Math 951 – Advanced PDE II
Homework 1 – Solutions!

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. Let $p \in [1, \infty)$ and $U \subset \mathbb{R}^n$ be open. Prove that translation is continuous in $L^p(U)$. More precisely, prove that for all $f \in L^p(U)$ and for every open V such that the closure $\bar{V}$ is compact and $\bar{V} \subset U$, denoted $V \Subset U$, we have

$$\lim_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(V)} = 0.$$

Is this result true for $p = \infty$? Either give a proof or a counterexample.

Solution: First, fix $1 \leq p < \infty$. Let $V \Subset U$ and first consider the case that $f \in C_c^\infty(U)$. By the mean value theorem, for all $|h|$ sufficiently small we have

$$\|f(\cdot + h) - f(\cdot)\|_{L^p(V)} \leq |V| \cdot \|f(\cdot + h) - f(\cdot)\|_{L^\infty(V)} \leq |V| \|f'\|_{L^\infty(U)} |h|,$$

which clearly converges to zero as $h \to 0$; here, $|V|$ denotes the Lebesgue measure of the set V. Alternatively, you could argue based on the fact that f is uniformly continuous on $\bar{V}$.

In the general case for that $f \in L^p(U)$ for some $p \in [1, \infty)$, let $\varepsilon > 0$ be given and note, by the density of $C_c^\infty(U)$ in $L^p(U)$, there exists a function $g \in C_c^\infty(U)$ such that $\|f - g\|_{L^p(U)} < \varepsilon$. It follows by the triangle inequality in $L^p(U)$ that for h sufficiently small we have

$$\begin{aligned}
\|f(\cdot + h) - f(\cdot)\|_{L^p(V)} &\leq \|f(\cdot + h) - g(\cdot + h)\|_{L^p(V)} + \|g(\cdot + h) - g(\cdot)\|_{L^p(V)} \\
&\quad + \|g - f\|_{L^p(V)} \\
&< 2\varepsilon + \|g(\cdot + h) - g(\cdot)\|_{L^p(V)}.
\end{aligned}$$

Since $g \in C_c^\infty(U)$, the first step above yields

$$\limsup_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(V)} \leq 2\varepsilon$$

and hence, since $\varepsilon > 0$ was arbitrary, we have $\lim_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{L^p(V)} = 0$ as claimed.

Notice this result is not true in the case $p = \infty$. Indeed, if $U = \mathbb{R}$ and f is the step function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

then $\|f(\cdot + h) - f(\cdot)\|_{L^\infty(V)} = 1$ for any $V \Subset U$ containing $x = 0$ and all $h \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

2. Suppose that $U \subset \mathbb{R}^n$ is open and bounded with smooth boundary and that $\Omega_1, \Omega_2 \subset U$ are two non-empty disjoint open sets with smooth boundaries which decompose U in the sense that $U = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is a smooth hypersurface in $\mathbb{R}^n$. Suppose $u_i \in H^1(\Omega_i)$, $i = 1, 2$ are such that u_1 restricted to Γ agrees with u_2 restricted to Γ (the restriction taken in the trace sense). Prove that the function

$$u(x) = \begin{cases} u_1(x), & \text{for } x \in \Omega_1 \\ u_2(x), & \text{for } x \in \Omega_2 \end{cases}$$

lies in the space $H^1(U)$, and identify the weak derivative Du.

Solution: First, we show that u defined above is weakly differentiable in U. To this end, fix $\phi \in C_c^\infty(U)$ and note that since u_i are weakly differentiable in U we have

$$\begin{aligned} \int_U u D\phi \, dx &= \int_{\Omega_1} u_1 D\phi \, dx + \int_{\Omega_2} u_2 D\phi \, dx \\ &= -\left(\int_{\Omega_1} Du_1 \phi \, dx + \int_{\Omega_2} Du_2 \phi \, dx \right) + \int_\Gamma (u_1 - u_2) \frac{\partial \phi}{\partial \nu} dS, \end{aligned}$$

where here ν denotes the unit *outer* normal vector along Γ: specifically note ν points in opposite directions when viewed from inside Ω_1 versus Ω_2. Defining

$$v(x) = Du_1(x)\chi_{\Omega_1}(x) + Du_2(x)\chi_{\Omega_2}(x)$$

it follows that

$$\int_U u D\phi \, dx = -\int_U v\phi \, dx$$

which, since $v \in L^1_{\text{loc}}(U)$, implies that u is weakly differentiable in U with weak derivative $Du = v$. Furthermore, since $Du_i \in L^2(\Omega_i)$ we clearly have that $v \in L^2(U)$, and hence $u \in H^1(U)$ as claimed.

