Math 951 – Advanced PDE II
Homework 3: Due Friday, March 27 at 4pm

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. In this exercise, we consider the solvability of the Laplacian operator equipped with *Neumann* boundary conditions. In the process, we will analyze the Neumann eigenvalues and eigenfunctions for the Laplacian operator on a bounded domain.

   To begin, let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary and consider the problem

   $$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{0.1}$$

   where $f \in L^2(U)$ is given and ν denotes the outer unit normal vector to ∂U. We say that $u \in H^1(U)$ is a weak solution of the above Neumann BVP if

   $$B[u,v] = \int_U fv \, dx \quad \forall v \in H^1(U) \tag{0.2}$$

   where $B : H^1(U) \times H^1(U) \to \mathbb{R}$ is the bilinear form $B[u,v] := \int_U Du \cdot Dv \, dx$.

   (a) Recall that when dealing with Dirichlet boundary conditions we searched for solutions in $H_0^1(U)$, thus imposing the desired boundary condition in the trial space. However, in the above Neumann problem, it is not immediately clear how the boundary condition $\frac{\partial u}{\partial \nu} = 0$ on ∂U is showing up in our problem: after all, we are not imposing it in the trial space $H^1(U)$.
   To reconcile this, assume that $u \in C^2(\bar{U})$ satisfies $B[u,v] = \int_U fv \, dx$ for all $v \in C^2(\bar{U})$, and that u satisfies $-\Delta u = f$ classically in U. Use the weak formulation to show that

   $$\int_{\partial U} v \frac{\partial u}{\partial \nu} dS = 0$$

   for all $v \in C^2(\bar{U})$. From this, conclude that $\frac{\partial u}{\partial \nu} = 0$ on ∂U. That is, although the boundary condition is not directly imposed in the trial space, it must inherently hold from the weak formulation. Boundary conditions that arise from the weak formulation in this way (and are not imposed explicitly by the Hilbert space) are called *natural* boundary conditions.

   (b) We now turn to studying the solvability of (0.1). First, show that a necessary[1] condition for the existence of weak solutions of the BVP (0.1) is that $\int_U f \, dx = 0$.

[1] We will see below that this condition is also a sufficient condition for existence, but that uniqueness fails: solutions are only unique up to additive constants!

(c) Prove that the bilinear form B defined in (0.2) is bounded, but *not* coercive on $H^1(U)$. (*Hint: To show it is not coercive, it is enough to find a nonzero $u \in H^1(U)$ such that $B[u,u] = 0$. Why?*)

(d) Prove that the bilinear form $B_1 : H^1(U) \times H^1(U) \to \mathbb{R}$ defined by

$$B_1[u,v] := B[u,v] + \int_U uv \, dx$$

is bounded and coercive on $H^1(U)$. Conclude then that for each $f \in L^2(U)$ there exists a unique $u \in H^1(U)$ such that

$$B_1[u,v] = \int_U fv \, dx \quad \forall v \in H^1(U). \tag{0.3}$$

(e) Continuing, for each $f \in L^2(U)$ denote the unique function $u \in H^1(U)$ satisfying (0.3) by $u = S_1(f)$. Prove that $S_1 : L^2(U) \to H^1(U)$ is a bounded linear operator.

(f) Prove that $S_1 : L^2(U) \to L^2(U)$ is compact and self-adjoint.

(g) Prove that S_1 has a countably infinite decreasing sequence of non-zero eigenvalues of finite multiplicity such that, when listed with respect to multiplicity,

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \to 0$$

and that $N(S_1 - \lambda_1 I) = \text{span}\{1\}$. Moreover, show that the corresponding eigenfunctions ϕ_k satisfying $S_1\phi_k = \lambda_k\phi_k$ may be chosen to form an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H^1(U)$ with respect to the inner product $B_1[\cdot,\cdot]$.

