Math 951 – Advanced PDE II
Homework 3 – Solutions!

Spring 2020

1. In this exercise, we consider the solvability of the Laplacian operator equipped with *Neumann* boundary conditions. In the process, we will analyze the Neumann eigenvalues and eigenfunctions for the Laplacian operator on a bounded domain.

   To begin, let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary and consider the problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial U \end{cases} \tag{0.1}$$

   where $f \in L^2(U)$ is given. We say that $u \in H^1(U)$ is a weak solution of the above Neumann BVP if

$$B[u,v] = \int_U fv\ dx \quad \forall v \in H^1(U) \tag{0.2}$$

   where $B : H^1(U) \times H^1(U) \to \mathbb{R}$ is the bilinear form $B[u,v] := \int_U Du \cdot Dv\ dx$.

   (a) When dealing with Dirichlet boundary conditions, we searched for solutions in $H_0^1(U)$, thus imposing the desired boundary conditions in the trial space. However, in the above Neumann problem, it is not immediately clear how the boundary condition $\frac{\partial u}{\partial \nu} = 0$ on ∂U is showing up in our problem: after all, we are not imposing it in the trial space $H^1(U)$.
   To reconcile this, assume that $u \in C^2(\bar{U})$ satisfies $B[u,v] = \int_U fv\ dx$ for all $v \in C^2(\bar{U})$, and that u satisfies $-\Delta u = f$ classically in U. Use the weak formulation to show that

$$\int_{\partial U} v \frac{\partial u}{\partial \nu} dS = 0$$

   for all $v \in C^2(\bar{U})$. From this, conclude that $\frac{\partial u}{\partial \nu} = 0$ on ∂U. That is, although the boundary condition is not directly imposed in the trial space, it must inherently hold from the weak formulation. Boundary conditions that arise from the weak formulation in this way (and are not imposed explicitly by the Hilbert space) are called *natural* boundary conditions.

   (b) We now turn to studying the solvability of (0.1). First, show that a necessary[1] condition for the existence of solutions of the BVP (0.1) is that $\int_U f\ dx = 0$.

   (c) Prove that the bilinear form B defined in (0.2) is bounded, but *not* coercive. (*Hint: To show it is not coercive, it is enough to find a nonzero $u \in H^1(U)$ such that $B[u,u] = 0$. Why?*)

[1] We will see below, among other things, that this condition is also a sufficient condition for existence, but that uniqueness fails: solutions are only unique up to additive constants!

(d) Prove that the bilinear form $B_1 : H^1(U) \times H^1(U) \to \mathbb{R}$ defined by

$$B_1[u,v] := B[u,v] + \int_U uv\ dx$$

is bounded and coercive on $H^1(U)$. Conclude then that for each $f \in L^2(U)$ there exists a unique $u \in H^1(U)$ such that

$$B_1[u,v] = \int_U fv\ dx \tag{0.3}$$

for all $v \in H^1(U)$.

(e) Continuing, for each $f \in L^2(U)$ denote the unique function $u \in H^1(U)$ satisfying (0.3) by $u = S_1(f)$. Prove that $S_1 : L^2(U) \to H^1(U)$ is a bounded linear operator.

(f) Prove that $S_1 : L^2(U) \to L^2(U)$ is compact and self-adjoint.

(g) Show then that S_1 has a countably infinite decreasing sequence of non-zero eigenvalues of finite multiplicity such that, when listed with respect to multiplicity,

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \to 0$$

and that $N(S_1 - \lambda_1 I) = \mathbb{R}span\{1\}$. Moreover, show that the corresponding eigenfunctions ϕ_k satisfying $S_1\phi_k = \lambda_k\phi_k$ form an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H^1(U)$ with respect to the inner product $B_1[\cdot,\cdot]$.

(h) Conclude that that the Neumann eigenvalues $\{\mu_j\}$ of the operator $-\Delta$ form an increasing sequence such that, when listed with respect to multiplicity,

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \to \infty$$

and that the corresponding eigenfunctions $\psi_k \in H^1(U)$ form an orthonormal basis of $L^2(U)$ and an orthogonal (with respect to inner product $B_1[\cdot,\cdot]$) of $H^1(U)$, where we are assuming $-\Delta\psi_j = \mu_j\psi_j$ weakly in U. Furthermore, show that we can take

$$\psi_1 = \frac{1}{|U|^{1/2}},$$

where $|U|$ denotes the Lebesgue measure of the domain U.

(i) Finally, let $f \in L^2(U)$ be such that $\int_U f\ dx = 0$. Prove that any weak solution of the Neumann problem

$$\begin{cases} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial U \end{cases} \tag{0.4}$$

can be expressed as

$$u = a_1 + \sum_{j=2}^{\infty} \frac{\langle f, \psi_j\rangle_{L^2(U)}}{\mu_j}\psi_j \tag{0.5}$$

with a_1 arbitrary and the series converging in $H^1(U)$. Moreover, show that any function of the form (0.5) is in fact a weak solution of (0.4).