3. (Evans 5.11) Suppose $U \subset \mathbb{R}^n$ is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \text{ a.e. in } U.$$

Prove that u is constant a.e. in U.

Solution: Let u^ε denote the standard mollification of u defined on the dialated domain U_ε and recall, for each $\varepsilon > 0$, that $u^\varepsilon \in C^\infty(U_\varepsilon)$ with

$$D(u^\varepsilon) = D(\eta_\varepsilon * u) = \eta_\varepsilon * Du.$$

Thus, we have that $D(u^\varepsilon) = 0$ in U_ε and since U is connected it follows that, for each $\varepsilon > 0$, $u^\varepsilon(x) = C_\varepsilon$ on U_ε for some constant C_ε, possibly depending on ε. Since $u^\varepsilon \to u$ a.e. in U as $\varepsilon \to 0$, we have for a.e. $x \in U$

$$\lim_{\varepsilon\to 0} C_\varepsilon = \lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x)$$

and hence the limit $\lim_{\varepsilon\to 0} C_\varepsilon$ exists. Setting $C = \lim_{\varepsilon \to 0} C_\varepsilon$ it follows that $u(x) = C$ for a.e. $x \in U$ as claimed.

4. (Evans 5.4 – **Suggested**) (a) Prove that if $u \in W^{1,p}(0,1)$ for some $1 \le p < \infty$, then u is equal almost everywhere (a.e.) to an absolutely continuous function, and that u' (which exists a.e.) belongs to $L^p(0,1)$.
(b) Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x-y|^{1-1/p} \left(\int_0^1 |u'(t)|^p dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$. Conclude that Sobolev functions in one-dimension are Hölder continuous[1], specifically that $W^{1,p}(0,1) \subset C^{0,1-1/p}(0,1)$.

Solution: (a) Notice that since $u \in W^{1,p}(0,1)$, its weak derivative u' exists and belongs to $L^p(U)$. Moreover, Hölder's inequality implies $u' \in L^1(0,1)$, so that the function $F : (0,1) \to \mathbb{R}$ defined by

$$F(x) = \int_{1/2}^x u'(t)dt$$

is well-defined. In fact, F is absolutely continuous since, by the Fundamental Theorem of calculus, we have that F' exists classically a.e. and equals u' and that

$$F(x) = F(1/2) + \int_{1/2}^x F'(t)dt.$$

It is enough then to show $u = F$ a.e. in $(0,1)$.

To this end, note for any $\phi \in C_c^\infty(0,1)$ we have

$$\int_0^1 F\phi' dx = -\int_0^1 F'\phi dx = -\int_0^1 v\phi dx = \int_0^1 u\phi' dx.$$

Therefore,

$$\int_0^1 (F-u)\,\phi' dx = 0$$

[1]See Section 6.5(a) of McOwen for definitions of the Hölder spaces.

for all $\phi \in C_c^\infty(0,1)$ so that $F-u$ is weakly differentiable on $(0,1)$ with $D(F-u)=0$ a.e. It now follows from Problem #3 above that $F-u$ is a.e. equal to a constant function on $(0,1)$, i.e. there exists a constant $C \in \mathbb{R}$ such that $u(x) = F(x) + C$ for a.e. $x \in (0,1)$. Therefore, u' exists classically a.e. and $u' \in L^p(0,1)$ as required.

For an alternate proof of part (a) using mollifiers, consider the following argument: Write u^ε for the standard mollification of u and fix $a \in (0,1)$. Then

$$u^\varepsilon(x) = u^\varepsilon(a) + \int_a^x (u^\varepsilon)'(t)dt$$

by the fundamental theorem of calculus (using that u^ε is smooth). Requiring a to be a Lebesgue point of u (recall, by the Lebesgue differentiation theorem, that a.e. point of $(0,1)$ is a Lebesgue point of u), so that

$$u^\varepsilon(a) \to u(a)$$

as $\varepsilon \to 0$. Moreover, recall from class that $(u^\varepsilon)' = (u')^\varepsilon$ and hence

$$(u^\varepsilon)' = (u')^\varepsilon \to u'$$

in $L^1(0,1)$ which implies that

$$\int_a^x (u^\varepsilon)'(t)dt \to \int_a^x u'(t)dt$$

as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ hence implies that

$$u(x) = u(a) + \int_a^x u'(t)dt$$

for a.e. $x \in (0,1)$, and hence $u \in AC([0,1])$.