(h) Conclude that that the Neumann eigenvalues $\{\mu_j\}$ of the operator $-\Delta$ form an increasing sequence such that, when listed with respect to multiplicity,

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \to \infty$$

and that the corresponding eigenfunctions $\psi_k \in H^1(U)$ may be chosen to form an orthonormal basis of $L^2(U)$ and an orthogonal (with respect to inner product $B_1[\cdot,\cdot]$) basis of $H^1(U)$, where we are assuming $-\Delta\psi_j = \mu_j\psi_j$ weakly in U. Furthermore, show that with the above normalizations we have

$$\psi_1 = \frac{1}{|U|^{1/2}},$$

where $|U|$ denotes the Lebesgue measure of the domain U.

(i) Finally, let $f \in L^2(U)$ be such that $\int_U f \, dx = 0$. Prove that any weak solution of the Neumann problem

$$\begin{cases} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial U \end{cases} \tag{0.4}$$

can be expressed as

$$u = a_1 + \sum_{j=2}^{\infty} \frac{\langle f, \psi_j \rangle_{L^2(U)}}{\mu_j} \psi_j \tag{0.5}$$

with a_1 arbitrary and the series converging in $H^1(U)$. Moreover, show that any function of the form (0.5) is in fact a weak solution of (0.4).

Remark: The lack of uniqueness in the above Neumann problem is due to the fact that $\mu_1 = 0$ is an eigenvalue of the Neumann Laplacian with constant eigenfunction. Thus, solutions are unique in the quotient space $\mathcal{H} := H^1(U)/\{\psi_1\} = \{g \in H^1(U) : \int_U g(x)dx = 0\}$. An alternate way to prove existence of solutions when $\int_U f = 0$ then is to prove the bilinear form $B[\cdot,\cdot]$ defined in (0.2) *is* coercive on $\mathcal{H}$, guaranteeing the existence of a unique weak solution in the quotient space $\mathcal{H}$, and then verifying this implies existence of solutions in $H^1(U)$ unique up to additive constants.

2. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $\{\phi_j\}_{j=1}^{\infty}$ denote the Dirichlet eigenfunctions of the operator $-\Delta$ defined on U. That is, for a fixed $j \in \mathbb{N}$, the functions $\phi_j \in H_0^1(U)$ satisfy the BVP

$$\begin{cases} -\Delta \phi_j = \lambda_j \phi_j, & \text{in } U, \\ \phi_j = 0 & \text{on } \partial U, \end{cases}$$

where λ_j is the j^{th} eigenvalue of $-\Delta$ on U. Furthermore, assume that the ϕ_j are normalized so that $\|\phi_j\|_{L^2(U)} = 1$. Fix $k \in \mathbb{N}$ and let $f \in L^2(U)$ be such that $\int_U f\phi_k \, dx \neq 0$. Given a real number $\varepsilon \neq 0$ sufficiently small, show there exists a unique weak solution $u_\varepsilon \in H_0^1(U)$ to the BVP

$$\begin{cases} -\Delta u = (\lambda_k + \varepsilon)u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{0.6}$$

and that this unique solution satisfies the estimate

$$\|u_\varepsilon\|_{L^2(U)} \geq \frac{\left|\int_U f\phi_k \, dx\right|}{|\varepsilon|}.$$

Remark: Note, by the Fredholm Alternative, the problem (0.6) has no solution when $\varepsilon = 0$. This exercise thus analyzes the behavior of the weak solution operator associated to (0.6) in the limit $\varepsilon \to 0$.

NOTE: In the next two exercises, we will extend the interior elliptic regularity theory developed in class to semilinear equations on open, bounded domains. For this, you need to be familiar with the Gagliardo–Nirenberg–Sobolev inequality and its consequences, which is contained in Theorem 5(a) of Section 6.5 in McOwen (alternatively, see Theorems 5.6.1 and 5.6.2 in Evans). The point of these theorems is that if $U \subset \mathbb{R}^n$ is open and bounded and $u \in W^{1,p}(U)$ with $1 \leq p < n$, then u has better than expected integrabilty: in particular, it belongs to $L^q(U)$ for all $1 \leq q < np/(n-p)$. In contrast to the linear case, where Poincaré is sufficient, these "Sobolev embedding theorems" are crucial elements of nonlinear PDE theory.