Solutions: **(a)** Suppose $u \in C^2(\bar{U})$ is a weak solution of our PDE (of course, in general, u is not so smooth). Starting from the weak formlation $\int_U Du \cdot Dv\ dx = \int_U fv\ dx$, we integrate by parts to obtain

$$-\int_U v\Delta u\ dx + \int_{\partial U} v\frac{\partial u}{\partial \nu} dS = \int_U vf\ dx$$

for all $v \in C^2(\bar{U})$. Since $u \in C^2(\bar{U})$, u satisfies $-\Delta u = f$ a.e. in U, and hence

$$\int_{\partial U} v\frac{\partial u}{\partial \nu}\ dS = 0$$

for all $v \in C^2(\bar{U})$. It follows that $\frac{\partial u}{\partial \nu} = 0$ on ∂U, which justifies our saying that the Neumann boundary condition holds in the weak sense, because we have shown that if u is smooth enough on $\bar{U}$ then the Neumann condition actually holds classically.

(b) If $u \in H^1(U)$ is a weak solution of the Neumann problem then

$$\int_U Du \cdot Dv\ dx = \int_U fv\ dx$$

for all $v \in H^1(U)$. In particular, taking $v = 1$ (since U is bounded) we get

$$\int_U f\ dx = 0.$$

Thus, if the Neumann problem has a weak solution then it must be the case that $\int_U f\ dx = 0$.

(c) The bilinear form $B[\cdot,\cdot]$ is clearly bounded on $H^1(U)$ since

$$|B[u,v]| \leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} \leq \|u\|_{H^1(U)}\|v\|_{H^1(U)}$$

for all $u, v \in H^1(U)$. However, since $1 \in H^1(U)$ and $B[1,1] = 0$ it follows that $B[\cdot,\cdot]$ can not be coercive on $H^1(U)$, i.e. there does not exist a constant $C > 0$ such that $B[u,u] \geq C\|u\|^2_{H^1(U)}$ for all $u \in H^1(U)$.

(d) Boundedness of B_1 on $H^1(U) \times H^1(U)$ follows as in part (a). Moreover, coercivity is clear since

$$B_1[u,u] = \int_U \left(u^2 + |Du|^2\right) dx = \|u\|^2_{H^1(U)}$$

for all $u \in H^1(U)$. Thus, by the Riesz-Representation Theorem (or Lax-Milgram if you prefer), for every $f \in L^2(U)$ there exists a unique weak solution $u \in H^1_0(U)$.

(e) The operator S_1 is clearly linear since $B[\cdot, v]$ is linear for each $v \in H^1(U)$. Moreover, by definition of S_1 for every $f \in L^2(U)$ we have

$$\|S_1(f)\|^2_{H^1(U)} = B_1[S_1(f), S_1(f)] = \int_U f S_1(f)\ dx \leq \|f\|_{L^2(U)} \|S_1(f)\|_{L^2(U)}$$

and hence it follows that

$$\|S_1(f)\|_{H^1(U)} \leq \|f\|_{L^2(U)}$$

for all $f \in L^2(U)$, i.e. $S_1 : L^2(U) \to H^1(U)$ is a bounded linear operator.

(f) First, we show that $S : L^2(U) \to L^2(U)$ is self adjoint. To this end, fix $f, g \in L^2(U)$ and notice that

$$\langle S_1(f), g\rangle_{L^2(U)} = \langle g, S_1(f)\rangle_{L^2(U)} = B_1[S_1(g), S_1(f)]$$

since $S_1(f) \in H^1(U)$ and $S_1(g)$, by definition, satisfies

$$B_1[S_1(g), v] = \int_U gv\ dx \quad \forall v \in H^1(U).$$

Since B is symmetric, it follows as above that

$$B_1[S_1(g), S_1(f)] = B_1[S_1(f), S_1(g)] = \langle f, S_1(g)\rangle_{L^2(U)}$$

and hence we have $\langle S_1(f), g\rangle_{L^2(U)} = \langle f, S_1(g)\rangle_{L^2(U)}$. Since $f, g \in L^2(U)$ were arbitrary, it follows that S_1 is self-adjoint on $L^2(U)$.
Next, to see that $S_1 : L^2(U) \to L^2(U)$ is compact, let $\{u_k\}$ be a bounded sequence in $L^2(U)$. Then since S_1 is a bounded map into $H^1(U)$ we have

$$\|S_1(u_k)\|_{H^1(U)} \leq \|u_k\|_{L^2(U)}$$

so that the sequence $\{S_1(u_k)\}$ is a bounded sequence in $H^1(U)$. Since $U \subset \mathbb{R}^n$ is open and bounded with a smooth boundary, Rellich-Kondrachov implies that there exists a subsequence $\{S_1(u_{k_j})\}$ which converges in $L^2(U)$, and hence S_1 is a compact operator from $L^2(U)$ into $L^2(U)$.