(b): Let $u \in W^{1,p}(0,1)$ and notice then that u is absolutely continuous with $u' \in L^p(0,1)$. Thus, given $x, y \in (0,1)$ with $y \leq x$, an application of Hölder's inequality yields

$$\begin{aligned}
|u(x) - u(y)| &\leq \int_y^x |u'(t)|dt \\
&\leq \left(\int_y^x 1dt\right)^{1-1/p} \left(\int_y^x |u'(t)|^p dt\right)^{1/p} \\
&\leq \|u'\|_{L^p(0,1)} |x-y|^{1-1/p}
\end{aligned}$$

as claimed. Now, since we identify functions in $W^{1,p}$ that agree a.e., we can assume that u is continuous and that the above inequality holds for all $x, y \in [0,1]$. It follows then that

$$[u]_{C^{0,1-1/p}(0,1)} = \sup_{x,y\in[0,1], x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-1/p}} \leq \|u'\|_{L^p(0,1)},$$

which is finite since $u \in W^{1,p}(0,1)$. Therefore, since u can be chosen to be continuous, it follows that $u \in C^{0,1-1/p}(0,1)$ as claimed.

5. Show that if $u \in H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$, and if u' denotes the weak derivative of u, then

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

where the limit is in the $L^2(\mathbb{R})$ sense.

Solution: Since $u \in H^1(\mathbb{R})$, we know that u is absolutely continuous and $u' \in L^2(\mathbb{R})$ exists a.e.. By the absolute continuity of u, it follows that the Fundamental Theorem of Calculus applies to u and hence

$$u(x+h) - u(x) = \int_0^1 u'(x+th)h \, dt,$$

which, by the triangle inequality, implies

$$\left| u'(x) - \frac{u(x+h) - u(x)}{h} \right| \leq \int_0^1 |u'(x) - u'(x+th)| dt.$$

Therefore, squaring and integrating gives

$$\begin{aligned} \left\| u'(x) - \frac{u(x+h) - u(x)}{h} \right\|_{L^2(\mathbb{R})}^2 &\leq \int_{\mathbb{R}} \int_0^1 |u'(x) - u'(x+th)|^2 dt \, dx \\ &= \int_0^1 \int_{\mathbb{R}} |u'(x) - u'(x+th)|^2 dx \, dt \quad \text{(by Tonelli)} \\ &\leq \sup_{\delta \in [-|h|, |h|]} \|u'(\cdot) - u'(\cdot + \delta)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Since translation is continuous in $L^2(\mathbb{R})$, it follows that

$$\frac{u(x+h) - u(x)}{h} \to u'(x) \quad \text{in } L^2(\mathbb{R}) \text{ as } h \to 0$$

as claimed.

6. This exercise introduces you to the so-called "Sobolev Embedding" theorems in the special case of Sobolev spaces of periodic functions. Such functions can be represented as Fourier series, which makes their analysis significantly more straightforward than for functions defined on arbitrary bounded domains we have been considering in class.

(a) Show that if $u \in H_p^s(Q)$ with $s > n/2$, then $u \in L^\infty(Q)$ with

$$\|u\|_{L^\infty(Q)} \leq C_s \|u\|_{H_p^s(Q)}$$

for some constant $C_s > 0$ independent of u. Conclude that $u \in C^0(Q)$.
Remark: Morally speaking, this shows that if a Sobolev function has "enough" derivatives in L^2, then it is in fact continuous. Using a simple induction, you could extend this result to show $H_p^s(Q) \subset C^k(Q)$ whenever $s > n/2 + k$.