3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $|f(t)| \leq |t|^3$ for all $t \in \mathbb{R}$ and let $U \subset \mathbb{R}^3$ be an open and bounded set. Prove that if $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE[2]

$$-\Delta u = f(u) \quad \text{in } U$$

then in fact we have $u \in H^2_{\text{loc}}(U)$. In particular, prove that for every $V \Subset U$ we have $u \in H^2(V)$ and that there exists a constant $C > 0$ independent of u such that

$$\|u\|_{H^2(V)} \leq C\|u\|_{H^1(U)} \left(1 + \|u\|^2_{H^1(U)}\right).$$

4. Let $U \subset \mathbb{R}^3$ be open and bounded with smooth boundary and let $f \in C^\infty(\mathbb{R})$ be given. Suppose that $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE

$$-\Delta u = f(u) \quad \text{in } U$$

and that, moreover, there exists a constant $M > 0$ such that $\|u\|_{L^\infty(U)} \leq M$. Prove that in fact one has $u \in C^2(U)$ and hence u is a classical solution of the PDE in the domain U.

5. (EXTRA CREDIT!!!) Consider the operator $K : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ defined by $K := (-\Delta + I)^{-1}$. The aim of this exercise is to prove that although K is a bounded linear operator from $L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ (as guaranteed by the first existence theorem), it is *NOT* a compact operator from $L^2(\mathbb{R}^n)$ into itself[3].

   (a) First, fix $\phi \in C_c^\infty(\mathbb{R}^n)$ and consider a sequence $\{a_j\} \subset \mathbb{R}^n$ such that $|a_j| \to \infty$ as $j \to \infty$. Prove that the sequence of functions $\{\phi_j\}$ defined by $\phi_j(x) := \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ and converges weakly[4] to zero to zero as $j \to \infty$.

[2] By a weak solution, we mean a function $u \in H^1(U)$ such that $\int_U Du \cdot Dv \, dx = \int_U f(u)v \, dx$ for every $v \in H^1(U)$.

[3] Thus, weak solution operators for uniformly elliptic PDE need not be compact if posed on an unbounded domain.

[4] Recall, a sequence $\{f_n\}$ in a Hilbert space H converges weakly to $f \in H$, denoted $f_n \rightharpoonup f$, if $F(f_n) \to F(f)$ as $n \to \infty$ for every $F \in H^*$. In the setting of problem #6(a) then, we have $\phi_j \rightharpoonup 0$ if $\int_U \phi_j v \, dx \to 0$ for every $v \in L^2(\mathbb{R}^n)$.

(b) Next, prove the following general theorem: if H is a Hilbert space and if M is a bounded, linear, compact operator then given any sequence $\{u_k\}$ which converges weakly to u in H, we have $Mu_k \to Mu$ strongly in H. That is, compact operators on Hilbert spaces upgrade weak convergence to strong (norm) convergence. (*Hint: A fundamental result of functional analysis, known as the Banach-Steinhaus Theorem, or principle of uniform boundedness, implies that all weakly convergent sequences in a Banach space are bounded.*)

(c) Prove that, if $K \in \mathcal{L}(L^2(\mathbb{R}^n))$ were compact, then $K\phi_j \to 0$ in $H^1(\mathbb{R}^n)$. (*Hint: Here, use the fact that, by construction, the range of* K *is actually contained in* $H^1(\mathbb{R}^n)$.)

(d) Still assuming that K is compact, use the fact that the bilinear form $B[u,v] := \int_U (Du \cdot Dv + uv)dx$ defined on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ is bounded and the result of part (c) to derive a contradiction.