Math 951 – Advanced PDE II
Homework 2 – Solutions!

Spring 2020

1. Let $U \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary, and consider the following Poincaré like inequality: Given any constant $\sigma > 0$, there exists a constant $C > 0$ such that
$$\int_U u^2 dx \leq C \left(\int_U |Du|^2 dx + \sigma \int_{\partial U} |u|^2 dS \right)$$
for all $u \in H^1(U)$, where here $u|_{\partial U}$ is interpreted in the trace sense.

   (a) Provide a direct proof of this fact, using that $C^\infty(\bar{U})$ is dense in $H^1(U)$. To receive full credit, it is not enough to simply prove for smooth functions on $\bar{U}$ and then just say "by density, it holds on $H^1(U)$". You must show the details of this final density argument.

   (b) Provide another proof of this inequality using a proof by contradiction. (*Hint: It may help to look over our proof of Poincaré on $W^{1,p}(U)$ here...*)

Solution: **(a)** Suppose that $u \in C^\infty(\bar{U})$. Since U is bounded, we have for any $\varepsilon > 0$,

$$\begin{aligned}
\int_U u^2 dx &= -\int_U x_j \frac{\partial}{\partial x_j}\left(u^2\right) dx + \int_{\partial U} u^2 x_j \nu_j \, dS \\
&\leq C \left(\int_U |u||Du| dx + \int_{\partial U} u^2 dS \right) \\
&\leq C \left(\|u\|_{L^2(U)} \|Du\|_{L^2(U)} + \|u\|^2_{L^2(\partial U)} \right) \\
&\leq C \left(\varepsilon \|u\|^2_{L^2(U)} + \frac{1}{4\varepsilon} \|Du\|^2_{L^2(U)} + \|u\|^2_{L^2(\partial U)} \right)
\end{aligned}$$

where the last inequality holds by the Cauchy with ε inequality $|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, valid for any $\varepsilon > 0$. Choosing ε so that $C\varepsilon < 1$, we see that

$$\|u\|^2_{L^2(U)} \leq C \left(\|Du\|^2_{L^2(U)} + \|u\|^2_{L^2(\partial U)} \right),$$

for some positive constant $C > 0$, which, by choosing $C > 0$ possibly larger, implies

$$\|u\|^2_{L^2(U)} \leq C \left(\|Du\|^2_{L^2(U)} + \sigma \|u\|^2_{L^2(\partial U)} \right).$$

For the general case $u \in H^1(U)$, recalling that $C^\infty(\bar{U})$ is dense in $H^1(U)$ we can find a sequence $\{u_j\}_{j=1}^\infty \subset C^\infty(\bar{U})$ such that $u_j \to u$ in $H^1(U)$. Then for each j, we have

$$\|u_j\|^2_{L^2(U)} \leq C \left(\|Du_j\|^2_{L^2(U)} + \|u_j\|^2_{L^2(\partial U)} \right)$$

for some constant $C > 0$ independent of j. Taking the limit as $j \to \infty$, we clearly have $\|u_j\|_{L^2(U)} \to \|u\|_{L^2(U)}$ and $\|Du_j\|_{L^2(U)} \to \|Du\|_{L^2(U)}$. Furthermore, by the Trace theorem, we have that $u_j = Tu_j$ on ∂U and, furthermore,

$$\|u_j - T(u)\|_{L^2(\partial U)} \leq C\|u_j - u\|_{H^1(U)}.$$

Hence, $u_j \to T(u)$ in $L^2(\partial U)$ as $j \to \infty$, which completes the proof.

(b) Suppose that no such constant $C > 0$ exists. Then for all $k \in \mathbb{N}$, there exists a non-zero function $v_k \in H^1(U)$ such that

$$\|v_k\|_{L^2(U)} > k\left(\|Dv_k\|_{L^2(U)} + \|T(v_k)\|_{L^2(\partial U)}\right).$$

Defining $w_k := \frac{v_k}{\|v_k\|_{L^2(U)}}$, it follows that $\|w_k\|_{L^2(U)} = 1$ and

$$\|Dw_k\|_{L^2(U)} + \|T(w_k)\|_{L^2(\partial U)} < \frac{1}{k}$$

for all $k \in \mathbb{N}$. In particular, the sequence $\{w_k\}$ is a bounded sequence in $H^1(U)$ and hence there exists a function $w \in L^2(U)$ and a subsequence $\{w_{k_j}\}$ such that $w_{k_j} \to w$ in $L^2(U)$ by Rellich-Kondrachov, and notice that since $\|w_{k_j}\|_{L^2(U)} = 1$ for all j, $\|w\|_{L^2(U)} = 1$.