(g) By part (f) we can apply the spectral theorem for self-adjoint compact operators to the operator $S_1 : L^2(U) \to L^2(U)$ to conclude that to get the existence of a countable number of eigenvalues $\{\lambda_j\}$ corresponding eigenfunctions ϕ_j. Indeed, $\sigma_p(S_1) \neq \emptyset$ since either $\pm\|S_1\| \in \sigma_p(S_1)$. Furthermore, $0 \in \sigma(S_1) \setminus \sigma_p(S_1)$ since if $S_1(\phi) = 0$ then by definition we have

$$0 = B_1[S_1(\phi), v] = \int_U \phi v\ dx$$

for all $v \in H^1(U)$, and hence we must have $\phi = 0$. Thus, 0 can not be an eigenvalue of S_1 and hence the set $\sigma_p(S_1)$ must be countably infinite since, if not, it would be finite and hence the corresponding total eigenspace would be finite-dimensional. However, this can not be since we have shown that $N(S_1 - \lambda_j I)$ is finite dimensional for each $\lambda_j \in \sigma_p(S_1)$ (since $0 \notin \sigma_p(S_1)$) and that, since S_1 is compact and self adjoint, the total eigenspace

$$\bigcup_j N\left(S_1 - \lambda_j I\right)$$

should be dense in $H^1(U)$, which is infinite dimensional. Thus, by the spectral theorem the set $\sigma_p(S_1)$ consists of a countably infinite sequence of non-zero real numbers $\{\lambda_j\}$ which form a sequence converging to zero as $j \to \infty$.

Next, we prove that $\sigma_p(S_1) \subset (0, 1]$. To this end, notice that if $S_1(\phi_j) = \lambda_j \phi_j$ then by the coercivity of B_1 found in part (b) we have

$$\|\phi_j\|^2_{H^1(U)} = B_1[\phi_j, \phi_j] = \frac{1}{\lambda_j} B_1[S_j \phi_j, \phi_j] = \frac{1}{\lambda_j} \int_U \phi_j^2 dx$$

and hence

$$0 < \lambda_j = \frac{\|\phi_j\|^2_{L^2(U)}}{\|\phi_j\|^2_{H^1(U)}} \leq 1$$

as claimed. Thus, we can order the eigenvalues of S_1 as

$$1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \to 0.$$

Next, I claim that $N(S_1 - I) = \text{span}\{1\}$ so that $\lambda_1 = 1$ is a simple eigenvalue of S_1, completing the proof. To see this, notice that $\lambda = 1$ is clearly an eigenvalue of S_1 since for any non-zero $\alpha \in \mathbb{R}$ we have

$$B_1[S_1(\alpha), v] = \alpha \int_U v \; dx = B_1[\alpha, v] \quad \forall v \in H^1(U).$$

and hence $S_1(\alpha) = \alpha$. Thus, $\text{span}\{1\} \subset N(S_1 - I)$. To prove the reverse inclusion, notice that if $S_1(\phi) = \phi$ then by above we have

$$\|\phi\|^2_{L^2(U)} = B_1[\phi, \phi] = B_1[S_1(\phi), \phi] = \|\phi\|^2_{H^1(U)}$$

and hence it must be that $\|D\phi\|_{L^2(U)} = 0$, i.e. ϕ must be a constant function. Thus, we conclude that $N(S_1 - I) = \text{span}\{1\}$ and hence the eigenvalue $\lambda_1 = 1$ is a simple eigenvalue of S_1.

Finally, it follows directly from the spectral theorem that we can choose the $\{\phi_j\}$ to be an orthonormal basis of $L^2(U)$. To see that it forms an orthogonal basis of $H^1(U)$ with respect to the inner product B_1, first notice that for all $j, k \in \mathbb{N}$

$$\lambda_j B_1[\phi_j, \phi_k] = B_1[S_1(\phi_j), \phi_k] = \int_U \phi_j \phi_k \; dx$$

and hence orthogonality with respect to $B_1[\cdot,\cdot]$ follows by orthogonality in $L^2(U)$. To show $\{\phi_j\}$ is a basis of $H^1(U)$, notice that if $u \in H^1(U)$ is such that $B[u,\phi_j] = 0$ for all $j \in \mathbb{N}$, then we must have

$$0 = B[u,\phi_j] = \frac{1}{\lambda_j} B[u, S_1(\phi_j)] = \frac{1}{\lambda_j}\int_U u\phi_j \, dx$$

for all $j \in \mathbb{N}$ and hence, since $\{\phi_j\}$ is a basis in $L^2(U)$ it follows that $u = 0$ in $L^2(U)$. Hence, if

$$M := \overline{\operatorname{span}\{\phi_j : j \in \mathbb{N}\}}^{H^1(U)}$$

it follows that $M^\perp = \{0\}$ so that $M = H^1(U)$, i.e. the set $\{\phi_j\}$ must be a basis for $H^1(U)$.

