Math 951 – Advanced PDE II
Homework 2 – Hints!

Spring 2020

1. Let $U \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary, and consider the following Poincaré like inequality: Given any constant $\sigma > 0$, there exists a constant $C > 0$ such that

$$\int_U u^2 dx \leq C \left(\int_U |Du|^2 dx + \sigma \int_{\partial U} |u|^2 dS \right)$$

for all $u \in H^1(U)$, where here $u|_{\partial U}$ is interpreted in the trace sense.

   (a) Provide a direct proof of this fact, using that $C^\infty(\bar{U})$ is dense in $H^1(U)$. To receive full credit, it is not enough to simply prove for smooth functions on $\bar{U}$ and then just say "by density, it holds on $H^1(U)$". You must show the details of this final density argument.

   (b) Provide another proof of this inequality using a proof by contradiction.

Hint: For part (a), let $u \in C^\infty(\bar{U})$ and note that, by integration by parts,

$$\int_U u^2 dx = \int_U u^2 \frac{d}{dx_1}(x_1) dx = -\int_U 2uu_{x_1} x_1 dx + \text{boundary terms}.$$

Note that if $u \in C_c^\infty(U)$, then the boundary terms would not be present (as in our proof of the Poincaré inequality on $C_c^\infty(U)$ from the first day of class). Here, however, there are some boundary terms to consider. Continuing with the above calculation, show that the desired inequality holds for all $u \in C^\infty(\bar{U})$: note at some point the Cauchy-with-epsilon inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad a, b > 0, \quad \varepsilon > 0$$

will be helpful. Now, use the density of $C^\infty(\bar{U})$ to show the inequality holds on $H^1(U)$. Here, be sure to carefully explain how $C_c^\infty(U) \ni u_j \to u$ in $H^1(U)$ implies that

$$\int_{\partial U} u_j^2 dS \to \int_{\partial U} u^2 dS.$$

For part (b), follow the proof of the Poincaré inequality on $W^{1,p}(U)$ given in class.

3. (Based on #4 in Section 6.2 of McOwen) Let $\mu \in \mathbb{R}$ be non-zero and consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned}$$

where $U \subset \mathbb{R}^n$ is open and bounded and $f \in L^2(U)$ is given.

(a) Derive the appropriate weak formulation of this problem for $u \in H_0^1(U)$.

(b) Set

$$\lambda_1 := \inf_{u \in H_0^1(U)} \frac{\int_U |Du|^2 dx}{\int_U u^2 dx}.$$

Prove that $\lambda_1 > 0$.

(c) Prove that if $\mu > -\lambda_1$, then the above BVP has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$.

Hint: For part (c), notice that by the definition of λ_1, we have

$$\int_U u^2 dx \leq \frac{1}{\lambda_1} \int_U |Du|^2 dx \quad \forall u \in H_0^1(U),$$

which is simply the Poincaré inequality on $H_0^1(U)$ with the "constant" equal to λ_1^{-1}. Now, follow the method for problem #2 to prove the existence of a unique weak solution for each $\mu > -\lambda_1$. At some point in your argument, you will probably want to try to use the Poincaré inequality on $H_0^1(U)$, at which point you should use the above form with the explicit λ_1 dependence.

4. Let $U \subset \mathbb{R}^n$ be a smooth, bounded, connected open set. Let Γ_1, Γ_2 be two disjoint subsets of ∂U of positive $(n-1)-$dimensional measure such that $\Gamma_1 \cup \Gamma_2 = \partial U$. (For example, in $\mathbb{R}^2$ U might be an annulus.) Define the set

$$\mathcal{H} := \left\{ \phi \in C^\infty(\bar{U}) : \text{dist}(\text{spt}\phi, \Gamma_1) > 0 \right\},$$

and define the Hilbert space $\widetilde{H}^1(U)$ as the closure of $\mathcal{H}$ in the standard $H^1(U)$ norm.

(a) Prove the following Poincaré inequality for functions in $\widetilde{H}^1(U)$: $\exists C > 0$ such that

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx \quad \forall u \in \widetilde{H}^1(U).$$

(b) Consider the following problem: Given $f \in L^2(U)$, find $u \in \widetilde{H}^1(U)$ such that

$$\int_U Du \cdot D\phi \; dx = \int_U f\phi \; dx \quad \forall \phi \in \widetilde{H}^1(U).$$

Prove the existence of a unique solution of this problem.

(c) Carefully explain what boundary value problem (i.e. PDE <u>and</u> boundary conditions) you solved in the weak sense in part (b)?

Hint: For part (a), I think it is easiest to try a proof by contradiction, similar to that used for #1(b). For part (b), follow the method of problem #2 again. For part (c), assume the weak solution is actually smooth and "undo" the integration by parts derivation of the weak solution. The boundary condition on Γ_2 will come from requiring the boundary terms coming from the integration by parts to vanish, while the boundary condition on Γ_1 should come from thinking through the nature of the trace operator $T : \widetilde{H}^1(U) \to L^2(\Gamma_1)$.