^1(\Omega_i)$, $i = 1, 2$ are such that u_1 restricted to Γ agrees with u_2 restricted to Γ (the restriction taken in the trace sense). Prove that the function
$$u(x) = \begin{cases} u_1(x), & \text{for } x \in \Omega_1 \\ u_2(x), & \text{for } x \in \Omega_2 \end{cases}$$
lies in the space $H^1(U)$, and identify the weak derivative Du.

3. (Evans 5.11) Suppose $U \subset \mathbb{R}^n$ is connected and $u \in W^{1,p}(U)$ satisfies
$$Du = 0 \text{ a.e. in } U.$$
Prove that u is constant a.e. in U. (*Hint: Mollify u, and use Theorem 3 from Section 6.5(b) from McOwen, or the "Properties of Mollifiers" theorem from Appendix C of Evans.*)

4. (Evans 5.4 – **Suggested**) (a) Prove that if $u \in W^{1,p}(0, 1)$ for some $1 \leq p < \infty$, then u is equal almost everywhere (a.e.) to an absolutely continuous function, and that u' (which exists a.e.) belongs to $L^p(0, 1)$.
Note: In measure theory, you are typically presented with a complicated definition of "absolutely continuous" functions. Here, you can use the following fact as a characterization[1]: Absolutely continuous functions are precisely those for which the Fundamental Theorem of Calculus applies. In particular, u is absolutely continuous on $[a, b]$

[1] In fact, this is equivalent to the complicated definition, and is sometimes even given as the definition.

if and only if the pointwise derivative exists a.e. in (a, b) and satisfies $u' \in L^1(a, b)$ and

$$u(x) = u(a) + \int_a^x u'(y)dy$$

for all $x \in [a, b]$.

(b) Prove directly that if $u \in W^{1,p}(0, 1)$ for some $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x - y|^{1-1/p} \left(\int_0^1 |u'(t)|^p dt \right)^{1/p}$$

for a.e. $x, y \in [0, 1]$. Conclude that Sobolev functions in one-dimension are Hölder continuous[2], specifically that $W^{1,p}(0, 1) \subset C^{0,1-1/p}(0, 1)$.

Note: This is a one-dimensional version of Morrey's inequality and is an example of a Sobolev embedding theorem. We will study these embeddings in detail later, as they form one of the cornerstones of the Sobolev theory.

5. Show that if $u \in H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$, and if u' denotes the weak derivative of u, then

$$u'(x) = \lim_{h \to 0} \frac{u(x + h) - u(x)}{h}$$

where the limit is in the $L^2(\mathbb{R})$ sense.

Hint: First, notice that u is absolutely continuous (by the above problem), and hence the fundamental theorem of calculus applies to u. Begin by showing that

$$u(x + h) - u(x) = \int_0^1 u'(x + th)hdt.$$

for all $x, h \in \mathbb{R}$, and then use this identity to show that

$$\left\| u' - \frac{u(\cdot + h) - u}{h} \right\|_{L^2(\mathbb{R})}^2 \to 0$$

as $h \to 0$. The result of Problem 1 above may be helpful for this last part: here, notice that the result of Problem 1 can be extended to show for all $f \in L^p(\mathbb{R})$, $p \in [1, \infty)$, we have

$$\lim_{h \to 0} \|f(\cdot + h) - f\|_{L^p(\mathbb{R})} = 0.$$

That is, boundedness of the domain was not necessary in Problem 1 above.

6. This exercise introduces you to the so-called "Sobolev Embedding" theorems in the special case of Sobolev spaces of periodic functions. Such functions can be represented as Fourier series, which makes their analysis significantly more straightforward than for functions defined on arbitrary bounded domains we have been considering in class.

[2]See Section 6.5(a) of McOwen for definitions of the Hölder spaces.

To begin, let $n \geq 1$ and $L > 0$. We say a function $u : \mathbb{R}^n \to \mathbb{R}$ is L-periodic if

$$u(x + Le_j) = u(x) \quad \forall\ j = 1, 2, \dots n,$$

where the e_j denote the standard unit basis vectors for $\mathbb{R}^n$. It is clear that such an L-periodic function is uniquely determined by its values on the n-dimensional cube $Q = [0, L]^n$ denote an n-dimensional cube. We define the space of square-integrable L-periodic functions as

$$L^2_p(Q) = \left\{ u \in L^2_{\text{loc}}(\mathbb{R}) : \ u \text{ is } L\text{-periodic} \right\}$$

equipped with the norm

$$\|u\|_{L^2_p(Q)} := \left(\int_Q |u|^2 dx \right)^{1/2}.$$

By basic Harmonic Analysis[3] any function $u \in L^2_p(Q)$ can be represented by a Fourier series, i.e. there exists constants $\{a_j\}_{j \in \mathbb{Z}^n}$ such that

$$u(x) = \sum_{j \in \mathbb{Z}^n} a_j e^{2\pi i j \cdot x / L} \quad \text{with} \quad a_{-j} = \bar{a}_j. \tag{0.1}$$

where this series converges in $L^2(Q)$. One can easily check (you do not need to show the details here) that

$$\int_Q |u|^2 dx = L^m \sum_{j \in \mathbb{Z}^n} |a_j|^2,$$

a result known as Parseval's identity.