(h) First, notice that $S_1(\phi) = \lambda\phi$ for some $\lambda \neq 0$ if and only if we have for all $v \in H^1(U)$

$$B[\phi, v] + \int_U \phi v \, dx = B_1[\phi, v] = \frac{1}{\lambda} B_1[S_1(\phi), v] = \frac{1}{\lambda}\int_U \phi v \, dx,$$

which in turn holds if and only if ϕ is a weak solution of the problem

$$\begin{cases} -\Delta u &= \left(\frac{1}{\lambda} - 1\right) u \quad \text{in } U \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial U. \end{cases}$$

Thus, it follows that the Neumann eigenvalues of $-\Delta$ are exactly the values

$$\mu_j := \left(\frac{1}{\lambda_j} - 1\right).$$

In particular, it follows that the Neumann eigenvalues of $-\Delta$ form a countable sequence $\{\mu_j\}$ such that

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \to \infty$$

with corresponding eigenfunctions $\{\phi_j\}$, i.e. the Neumann eigenfunction of $-\Delta$ corresponding to the eigenvalues μ_j is precisely the eigenfunction ϕ_j corresponding to the eigenvalue λ_j of the operator S_1. The proof of this part is complete by noting that the spectral theorem implies the set $\{\phi_j\}$ is an orthonormal basis of $L^2(U)$. As such, the constant function ϕ_1 must be equal to $|U|^{-1/2}$.

(i) Let $f \in L^2(U)$ satisfy $\int_U f \, dx = 0$ and let $u \in H^1(U)$ be a weak solution to the problem

$$\begin{cases} -\Delta u &= f \quad \text{in } U \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial U. \end{cases} \tag{0.6}$$

Since $u \in H^1(U)$ and since $\{\phi_j\}$ is an orthogonal basis of $H^1(U)$ with respect to the inner product $B_1[\cdot,\cdot]$, it follows that we can find constants $\{a_j\}$ and $\{d_j\}$ so that

$$u = \sum_{j=1}^{\infty} a_j \phi_j, \quad \text{and} \quad f = \sum_{j=1}^{\infty} d_j \phi_j$$

with the series for u converging in $H^1(U)$ and the series for f converging in $L^2(U)$. Since u is a weak solution of (0.6) it follows that

$$B[u,v] = \int_U fv \; dx \quad \forall v \in H^1(U)$$

and hence that

$$B_1[u,v] = \int_U (f+u)v \; dx \quad \forall v \in H^1(U). \tag{0.7}$$

Using the above expansion then, we have

$$B_1[u,v] = \sum_{j=1}^{\infty} a_j B_1[\phi_j, v] = \sum_{j=1}^{\infty} \frac{a_j}{\lambda_j} B_1[S_1(\phi_j), v] = \sum_{j=1}^{\infty} \frac{a_j}{\lambda_j} \int_U \phi_j v \; dx$$

and, similarly, that

$$\int_U (f+u)\, v \; dx = \sum_{j=1}^{\infty} (d_j + a_j) \int_U \phi_j v \; dx$$

Therefore, using (0.9) it follows that

$$\sum_{j=1}^{\infty} (a_j - \lambda_j(d_j + a_j)) \int_U \phi_j v \; dx = 0$$

for all $v \in H^1(U)$ and hence for each $j \in \mathbb{N}$ we must have

$$(1-\lambda_j) a_j = \lambda_j d_j.$$

Since $\lambda_1 = 1$ and $d_1 = 0$ it follows that a_1 can be chosen arbitrary. Furthermore, since

$$1 > \lambda_2 \geq \lambda_2 \geq \ldots \to 0$$

it follows that the solution u must take the form

$$u = a_1 + \sum_{j=2}^{\infty} \frac{\lambda_j d_j}{1-\lambda_j} \phi_j = a_1 + \sum_{j=2}^{\infty} \frac{d_j}{\mu_j} \phi_j \tag{0.8}$$

with a_1 arbitrary and the series converging in $H^1(U)$, as claimed.

