Math 951 – Advanced PDE II
Homework 2: Due Monday, March 2 at 4pm

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. Let $U \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary, and consider the following Poincaré like inequality: Given any constant $\sigma > 0$, there exists a constant $C > 0$ such that
$$\int_U u^2 dx \leq C \left(\int_U |Du|^2 dx + \sigma \int_{\partial U} |u|^2 dS \right)$$
for all $u \in H^1(U)$, where here $u|_{\partial U}$ is interpreted in the trace sense.

   (a) Provide a direct proof of this fact, using that $C^\infty(\bar{U})$ is dense in $H^1(U)$. To receive full credit, it is not enough to simply prove for smooth functions on $\bar{U}$ and then just say "by density, it holds on $H^1(U)$". You must show the details of this final density argument.

   (b) Provide another proof of this inequality using a proof by contradiction. (*Hint: It may help to look over our proof of Poincaré on $W^{1,p}(U)$ here...*)

2. (Based on #5 in Section 6.6 of Evans) Let $\sigma > 0$ be a fixed constant and suppose $U \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary. Given $f \in L^2(\mathbb{R}^n)$, consider Poisson's equation with Robin boundary conditions:
$$\begin{cases} -\Delta u = f \text{ in } U, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 \text{ on } \partial U. \end{cases}$$
The goal of this exercise is to verify the existence and uniqueness of a "weak" solution of the above BVP.

   (a) We say a function $u \in H^1(U)$ is a weak solution of the given BVP if
$$\int_U Du \cdot D\phi \; dx + \sigma \int_{\partial U} u\phi \; dS = \int_U f\phi \; dx$$
for all $\phi \in H^1(U)$. Justify that this is a reasonable definition of a weak solution for the given BVP by first supposing $u \in C^\infty(\bar{U})$, and multiplying the PDE by an arbitrary $\phi \in C^\infty(\bar{U})$, and integrating over U.

   (b) Define the bilinear form $B : H^1(U) \times H^1(U) \to \mathbb{R}$ by
$$B[v_1, v_2] := \int_U Dv_1 \cdot Dv_2 \; dx + \sigma \int_{\partial U} v_1 v_2 \; dS.$$
Show that $B[\cdot, \cdot]$ defines an inner product on $H^1(U)$. (*Hint: Problem # 1 above will be helpful here...*)

(c) Verify that the inner product $B[\cdot,\cdot]$ generates a norm on $H^1(U)$ that is equivalent to the standard one, i.e. show there exists a $C > 1$ such that

$$C^{-1}\|v\|^2_{H^1(U)} \leq B[v,v] \leq C\|v\|^2_{H^1(U)}$$

for all $v \in H^1(U)$. Show then that the set $H^1(U)$ equipped with the norm $\|\cdot\|_* := \sqrt{B[\cdot,\cdot]}$ is a Hilbert space. (*Hint: For this last part, all you really need to check is that Cauchy sequences in* $(H^1(U), \|\cdot\|_*)$ *converge in* $(H^1(U), \|\cdot\|_*)$.)

(d) Given $f \in L^2(U)$, show that the map $H^1(U) \ni \phi \mapsto \int_U f\phi\, dx$ defines a continuous linear functional on the Hilbert space $(H^1(U), \|\cdot\|_*)$.

(e) Using the Riesz-Representation Theorem, verify that for every $f \in L^2(U)$, there exists a unique weak solution $u \in H^1(U)$ of the given BVP.

3. (Based on #4 in Section 6.2 of McOwen) Let $\mu \in \mathbb{R}$ be non-zero and consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned}$$

where $U \subset \mathbb{R}^n$ is open and bounded and $f \in L^2(U)$ is given.

(a) Derive the appropriate weak formulation of this problem for $u \in H^1_0(U)$.

(b) Set

$$\lambda_1 := \inf_{u \in H^1_0(U)} \frac{\int_U |Du|^2 dx}{\int_U u^2 dx}.$$

Prove that $\lambda_1 > 0$.

(c) Prove that if $\mu > -\lambda_1$, then the above BVP has a unique weak solution $u \in H^1_0(U)$ for each $f \in L^2(U)$.

Remark: As we will see later, the number λ_1 is precisely the smallest eigenvalue of the operator $-\Delta$ on $H^1_0(U)$. Thus, this result says that as long as the number $-\mu$ is less than this smallest eigenvalue of $-\Delta$, the given BVP has a unique weak solution. This should be more or less what you expect from your experience with finite-dimensional linear algebra.

4. Let $U \subset \mathbb{R}^n$ be a smooth, bounded, connected open set. Let Γ_1, Γ_2 be two disjoint subsets of ∂U of positive $(n-1)-$dimensional measure such that $\Gamma_1 \cup \Gamma_2 = \partial U$. (For example, in $\mathbb{R}^2$ U might be an annulus.) Define the set

$$\mathcal{H} := \left\{\phi \in C^\infty(\bar{U}) : \text{dist}(\text{spt}\phi, \Gamma_1) > 0\right\},$$

and define the Hilbert space $\widetilde{H}^1(U)$ as the closure of $\mathcal{H}$ in the standard $H^1(U)$ norm.

(a) Prove the following Poincaré inequality for functions in $\widetilde{H}^1(U)$: $\exists C > 0$ such that

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx \quad \forall u \in \widetilde{H}^1(U).$$

(b) Consider the following problem: Given $f \in L^2(U)$, find $u \in \widetilde{H}^1(U)$ such that

$$\int_U Du \cdot D\phi \, dx = \int_U f\phi \, dx \quad \forall \phi \in \widetilde{H}^1(U).$$

Prove the existence of a unique solution of this problem.

(c) Carefully explain what boundary value problem (i.e. PDE <u>and</u> boundary conditions) you solved in the weak sense in part (b)?