However, it also follows that $Dw_{k_j} \to 0$ in $L^2(U)$, and hence I claim that w must be locally constant. To see this, simply notice that for all $\phi \in C_c^\infty(U)$

$$\int_U wD\phi \, dx = \lim_{j\to\infty} \int_U w_{k_j} D\phi \, dx = 0$$

from which it follows that $Dw = 0$ in U, and hence w is a.e. equal to a constant function on each connected component of U. Moreover, we also have that $T(w_{k_j}) \to 0$ in $L^2(U)$ and hence, since $T : H^1(U) \to L^2(\partial U)$ is continuous and since clearly $w \in H^1(U)$ and $w_{k_j} \to w$ in $H^1(U)$, we have

$$T(w) = \lim_{j\to\infty} T(w_{k_j}) = 0$$

and hence $w \in H_0^1(U)$. Recalling that w is a.e. equal to a constant function on each connected component of U it follows that $w = 0$ a.e. in U, which contradicts the fact that $\|w\|_{L^2(U)} = 1$.

2. (Based on #5 in Section 6.6 of Evans) Let $\sigma > 0$ be a fixed constant and suppose $U \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary. Given $f \in L^2(\mathbb{R}^n)$, consider Poisson's equation with Robin boundary conditions:

$$\begin{cases} -\Delta u = f \text{ in } U, \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0 \text{ on } \partial U. \end{cases}$$

The goal of this exercise is to verify the existence and uniqueness of a "weak" solution of the above BVP.

(a) We say a function $u \in H^1(U)$ is a weak solution of the given BVP if

$$\int_U Du \cdot D\phi \; dx + \sigma \int_{\partial U} u\phi \; dS = \int_U f\phi \; dx$$

for all $\phi \in H^1(U)$. Justify that this is a reasonable definition of a weak solution for the given BVP by first supposing $u \in C^\infty(\bar{U})$, and multiplying the PDE by an arbitrary $\phi \in C^\infty(\bar{U})$, and integrating over U.

(b) Define the bilinear form $B : H^1(U) \times H^1(U) \to \mathbb{R}$ by

$$B[v_1, v_2] := \int_U Dv_1 \cdot Dv_2 \; dx + \sigma \int_{\partial U} v_1 v_2 \; dS.$$

Show that $B[\cdot, \cdot]$ defines an inner product on $H^1(U)$. (*Hint: Problem # 1 above will be helpful here...*)

(c) Verify that the inner product $B[\cdot, \cdot]$ generates a norm on $H^1(U)$ that is equivalent to the standard one, i.e. show there exists a $C > 1$ such that

$$C^{-1}\|v\|^2_{H^1(U)} \le B[v, v] \le C\|v\|^2_{H^1(U)}$$

for all $v \in H^1(U)$. Show then that the set $H^1(U)$ equipped with the norm $\|\cdot\|_* := \sqrt{B[\cdot,\cdot]}$ is a Hilbert space. (*Hint: For this last part, all you really need to check is that Cauchy sequences in* $(H^1(U), \|\cdot\|_*)$ *converge in* $(H^1(U), \|\cdot\|_*)$.)

(d) Given $f \in L^2(U)$, show that the map $H^1(U) \ni \phi \mapsto \int_U f\phi \, dx$ defines a continuous linear functional on the Hilbert space $(H^1(U), \|\cdot\|_*)$.

(e) Using the Riesz-Representation Theorem, verify that for every $f \in L^2(U)$, there exists a unique weak solution $u \in H^1(U)$ of the given BVP.

Solution: **(a)** Suppose $u \in C^\infty(\bar{U})$ is a smooth solution of the given BVP, and let $\phi \in C^\infty(\bar{U})$ be arbitrary. Multiplying the PDE by ϕ and integrating over U gives

$$\int_U f\phi \; dx = -\int_U \Delta u \; \phi \; dx = \int_U Du \cdot D\phi - \int_{\partial U} \phi Du \cdot \nu \; dS.$$

From the boundary conditions, $Du \cdot \nu = -\sigma u$ on ∂U, which gives

$$\int_U f\phi \, dx = \int_U Du \cdot D\phi + \sigma \int_{\partial U} u\phi dS.$$

Since $C^\infty(\bar{U})$ is dense in $H^1(U)$, this justifies the given notion of a weak solution for this problem.