To complete the proof, we must show that any function of the form (0.8) is in fact a weak solution of (0.6). To this end, let $v \in H^1(U)$ be arbitrary and notice by continuity of the map

$$H^1(U) \ni w \mapsto B[w, v]$$

we find that

$$B[u, v] = a_1 B[1, v] + \sum_{j=1}^{\infty} \frac{d_j}{\mu_j} B[\phi_j, v]$$

Since $B[1, v] = 0$ and for $j \geq 2$

$$\begin{aligned} B[\phi_j, v] &= B_1[\phi_j, v] - \int_U \phi_j v \, dx \\ &= \frac{1}{\lambda_j} B_1[S_1(\phi_j), v] - \int_U \phi_j v \, dx \\ &= \left(\frac{1}{\lambda_j} - 1 \right) \int_U \phi_j v \, dx \\ &= \mu_j \int_U \phi_j v \, dx, \end{aligned}$$

it follows then that

$$B[u, v] = \sum_{j=1}^{\infty} d_j \int_U \phi_j v \, dx = \int_U \left(\sum_{j=1}^{\infty} d_j \phi_j \right) v \, dx = \int_U f v \, dx$$

for all $v \in H^1(U)$. Therefore, u must be a weak solution of (0.6) as claimed.

2. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $\{\phi_j\}_{j=1}^{\infty}$ denote the Dirichlet eigenfunctions of the operator $-\Delta$ defined on U. That is, for a fixed $j \in \mathbb{N}$, the functions $\phi_j \in H_0^1(U)$ satisfy the BVP

$$\begin{cases} -\Delta \phi_j = \lambda_j \phi_j, & \text{in } U, \\ \phi_j = 0 & \text{on } \partial U, \end{cases}$$

where λ_j is the j^{th} eigenvalue of $-\Delta$ on U. Furthermore, assume that the ϕ_j are normalized so that $\|\phi_j\|_{L^2(U)} = 1$. Fix $k \in \mathbb{N}$ and let $f \in L^2(U)$ be such that $\int_U f \phi_k \, dx \neq 0$. Given a real number $\varepsilon \neq 0$ sufficiently small, show there exists a unique weak solution $u_\varepsilon \in H_0^1(U)$ to the BVP

$$\begin{cases} -\Delta u = (\lambda_k + \varepsilon) u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{0.9}$$

and that this unique solution satisfies the estimate

$$\|u_\varepsilon\|_{L^2(U)} \geq \frac{\left| \int_U f \phi_k \, dx \right|}{|\varepsilon|}.$$

Solution: First, note that since $\sigma(-\Delta)$ is a discrete subset of $\mathbb{R}$, for a fixed $k \in \mathbb{N}$ there is an $\varepsilon_0 > 0$ such that $\lambda_k + \varepsilon \in \rho(-\Delta)$ for all $0 < |\varepsilon| < \varepsilon_0$. That is, for ε sufficiently small, the stated BVP has a unique weak solution for each $f \in L^2(U)$.

Denoting this weak solution by u, it follows that

$$\int_U Du \cdot D\phi_k \, dx = (\lambda_k + \varepsilon) \int_U u\phi_k \, dx + \int_U f\phi_k \, dx.$$

Since ϕ_k solves $-\Delta\phi_k = \lambda_k\phi_k$, it follows that $\int_U Du \cdot D\phi \, dx = \lambda_k \int_U u\phi_k \, dx$ so that the above equality implies

$$\varepsilon \int_U u\phi_k \, dx = -\int_U f\phi_k \, dx.$$

Since $\|\phi_k\|_{L^2(U)} = 1$, it follows by Cauchy-Schwarz that

$$\left| \int_U f\phi_k \, dx \right| \leq |\varepsilon| \cdot \|u\|_{L^2(U)}$$

as claimed.

3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $|f(t)| \leq |t|^3$ for all $t \in \mathbb{R}$ and let $U \subset \mathbb{R}^3$ be an open and bounded set. Prove that if $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE[2]

$$-\Delta u = f(u) \quad \text{in } U$$

then in fact we have $u \in H^2_{\mathbb{R}loc}(U)$. In particular, prove that for every $V \Subset U$ we have $u \in H^2(V)$ and that there exists a constant $C > 0$ independent of u such that

$$\|u\|_{H^2(V)} \leq C\|u\|_{H^1(U)} \left(1 + \|u\|^2_{H^1(U)}\right).$$

Solution: Fix $V \Subset U$ and let $W \subset U$ be an open set with smooth boundary such that

$$V \Subset W \Subset U.$$

If $u \in H^1(U)$, then $f(u) \in L^2(W)$ since

$$\int_W |f(u)|^2 dx \leq \int_W |u|^6 dx$$

[2]By a weak solution, we mean a function $u \in H^1(U)$ such that $\int_U Du \cdot Dv \, dx = \int_U f(u)v \, dx$ for every $v \in H^1(U)$.

which is finite since $H^1(W) \subset L^6(W)$ by Sobolev embedding. Thus, by the interior H^2-regularity theory it follows that $u \in H^2(V)$ with

$$\begin{aligned}
\|u\|_{H^2(V)} &\leq C\left(\|f(u)\|_{L^2(W)} + \|u\|_{H^1(W)}\right) \\
&\leq C\left(\|u\|^3_{L^6(W)} + \|u\|_{H^1(W)}\right) \\
&\leq C\|u\|_{H^1(U)}\left(\|u\|^2_{H^1(U)} + 1\right).
\end{aligned}$$

Since $V \Subset U$ was arbitrary, we are done.

