Math 951 – Advanced PDE II
Homework 1 – Hints!!

Spring 2020

3. (Evans 5.11) Suppose $U \subset \mathbb{R}^n$ is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \ \text{ a.e. in } U.$$

Prove that U is constant a.e. in U.

Hint: Consider the mollification of u as done in class and recall that when $u \in W^{1,p}(U)$, we have

$$Du_\varepsilon = \eta_\varepsilon * (Du)$$

where the "D" on the left and right hand sides represent the classical and weak derivatives, respectively. Use this fact to calculate Du_ε and explain why taking $\varepsilon \to 0^+$ gives the desired result.

3. (Evans 5.4) (a) Prove that if $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$, then u is equal almost everywhere (a.e.) to an absolutely continuous function, and that u' (which exists a.e.) belongs to $L^p(0,1)$.

Hint: The fact that $u \in W^{1,p}(0,1)$ implies that the weak derivative u' exists a.e. and belongs to $L^p(0,1)$. One approach to this problem verifies that the function $v : (0,1) \to \mathbb{R}$ defined by

$$v(x) = \int_{1/2}^{x} u'(y)dy$$

is well defined and is absolutely continuous. It is enough to then show that $u - v$ is equal to a constant almost everywhere. For this last part, #1 above may be helpful.

5. Show that if $u \in H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ and if u' denotes the weak derivative of u, then

$$u' = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

where the limit is in the $L^2(\mathbb{R})$ sense.

Hint: Show that Cauchy-Schwartz implies the inequality

$$\left(\int_0^1 g(x)dx \right)^2 \leq \int_0^1 g(x)^2 dx$$

for all $g \in L^2(0,1)$. Use this to show

$$\left\| u' - \frac{u(\cdot + h) - u}{h} \right\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \int_0^1 |u'(x) - u'(x+th)|^2 dx \ dt.$$

Note: A result from Measure theory (Tonelli's Theorem) tells you you can freely interchange orders of integration when integrating and integrable non-negative function. Look this result up and apply it to the above problem.

6. Parts (a) and (b) follow by playing with appropriate "summable weights". Essentially, you are going to do tricks like in the proof the following elementary result:

   Theorem 0.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$*. If* $\sum_{n=1}^{\infty} n^2 a_n^2 < \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

   Proof: By Hölder's inequality, we have

   $$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} \cdot n|a_n| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} n^2 a_n^2\right)^{1/2},$$

   which is finite by assumption since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

   For part (c), follow the hint in the HW...