With this in mind, we can formally start taking derivatives and find that, given a multi-index $\alpha \in \mathbb{N}^n$ we have

$$D^\alpha u(x) = \left(\frac{2\pi i}{L} \right)^{|\alpha|} \sum_{j \in \mathbb{Z}^n} a_j j^\alpha e^{2\pi i j \cdot x / L}$$

so that

$$\int_Q |D^\alpha u|^2 dx = L^m \left(\frac{2\pi}{L} \right)^{2|\alpha|} \sum_{j \in \mathbb{Z}^n} |a_j|^2 |j^{2\alpha}|$$

Given any $s \in \mathbb{N}$ we can define the periodic Sobolev space $H^s_p(Q)$ as the collection of all functions in $L^2_p(Q)$ such that the norm

$$\|u\|_{H^s_p(Q)} := \left(\sum_{j \in \mathbb{Z}^n} (1 + |j|^{2s}) |a_j|^2 \right)^{1/2}$$

[3] If there is such a thing :-)

is finite. An important observation[4] is then that if $u \in H_p^s(Q)$ then the Fourier series in (0.1) converges to u in $H_p^s(Q)$. Notice, in particular, that this definition does not actually require that s be a positive integer. As a result, we can consider the periodic Sobolev spaces[5] $H_p^s(Q)$ for any $s \geq 0$.

With the above set up, complete the following exercises:

(a) Show that if $u \in H_p^s(Q)$ with $s > n/2$, then $u \in L^\infty(Q)$ with

$$\|u\|_{L^\infty(Q)} \leq C_s \|u\|_{H_p^s(Q)}$$

for some constant $C_s > 0$ independent of u. Conclude that $u \in C^0(Q)$.
Remark: Morally speaking, this shows that if a Sobolev function has "enough" derivatives in L^2, then it is in fact continuous. Using a simple induction, you could extend this result to show $H_p^s(Q) \subset C^k(Q)$ whenever $s > n/2 + k$.

(b) Show that if $u \in H_p^s(Q)$ with $0 < s < n/2$, then $u \in L^q(Q)$ for all

$$q \in \left[2, \frac{n}{(n/2) - s}\right).$$

Hint: You may use, without proof[6], the fact that if $\{a_j\}_{j \in \mathbb{Z}^n} \in \ell^r$ for some $r \in [1, 2]$ then the function

$$u(x) = \sum_{j \in \mathbb{Z}^n} a_j e^{2\pi i j \cdot x / L}$$

belongs to $L^q(Q)$ with $r^{-1} + q^{-1} = 1$ and

$$\|u\|_{L^q(Q)} \leq C \|\{a_j\}\|_{\ell^r}$$

for some constant $C > 0$ independent of u. Note in particular the restriction on r implies $q \geq 2$.
Remark: Morally speaking, this shows that Sobolev functions always have better than expected integrability properties...

(c) (**Suggested**) Show that $H_p^s(Q)$ is compactly embedded in $L_p^2(Q)$ for all $0 < s < \infty$. That is, a bounded sequence $\{u_k\}_{k=1}^\infty$ in $H_p^s(Q)$ has a subsequence that converges in $L_p^2(Q)$.
Hint: First show that if $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $H_p^s(Q)$, then each Fourier

[4] You may use this without proof.

[5] This should be of interest to any student of Harmonic analysis, as it allows you to define fractional order Sobolev spaces, which leads naturally to a basic class of Fourier multipliers and pseudo-differential operators.

[6] The proof is nontrivial, relying on methods from complex analysis.

coefficient of the sequence u_k is uniformly bounded in k. Once you establish this, a diagonal argument might be useful...
Remark: This is a periodic version of the Rellich-Kondrachov compactness theorem. Note, however, that you can not directly invoke the Rellich-Kondrachov theorem from class since here we allow non-integer values of s.

7. (**Suggested**) Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Given $u \in W^{1,p}(U)$, define $E(u) : \mathbb{R}^n \to \mathbb{R}$ by

$$E(u)(x) = \begin{cases} u(x), & \text{for } x \in U \\ 0, & \text{for } x \notin U. \end{cases}$$

Show that the mapping $E : W_0^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ is well defined. That is, verify that for $u \in W_0^{1,p}(U)$ one has $E(u) \in W^{1,p}(\mathbb{R}^n)$. Is the map E well defined from $W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$? Explain.
Discussion: Thus, by extending by zero outside U, any function in $u \in W_0^{1,p}(U)$ can be considered as a function in $W^{1,p}(\mathbb{R}^n)$. This is an example of an "Extension" theorem.

8. (**Suggested**) Give an example of a continuous function on $[0,1]$ which has classical derivative defined almost everywhere, but which is not weakly differentiable. (*Hint: Consider functions which are continuous on* $[0,1]$ *but not absolutely continuous on* $[0,1]$.)