4. Let $U \subset \mathbb{R}^3$ be open and bounded with smooth boundary and let $f \in C^\infty(\mathbb{R})$ be given. Suppose that $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE

$$-\Delta u = f(u) \quad \text{ in } U$$

and that, moreover, there exists a constant $M > 0$ such that $\|u\|_{L^\infty(U)} \leq M$. Prove that in fact one has $u \in C^2(U)$ and hence u is a classical solution of the PDE in the domain U.

Solution: First, notice that since $f \in C^2(\mathbb{R})$ and $u \in L^\infty(U)$ we immediately have that $f(u) \in L^2(U)$ and hence, by our H^2-interior regularity result we have that $u \in H^2_{\mathbb{R}loc}(U)$. Now, fix $V \Subset U$ and note for each $v \in H^2(V) \cap H^1_0(V)$ and $i \in \{1, 2, \ldots n\}$ we have

$$\int_V Du \cdot Dv_{x_i} dx = \int_V f(u) v_{x_i} dx$$

which, using that each component of Du belongs to $H^1(V)$, implies

$$\int_V Du_{x_i} \cdot Dv \; dx = \int_V f(u)_{x_i} v \; dx$$

for all $v \in H^2(V) \cap H^1_0(V)$. It follows that u_{x_i} solves the PDE $-\Delta v = f(u)_{x_i}$, with appropriate boundary conditions, weakly in V. Since $f(u) \in L^2(V)$ and $f(u)_{x_j} = f'(u)u_{x_j} \in L^2(V)$, we can apply the interior H^2 regularity theorem to see that $u_{x_i} \in H^2(W)$ for every $W \Subset V$. Since i and$V \Subset U$ were arbitrary, it follows that $u \in H^3_{loc}(U)$. Thus, by the regularity theorem it follows that we can take $u \in C^1(U)$ with $u, Du \in L^\infty(U)$. With this observation, following as above we find that for each $i, j \in \{1, \ldots, k\}$ and $V \Subset U$ the function $u_{x_i x_j}$ solves the PDE $-\Delta v = f(u)_{x_i x_j}$, with appropriate boundary conditions, weakly in V. Since

$$f(u)_{x_i x_j} = f''(u) u_{x_i} u_{x_j} + f'(u) u_{x_i x_j} \in L^2(W)$$

it follows by the interior H^2 regularity theorem that $u_{x_i x_j} \in H^2(W)$ for every $W \Subset V$. As above, it follows that $u \in H^4_{loc}(U)$ so that $u \in C^2(U)$ with $u, Du, D^2u \in L^\infty(U)$, as desired. so that $f \in H^2(W)$ and hence, by the regularity theorem, we have $u \in H^4(V)$.

5. (Suggested) Consider the multiplication operator $A : L^2(0,1) \to L^2(0,1)$ defined by $Af(x) = xf(x)$.

   (a) Show that $A \in \mathcal{L}\left(L^2(0,1)\right)$ and that $\sigma_p(A) = \emptyset$.

   (b) Show that $\mathbb{C} \setminus [0,1] \subset \rho(A)$.

   (c) Show that if $\lambda \in [0,1]$, then non-zero constants are not in the range of $A - \lambda I$. Conclude that $\sigma(A) = [0,1]$.

   Solution: (a) Clearly we have

$$\|A(f)\|^2_{L^2(0,1)} = \int_0^1 (xf(x))^2\,dx \leq \int_0^1 f^2 dx$$

   so that $A \in \mathcal{L}(L^2(0,1))$ with $\|A\| \leq 1$. To show there are no eigenvalues, suppose $f \in L^2(0,1)$ satisfies

$$Af = \lambda f$$

   in $L^2(0,1)$ for some $\lambda \in \mathbb{C}$. This implies that $(x-\lambda)f(x) = 0$ for a.e. $x \in (0,1)$ which clearly is only satisfied if $f(x) = 0$ for a.e. $x \in (0,1)$. Thus, for every $\lambda \in \mathbb{C}$ the kernel of the opeartor $A - \lambda I$ is trivial. It immediately follows that $\sigma_p(A) = \emptyset$.