Solution: Using the representation $u(x) = \sum_{j\in\mathbb{Z}^n} a_j e^{2\pi i j\cdot x/L}$ we find by the triangle inequality that for any $x \in Q$

$$|u(x)| \leq \sum_{j\in\mathbb{Z}^n} |a_j|.$$

Now, Hölder's inequality implies that

$$\sum_{j\in\mathbb{Z}^n} |a_j| = \sum_{j\in\mathbb{Z}^n} \frac{\sqrt{1+|j|^{2s}}}{\sqrt{1+|j|^{2s}}}\,|a_j| \leq \left(\sum_{j\in\mathbb{Z}^n} \frac{1}{1+|j|^{2s}}\right)^{1/2} \left(\sum_{j\in\mathbb{Z}^n} (1+|j|^{2s})|a_j|^2\right)^{1/2}.$$

Since the n-dimensional p-series test implies that

$$\sum_{j\in\mathbb{Z}^n} \frac{1}{1+|j|^{2s}} < \infty$$

provided $2s > n$, it follows that

$$\|u\|_{L^\infty(Q)} \leq C\|u\|_{H_p^s(Q)}$$

for some constant $C > 0$ provided that $s > n/2$, as claimed.

To conclude that $H_p^s(Q) \subset C^0(Q)$ for $s > n/2$, notice that the above inequality implies that convergence in $H_p^s(Q)$ implies uniform convergence provided $s > n/2$. Since $u \in H_p^s(Q)$ implies the Fourier series for u is itself a limit of continuous functions, convergences in $H_p^s(Q)$ to u, it follows that u is the uniform limit of continuous functions and hence continuous.

(b) Show that if $u \in H_p^s(Q)$ with $0 < s < n/2$, then $u \in L^q(Q)$ for all

$$q \in \left[2, \frac{n}{(n/2)-s}\right).$$

Solution: First, notice by the Hausdorff-Young inequality (given in the hints) we have that if $u(x) = \sum_{j\in\mathbb{Z}^n} a_j e^{2\pi i j\cdot x/L}$ then for any $q \geq 2$ we have

$$\|u\|_{L^q(Q)} \leq \|\{a_j\}\|_{\ell^r}, \quad \frac{1}{q} + \frac{1}{r} = 1;$$

in particular, notice that $1 \leq r \leq 2$. Similarly as in part (a), we introduce a weight $(1+|j|^{2s})^m$ for some $m > 0$, to be determined later, and use Hölder's inequality to obtain

$$\begin{aligned}
\|\{a_j\}\|_{\ell^r}^r = \sum_{j\in\mathbb{Z}^n} |a_j|^r &= \sum_{j\in\mathbb{Z}^n} \frac{(1+|j|^{2s})^m}{(1+|j|^{2s})^m}\ |a_j|^r \\
&\leq \left(\sum_{j\in\mathbb{Z}^n} \frac{1}{(1+|j|^{2s})^{m(2/r)'}}\right)^{1/(2/r)'} \left(\sum_{j\in\mathbb{Z}^n} (1+|j|^{2s})^{2m/r}|a_j|^2\right)^{r/2},
\end{aligned}$$

where $(2/r)'$ is the Hölder conjugate of the number $2/r > 1$, i.e. $1/(r/2)+1/(r/2)' = 1$. Choosing $m = r/2$ it follows that

$$\|u\|_{L^q(Q)} \leq \left(\sum_{j\in\mathbb{Z}^n} \frac{1}{(1+|j|^{2s})^{(r/2)\cdot(2/r)'}}\right)^{1/(2/r)'} \|u\|_{H^s_p(Q)}.$$

Since $(2/r)' = \frac{2}{2-r}$, it follows from the n-dimensional p-series test that

$$\sum_{j\in\mathbb{Z}^n} \frac{1}{(1+|j|^{2s})^{(r/2)\cdot(2/r)'}} < \infty$$

provided

$$(2s)\cdot(r/2)\cdot\left(\frac{2}{2-r}\right) > n, \quad \text{i.e.} \quad r > \frac{2n}{2s+n}.$$

Recalling that $r = q' = \frac{q}{q-1}$ it follows that $r > \frac{2n}{2s+n}$ precisely when $q < \frac{2n}{n-2s}$, giving the desired result.

(c) Show that $H^s_p(Q)$ is compactly embedded in $L^2_p(Q)$ for all $0 < s < \infty$. That is, a bounded sequence $\{u_k\}_{k=1}^\infty$ in $H^s_p(Q)$ has a subsequence that converges in $L^2_p(Q)$.