(b) Clearly $B[\cdot,\cdot]$ defines a symmetric bilinear map on $H^1(U) \times H^1(U)$ and, furthermore, it is clear that $B[u,u] \geq 0$ for all $u \in H^1(U)$. To see this defines an inner product, we must verify that $B[u,u] = 0$ if and only if $u = 0$ in $H^1(U)$. Recalling from Problem # 1 above that there exists a $C > 0$ such that $B[u,u] \geq C\|u\|_{L^2(U)}$ for all $u \in H^1(U)$, it follows that if $B[u,u] = 0$ then $u = 0$ in $L^2(U)$, and hence $u = 0$ in $H^1(U)$. Thus, $B[\cdot,\cdot]$ defines an inner product on $H^1(U)$.

(c) Using the Poincaré inequality in Problem # 1 again, we have by the definition of $B[\cdot,\cdot]$ that

$$\|v\|_{H^1(U)}^2 = \|v\|_{L^2(U)}^2 + \|Dv\|_{L^2(U)}^2 \leq CB[v,v] + \|Dv\|_{L^2(U)}^2 \leq (C+1)B[v,v]$$

for some constant $C > 0$. To verify the other direction, notice that by the Trace Theorem and the definition of the $H^1(U)$ norm we have

$$B[v,v] \leq \int_U |Dv|^2 dx + C\|v\|_{H^1(U)}^2 \leq (C+1)\|v\|_{H^1(U)}^2.$$

It follows that the norm $\|\cdot\|_*$ defines a norm on $H^1(U)$ that is equivalent to the standard one. To see then that $(H^1(U), \|\cdot\|_*)$ is a Hilbert space then, we must verify that it is complete. To this end, let $\{v_j\}$ be a Cauchy sequence in $(H^1(U), \|\cdot\|_*)$ and note, since $\|\cdot\|_* \geq C\|\cdot\|_{H^1(U)}$, it follows that $\{v_j\}$ is Cauchy in $(H^1(U), \|\cdot\|_{H^1(U)})$. Since $H^1(U)$ is complete with respect to its usual norm, it follows that the sequence $\{v_j\}$ must converge in $H^1(U)$. However, the inequality $\|\cdot\|_* \leq C\|\cdot\|_{H^1(U)}$ implies that convergence in $(H^1(U), \|\cdot\|_{H^1(U)})$ implies convergence in $(H^1(U), \|\cdot\|_*)$ and thus, the sequence $\{v_j\}$ must converge in $(H^1(U), \|\cdot\|_*)$. Hence, the space $(H^1(U), \|\cdot\|_*)$ is a complete inner product space, i.e. a Hilbert space.

(d) Using the equivalence of the norm $\|\cdot\|_*$ with the standard $\|\cdot\|_{H^1(U)}$ norm on $H^1(U)$, given $f \in L^2(U)$ it follows that

$$\left|\int_U f\phi \, dx\right| \leq C\|f\|_{L^2(U)}\|\phi\|_{H^1(U)} \leq C\|f\|_{L^2(U)}\|\phi\|_*$$

for all $\phi \in (H^1(U), \|\cdot\|_*)$. Thus, the given mapping defines a bounded linear functional on $(H^1(U), \|\cdot\|_*)$.

(e) By the Riesz-representation theorem (or, equivalently in this case, Lax-Milgram), it follows that for each $f \in L^2(U)$ there exists a unique $u \in (H^1(U), \|\cdot\|_*)$ such that

$$B[u,\phi] = \int_U f\phi \, dx$$

for all $\phi \in (H^1(U), \|\cdot\|_*)$. Using the equivalence of the norms $\|\cdot\|_{H^1(U)}$ and $\|\cdot\|_*$ one last time, it follows that the above function $u \in H^1(U)$ is a weak solution of the given BVP.