   (b) Let $\lambda \in \mathbb{C} \setminus [0,1]$ and note that the function $(x-\lambda)^{-1}$ is well-defined and belongs to $L^\infty(0,1)$. Defining $B_\lambda : L^2(0,1) \to L^2(0,1)$ by $Bf(x) = (x-\lambda)^{-1}$, it follows that $B_\lambda \in \mathcal{L}(L^2(0,1))$ and that

$$B_\lambda\left((A-\lambda I)f\right)(x) = f(x) = (A-\lambda I)\left(B_\lambda f\right)(x)$$

   for every $f \in L^2(0,1)$. Therefore, $A - \lambda I$ is bijective with $(A-\lambda I)^{-1} = B_\lambda \in \mathcal{L}(L^2(0,1))$, from which it immediately follows that $\lambda \in \rho(A)$.

   (c) A constant $g \in \mathbb{R}$ belongs to the range of $A - \lambda I$ if there exists a $f \in L^2(0,1)$ such that $Af(x) = g$ in $L^2(0,1)$. By the definition of A, the only candidate for f is $f(x) = g(x-\lambda)^{-1}$, which does not belong to $L^2(0,1)$ if $\lambda \in [0,1]$. Thus, $A - \lambda I$ is not surjective for $\lambda \in [0,1]$, and hence $[0,1] \subset \sigma(A)$. By part (b) above, it follows that $[0,1] = \sigma(A)$, as claimed.

6. (EXTRA CREDIT!!!) Consider the operator $K : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ defined by $K := (-\Delta + I)^{-1}$. The aim of this exercise is to prove that although K is a bounded linear operator from $L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ (as guaranteed by the first existence theorem), it is *NOT* a compact operator from $L^2(\mathbb{R}^n)$ into itself[3].

   (a) First, fix $\phi \in C_c^\infty(\mathbb{R}^n)$ and consider a sequence $\{a_j\} \subset \mathbb{R}^n$ such that $|a_j| \to \infty$ as $j \to \infty$. Prove that the sequence of functions $\{\phi_j\}$ defined by $\phi_j(x) := \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ and converges weakly[4] to zero to zero as $j \to \infty$.

[3]Thus, weak solution operators for uniformly elliptic PDE need not be compact if posed on an unbounded domain.

[4]Recall, a sequence $\{f_n\}$ in a Hilbert space H converges weakly to $f \in H$, denoted $f_n \rightharpoonup f$, if $F(f_n) \to F(f)$ as $n \to \infty$ for every $F \in H^*$. In the setting of problem #6(a) then, we have $\phi_j \rightharpoonup 0$ if $\int_U \phi_j v\,dx \to 0$ for every $v \in L^2(\mathbb{R}^n)$.

(b) Next, prove the following general theorem: if H is a Hilbert space and if M is a bounded, linear, compact operator then given any sequence $\{u_k\}$ which converges weakly to u in H, we have $Mu_k \to Mu$ strongly in H. That is, compact operators on Hilbert spaces upgrade weak convergence to strong (norm) convergence. (*Hint: A fundamental result of functional analysis, known as the Banach-Steinhaus Theorem, or principle of uniform boundedness, implies that all weakly convergent sequences in a Banach space are bounded.*)

(c) Prove that, if $K \in \mathcal{L}(L^2(\mathbb{R}^n))$ were compact, then $K\phi_j \to 0$ in $H^1(\mathbb{R}^n)$. (*Hint: Here, use the fact that, by construction, the range of K is actually contained in $H^1(\mathbb{R}^n)$.*)

(d) Still assuming that K is compact, use the fact that the bilinear form $B[u,v] := \int_U (Du \cdot Dv + uv)dx$ defined on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ is bounded and the result of part (c) to derive a contradiction.

Solution: (a) The sequence $\{\phi_j\}$ is clearly bounded in $L^2(\mathbb{R}^n)$ since $\|\phi_j\|_{L^2(\mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)}$ for all $j \in \mathbb{N}$. To show $\phi_j \rightharpoonup 0$ in $L^2(\mathbb{R}^n)$, first notice that if $v \in C_c^\infty(\mathbb{R}^n)$ is fixed then there exists a $J \in \mathbb{N}$ such that

$$\operatorname{spt}(\phi_j) \cap \operatorname{spt}(v) = \emptyset$$

for all $j \geq J$ and hence

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \phi_j v \; dx = 0 \quad \forall v \in C_c^\infty(\mathbb{R}^n).$$

Now, for $v \in L^2(\mathbb{R}^n)$ let $\varepsilon > 0$ and note we can find a function $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|v - g\|_{L^2(\mathbb{R}^n)} < \varepsilon$. Thus, for all $j \in \mathbb{N}$ we have

$$\left|\int_{\mathbb{R}^n} \phi_j v \; dx\right| \leq \|\phi_j\|_{L^2(\mathbb{R}^n)} \|v - g\|_{L^2(\mathbb{R}^n)} + \left|\int_{\mathbb{R}^n} \phi_j g \; dx\right|$$

from which it follows that

$$\limsup_{j\to\infty} \left|\int_{\mathbb{R}^n} \phi_j v \; dx\right| \leq \|\phi\|_{L^2(\mathbb{R}^n)} \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\int_{\mathbb{R}^n} \phi_j v \; dx = 0$ for all $v \in L^2(\mathbb{R}^n)$, i.e. $\phi_j \rightharpoonup 0$ in $L^2(\mathbb{R}^n)$.