Solution Let $\{u_k\}_{k=1}^\infty$ be a bounded sequence in $H^s_p(Q)$ and, for each $k = 1, 2, \ldots$ write

$$u_k(x) = \sum_{j\in\mathbb{Z}^n} a_{k,j} e^{2\pi i j\cdot x/L}.$$

Since the sequence $\{u_k\}$ is bounded, it follows that there exists a constant $M > 0$ such that

$$\sum_{j\in\mathbb{Z}^n} (1+|j|^{2s})|a_{k,j}|^2 \leq M \tag{0.1}$$

for all $k = 1, 2, \ldots$. It follows for each $j \in \mathbb{Z}^n$ that the sequence $\{a_{k,j}\}_{k=1}^\infty$ is uniformly bounded. In the sequel, for notational simplicity we shall enumerate $\mathbb{Z}^n$ by the natural numbers $\mathbb{N}$ (which is clearly possible since $\mathbb{Z}^n$ is a countable set.

Since, by above, the sequence $\{a_{k,1}\}_{k=1}^{\infty}$ is bounded, we may select a subsequence $\{u_{k_{1_r}}\}_{r=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ such that the Fourier coefficients $a_{k_{1_r},1}$ converge as $r \to \infty$. From this, we may extract yet another subsequence $\{u_{k_{2_r}}\}_{r=1}^{\infty}$ of $\{u_{k_{1_r}}\}_{r=1}^{\infty}$ such that both $a_{k_{1_r},1}$ and $a_{k_{2_r},2}$ converge as $r \to \infty$. Continuing in this way, we may obtain for each $m = 1, 2, \ldots$ a subsequence $\{u_{k_{m_r}}\}_{r=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ such that the coefficients $a_{k_{m_r},j}$ for $j = 1, 2, \ldots m$ all converge as $r \to \infty$. Now, taking the diagonal sequence $\tilde{u}_m = u_{k_{m_m}}$ and writing $a_{m,j} := a_{k_{m_m},j}$ for the corresponding Fourier coefficients, it follows for each $j \in \mathbb{Z}^n$ that the $a_{m,j}$ converge as $m \to \infty$ to a limit $a_j^* \in \mathbb{C}$, from which we may define the function

$$u^*(x) = \sum_{j \in \mathbb{Z}^n} a_j^* e^{2\pi i j \cdot x/L}.$$

Note that $u^* \in H_p^s(Q)$ since, by taking limits in (0.1) we have

$$\sum_{j \in \mathbb{Z}^n} (1 + |j|^{2s})|a_j^*|^2 \le M \tag{0.2}$$

I claim that $\tilde{u}_m \to u^*$ in $L_p^2(Q)$ as $m \to \infty$. To see this, let $K > 0$ be arbitrary (for the moment) and observe that Parseval's inequality gives

$$\begin{aligned} \|\tilde{u}_m - u^*\|_{L_p^2(Q)}^2 &= \sum_{j \in \mathbb{Z}^n} |a_{m,j} - a_j^*|^2 \\ &\le \sum_{|j| \le K} |a_{m,j} - a_j^*|^2 + \frac{1}{K^{2s}} \sum_{|j| \ge K} |a_{m,j} - a_j^*|^2 |k|^{2s}. \end{aligned}$$

Since (0.1) and (0.2) imply

$$\sum_{|j| \ge K} |a_{m,j} - a_j^*|^2 |k|^{2s} \le \sum_{|j| \ge K} |a_{m,j} - a_j^*|^2 (1 + |k|^{2s}) \le 2M,$$

it follows that

$$\|\tilde{u}_m - u^*\|_{L_p^2(Q)}^2 \le \sum_{|j| \le K} |a_{m,j} - a_j^*|^2 + \frac{2M}{K^{2s}}.$$

Now, letting $\varepsilon > 0$ be given, we may choose $m = 1, 2, \ldots$ large enough such that the first

$$\sum_{|j| \le K} |a_{m,j} - a_j^*|^2 < \epsilon$$

and choose $K > 0$ large enough that $\frac{2M}{K^{2s}} < \varepsilon$, it follows that

$$\|\tilde{u}_m - u^*\|_{L_p^2(Q)}^2 < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that the subsequence $\tilde{u}_m$ of the original bounded sequence u_k converges in $L_p^2(Q)$. Since u_k was an arbitrary bounded subsequence in $H_p^s(Q)$, it follows that $H_p^s(Q)$ is compactly embedded in $L_p^2(Q)$, as claimed.

7. (**Suggested**) Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. Given $u \in W^{1,p}(U)$, define $E(u) : \mathbb{R}^n \to \mathbb{R}$ by

$$E(u)(x) = \begin{cases} u(x), & \text{for } x \in U \\ 0, & \text{for } x \notin U. \end{cases}$$

Show that the mapping $E : W_0^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ is well defined. That is, verify that for $u \in W_0^{1,p}(U)$ one has $E(u) \in W^{1,p}(\mathbb{R}^n)$. Is the map E well defined from $W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$? Explain.