3. (Based on #4 in Section 6.2 of McOwen) Let $\mu \in \mathbb{R}$ be non-zero and consider the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u = f \quad & \text{in } \; U \\ u = 0 \quad & \text{on } \; \partial U \end{aligned}$$

where $U \subset \mathbb{R}^n$ is open and bounded and $f \in L^2(U)$ is given.

(a) Derive the appropriate weak formulation of this problem for $u \in H_0^1(U)$.

(b) Set

$$\lambda_1 := \inf_{u \in H_0^1(U)} \frac{\int_U |Du|^2 dx}{\int_U u^2 dx}.$$

Prove that $\lambda_1 > 0$.

(c) Prove that if $\mu > -\lambda_1$, then the above BVP has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$.

Solution: (a) Suppose u is a smooth solution of the given BVP. Given $\phi \in C_c^\infty(U)$, multiplying the PDE by ϕ and integrating by parts implies that u satisfies

$$\int_U Du \cdot D\phi \; dx + \mu \int_U u\phi \; dx = \int_U f\phi \; dx$$

for all $\phi \in C_c^\infty(U)$. Since the left hand side is well defined for all $u, \phi \in H_0^1(U)$, and the right hand side is defined for $f, \phi \in L^2(U)$ the weak formulation is as follows: given $f \in L^2(U)$, find a $u \in H_0^1(U)$ such that

$$\int_U Du \cdot D\phi \; dx + \mu \int_U u\phi \; dx = \int_U f\phi \; dx$$

for all $\phi \in H_0^1(U)$.

(b) The Poincaré inequality on $H_0^1(U)$ implies there exists a constant $C > 0$ such that

$$\int_U u^2 dx \leq C \int_U |Du|^2 dx$$

for all $u \in H_0^1(U)$. Rewriting, it follows that

$$\frac{1}{C} \leq \frac{\int_U |Du|^2 dx}{\int_U u^2 dx}$$

for all $u \in H_0^1(U)$. It follows that $\lambda_1 \geq \frac{1}{C} > 0$.

(c) Define the bilinar form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ by

$$B[u, v] := \int_U Du \cdot Dv \; dx + \mu \int_U uv \; dx.$$

Using Cauchy-Schwartz, we find that

$$|B[u, v]| \leq \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + \mu \|u\|_{L^2(U)} \|v\|_{L^2(U)} \leq (1 + \max(\mu, 0)) \|u\|_{H^1(U)} \|v\|_{H^1(U)}$$

so that B is bounded. To see that B is coercive, first observe that if $\mu > 0$ then we clearly have

$$B[u, u] \geq \min\{1, \mu\} \|u\|_{H^1(U)}^2$$

so that B is coercive for all $\mu > 0$: note all such μ trivially satisfy $\mu + \lambda_1 > 0$. For $\mu \leq 0$, notice the definition of λ_1 and the fact that $\lambda_1 > 0$ implies that

$$\int_U u^2 \; dx \leq \frac{1}{\lambda_1} \int_U |Du|^2 dx$$

for all $u \in H_0^1(U)$. Thus, if $\mu \leq 0$ then for a given $u \in H_0^1(U)$ we have

$$\begin{aligned} B[u, u] &= \int_U |Du|^2 dx + \mu \int_U u^2 dx \\ &\geq \left(1 + \frac{\mu}{\lambda_1}\right) \int_U |Du|^2 dx \quad (\text{since} \;\; \mu \leq 0) \\ &\geq C \left(1 + \frac{\mu}{\lambda_1}\right) \|u\|_{H^1(U)}^2 \end{aligned}$$

for some $C > 0$. It follows that B is coercive for all $\mu \geq -\lambda_1$.

Finally, noting that Cauchy-Schwartz implies

$$H_0^1(U) \ni \phi \mapsto \int_U f\phi \; dx \in \mathbb{R}$$

is a bounded linear functional on $H_0^1(U)$, it follows from the Lax-Milgram theorem that if $\mu > -\lambda_1$, then for each $f \in L^2(U)$ there exists a unique $u \in H_0^1(U)$ such that

$$B[u, \phi] = \int_U f\phi \; dx$$

for all $\phi \in H_0^1(U)$, and hence there exists a unique weak solution of the given BVP.