(b) Since $\{u_k\}$ is assumed to be weakly convergent, it is a bounded sequence in H by the Banach-Steinhaus theorem (or the principle of uniform boundedness). Thus, since the operator $M : H \to H$ is compact, given any subsequence $\{u_{k_j}\}$ there exists a subsequence $\{u_{k_{j_l}}\}$ and a $w \in H$ such that $Mu_{k_{j_l}} \to w$ in M. Next, I claim that $w = Mu$ where u is the weak limit of $\{u_{k_{j_l}}\}$ in H.

To see this, first notice that $Mu_{k_{j_l}} \rightharpoonup w$ in H since, for all $F \in H^*$,

$$\left|F\left(Mu_{k_{j_l}} - w\right)\right| \leq C\|Mu_{k_{j_l}} - w\| \to 0$$

as $j \to \infty$. But, we also have that $Mu_k \rightharpoonup Mu$ since, for all $v \in H$,

$$\langle Mu_k, v\rangle = \langle u_k, M^*v\rangle \to \langle u, M^*v\rangle = \langle Mu, v\rangle .$$

By uniqueness of weak limits (CHECK!) it follows that $Mu = w$ and hence $Mu_{k_{j_l}} \to Mu$. Thus, every subsequence of Mu_k has a subsequence which converges to Mu, and hence it follows that $Mu_k \to Mu$ as claimed.

(c) By parts (a) and (b), if K were compact then the sequence $\{\psi_j\}$ defined by $\psi_j := K\phi_j$ converges strongly to zero in $L^2(\mathbb{R}^n)$. Moreover, notice that $\psi_j \in H^1(\mathbb{R}^n)$ by construction and, furthermore, for each $j \in \mathbb{N}$ the function ψ_j satisfies

$$\int_{\mathbb{R}^n} (D\psi_j \cdot Dv + \psi_j v)\, dx = \int_{\mathbb{R}^n} \phi_j v\ dx \quad \forall v \in H_0^1(U).$$

I claim that choosing $v = \psi_j$ and taking limits as $j \to \infty$ implies that $D\psi_j \to 0$ in $L^2(\mathbb{R}^n)$. To see this, notice

$$\int_{\mathbb{R}^n} \psi_j^2\ dx \to 0$$

as $j \to \infty$ by above and, since $\{\phi_j\}$ is a bounded in $L^2(\mathbb{R}^n)$,

$$\left|\int_{\mathbb{R}^n} \phi_j\psi_j\ dx\right| \leq \|\phi_j\|_{L^2(\mathbb{R}^n)}\|\psi_j\|_{L^2(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. Therefore, it follows that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n} |D\psi_j|^2 dx = \lim_{j\to\infty}\int_{\mathbb{R}^n} \left(\phi_j\psi_j - \psi_j^2\right) dx = 0$$

and hence $D\psi_j \to 0$ in $L^2(\mathbb{R}^n)$. Together then this verifies that $K\phi_j \to 0$ strongly in $H^1(\mathbb{R}^n)$, as claimed.

(d) Finally, since $\psi_j = K\phi_j$ it follows for each $j \in \mathbb{N}$ that

$$B[\psi_j, w] = \int_U \phi_j w\ dx \quad \forall w \in H_0^1(\mathbb{R}^n).$$

Choosing $w = \phi_j$ then, and using the boundedness of $B[\cdot,\cdot]$ it follows that

$$\|\phi_j\|^2_{L^2(\mathbb{R}^n)} = B[\psi_j, \phi_j] \leq C\|\phi_j\|_{H^1(\mathbb{R}^n)}\|\psi_j\|_{H^1(\mathbb{R}^n)}$$

and hence, for all $j \in \mathbb{N}$.

$$\|\psi_j\|_{H^1(\mathbb{R}^n)} \geq \frac{\|\phi_j\|_{L^2(\mathbb{R}^n)}}{C\|\phi_j\|_{H^1(\mathbb{R}^n)}} = \frac{\|\phi\|_{L^2(\mathbb{R}^n)}}{C\|\phi\|_{H^1(\mathbb{R}^n)}} > 0.$$

However, this contradicts the fact that $\psi_j \to 0$ in $H^1(\mathbb{R}^n)$ by part (c). Thus, our assumption that the operator $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact must be false.