Discussion: Thus, by extending by zero outside U, any function in $u \in W_0^{1,p}(U)$ can be considered as a function in $W^{1,p}(\mathbb{R}^n)$. This is an example of an "Extension" theorem.

Solution: First, for any $u \in W^{1,p}(U)$ we clearly have that $E(u) \in L^p(\mathbb{R}^n)$. Next, we verify that $E(u)$ is weakly differentiable in $\mathbb{R}^n$. To this end, let $\phi \in C_c^\infty(\mathbb{R}^n)$ and note that since u is weakly differentiable in U we have

$$\int_{\mathbb{R}^n} E(u) D\phi \, dx = \int_U uD \, dx = -\int_U Du\phi \, dx + \int_{\partial U} u\phi \, dS,$$

where here Du denotes the weak derivative of u in U. Defining the function $\widetilde{E}(Du) : \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{E}(Du)(x) = \begin{cases} Du(x), & \text{for } x \in U \\ 0, & \text{for } x \notin U, \end{cases}$$

it follows that for all $\phi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} E(u) D\phi \, dx = -\int_{\mathbb{R}^n} \widetilde{E}(Du)\phi \, dx + \int_{\partial U} u\phi \, dS.$$

Furthermore, since $\widetilde{E}(Du) \in L^1_{\text{loc}}(\mathbb{R}^n)$, if $u = 0$ on ∂U (in the trace sense) we find that $E(u)$ is weakly differentiable in $\mathbb{R}^n$ with weak derivative $\widetilde{E}(Du)$. Since we clearly have $\widetilde{E}(Du) \in L^p(\mathbb{R}^n)$, it follows then that $E(u) \in W^{1,p}(\mathbb{R}^n)$. Since u was arbitrary, it follows that $E(u) \in W^{1,p}(\mathbb{R}^n)$ for all $u \in W_0^{1,p}(U)$, i.e. the map $E : W_0^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ is well defined.

Note that if $u \in W^{1,p}(U)$ then it does not necessarily follow that $E(u)$ is weakly differentiable on $\mathbb{R}^n$. Indeed, weak differentiability of $E(u)$ implies the linear functional

$$\phi \mapsto \int_{\partial U} u\phi \, dS \in \mathbb{R}$$

is identically zero on $C_c^\infty(\mathbb{R}^n)$, which clearly does not hold for the function $u = 1$.

8. (**Suggested**) Give an example of a continuous function on $[0, 1]$ which has classical derivative defined almost everywhere, but which is not weakly differentiable.

Solution: Let $f \in C(\mathbb{R})$ be the Cantor function, which may be constructed as a uniform limit of piecewise constant functions defined on the standard "middle-thirds" Cantor set C. For example, $f(x) = \frac{1}{2}$ for $\frac{1}{3} \leq x \leq \frac{2}{3}$, $f(x) = \frac{1}{4}$ for $\frac{1}{9} \leq x \leq \frac{2}{9}$, $f(x) = \frac{3}{4}$ for $\frac{7}{9} \leq x \leq \frac{8}{9}$, and so on. Then f is differentiable a.e. with $f' = 0$ a.e. on $[0,1]$. However, f is not weakly differentiable. To see this, suppose that there exists a $g \in L^1_{\text{loc}}([0,1])$ such that $f' = g$ weakly, i.e. such that

$$\int_{[0,1]} g\phi \; dx = -\int_{[0,1]} f\phi' \; dx$$

for all $\phi \in C_c^\infty([0,1])$. The compliment of the Cantor set in $[0,1]$ is a union of open intervals

$$[0,1] \setminus C = \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots,$$

whose measure is equal to one. Taking test functions ϕ whose supports are compactly supported in one of these intervals, call it I, and using the fact that $f = c_I$ is constant on I, we find that

$$\int_{[0,1]} g\phi \; dx = -\int_I f\phi' \; dx = -c_I \int_I \phi' dx = 0.$$

It follows that $g = 0$ pointwise a.e. on $[0,1] \setminus C$, and hence if f is weakly differentiable, $f' = 0$. However, this would imply that f is a constant function on $[0,1]$, which is clearly not true. Thus, f can not be weakly differentiable on $[0,1]$ even though it has classical derivative defined a.e. on $[0,1]$.