4. Let $U \subset \mathbb{R}^n$ be a smooth, bounded, connected open set. Let Γ_1, Γ_2 be two disjoint subsets of ∂U of positive $(n-1)-$dimensional measure such that $\Gamma_1 \cup \Gamma_2 = \partial U$. (For example, in $\mathbb{R}^2$ U might be an annulus.) Define the set

$$\mathcal{H} := \left\{ \phi \in C^\infty(\bar{U}) : \text{dist}(\text{spt}\phi, \Gamma_1) > 0 \right\},$$

and define the Hilbert space $\widetilde{H}^1(U)$ as the closure of $\mathcal{H}$ in the standard $H^1(U)$ norm.

(a) Prove the following Poincaré inequality for functions in $\widetilde{H}^1(U)$: $\exists C > 0$ such that

$$\int_U u^2 dx \le C \int_U |Du|^2 dx \quad \forall u \in \widetilde{H}^1(U).$$

(b) Consider the following problem: Given $f \in L^2(U)$, find $u \in \widetilde{H}^1(U)$ such that

$$\int_U Du \cdot D\phi \; dx = \int_U f\phi \; dx \quad \forall \phi \in \widetilde{H}^1(U).$$

Prove the existence of a unique solution of this problem.

(c) Carefully explain what boundary value problem (i.e. PDE <u>and</u> boundary conditions) you solved in the weak sense in part (b)?

Solution: (a) Suppose the stated inequality is false. Then for each $k \in \mathbb{N}$ there exists a non-zero $u_k \in \widetilde{H}^1(U)$ such that

$$\int_U u_k^2 dx > k \int_U |Du_k|^2 dx.$$

Defining $w_k := \frac{u_k}{\|u_k\|_{L^2(U)}}$ it follows that $\{w_k\}$ is a bounded sequence in $\widetilde{H}^1(U)$ and satisfies

$$\int_U |Dw_k|^2 dx < \frac{1}{k}$$

for each $k \in \mathbb{N}$. Since $\widetilde{H}^1 \subset H^1(U)$, there exists a subsequence $\{w_{k_j}\}$ which converges in $L^2(U)$ to some function $w \in L^2(U)$ with $\|w\|_{L^2(U)} = 1$.

Next, I claim that w is weakly differentiable with $Dw = 0$ in U. To see this, notice that since $w_{k_j} \to w$ in $L^2(U)$ we have for all $\phi \in C_c^\infty(U)$

$$\left| \int_U vD\phi \; dx \right| = \lim_{j\to\infty} \left| \int_U Dw_{k_j}\phi \; dx \right| \le \lim_{j\to\infty} \frac{\|\phi\|_{L^2(U)}}{k_j} = 0.$$

Thus, since U is connected it follows that w is constant in U. But, since $\widetilde{H}^1(U)$ is closed it follows that $T(w) = 0$ on Γ_1 and hence $w = 0$ in U. This is a contradiction since we have already shown that $\|w\|_{L^2(U)} = 1$.

(b) By part (a), the functional $(u,v) := \int_U Du \cdot Dv \; dx$ induces a norm on $\widetilde{H}^1(U)$ which is equivalent to the standard $H^1(U)$ norm. In particular, since the map

$$H^1(U) \ni v \mapsto \int_U fv \; dx \in \mathbb{R}$$

is a bounded linear functional on $H^1(U)$, it follows that it is a bounded linear functional on $\widetilde{H}^1(U)$ and hence the Riesz-Representation Theorem implies the existence of a unique $u \in \widetilde{H}^1(U)$ such that

$$\int_U fv \; dx = \int_U Du \cdot Dv \; dx \quad \forall v \in \widetilde{H}^1(U).$$

Thus, for each $f \in L^2(U)$ there exists a unique weak solution of the given problem.

(c) First, notice by continuity of the trace operator that $T(u) = 0$ on Γ_1 for all $u \in \widetilde{H}^1(U)$. Furthermore, supposing that $u \in C^2(\bar{U})$ an application of integration by parts yields

$$\int_U (-\Delta u + f)\, \phi \; dx + \int_{\Gamma_2} \phi \frac{\partial u}{\partial n} dS = 0$$

for every $\phi \in C^\infty(\bar{U})$, say. Setting both integrals to zero independently implies that u any smooth solution u of the problem in part (b) must satisfy the boundary value problem

$$\begin{cases} -\Delta u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_2, \end{cases}$$

i.e. $u \in H^1(U)$ is a weak solution of Poisson's problem with mixed Dirichlet–Neumann boundary conditions.