Math 951 – Advanced PDE II
Homework 3 – Hints!

Spring 2020

1. A few comments on this problem:

   - The point of this problem is to get you to go through your notes again and adapt all the theory we developed for uniformly elliptic PDE on bounded domains with *Dirichlet* boundary conditions to the setting with *Neumann* boundary conditions. Honestly, the majority of the solutions to this problem are contained in your notes already, you just have to find how (or if) some of the proofs need to be modified to allow for the new set of boundary conditions.
   - At first, this looks like a VERY long exercise (and it is!). But, the reason it is so long is that I have tried very hard to outline each main step needed to complete the program. Many of these steps have very short solutions and follow closely the analogous steps carried out in your lecture notes in our study of the analogous problem with Dirichlet boundary conditions.

2. Let $U \subset \mathbb{R}^n$ be open and bounded, and let $\{\phi_j\}_{j=1}^{\infty}$ denote the Dirichlet eigenfunctions of the operator $-\Delta$ defined on U. That is, for a fixed $j \in \mathbb{N}$, the functions $\phi_j \in H_0^1(U)$ satisfy the BVP

$$\begin{cases} -\Delta\phi_j = \lambda_j\phi_j, & \text{in } U, \\ \phi_j = 0 & \text{on } \partial U, \end{cases}$$

where λ_j is the j^{th} eigenvalue of $-\Delta$ on U. Furthermore, assume that the ϕ_j are normalized so that $\|\phi_j\|_{L^2(U)} = 1$. Fix $k \in \mathbb{N}$ and let $f \in L^2(U)$ be such that $\int_U f\phi_k \, dx \neq 0$. Given a real number $\varepsilon \neq 0$ sufficiently small, show there exists a unique weak solution $u_\varepsilon \in H_0^1(U)$ to the BVP

$$\begin{cases} -\Delta u = (\lambda_k + \varepsilon)u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{0.1}$$

and that this unique solution satisfies the estimate

$$\|u_\varepsilon\|_{L^2(U)} \geq \frac{\left|\int_U f\phi_k \, dx\right|}{|\varepsilon|}.$$

Hint: At some point, you will need to use the defining fact that ϕ_k satisfies $-\Delta\phi_k = \lambda_k\phi_k$ weakly, i.e. that

$$\int_U DvD\phi_k \, dx = \lambda_k \int_U v\phi_k \, dx$$

for all $v \in H_0^1(U)$.

3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition $|f(t)| \leq |t|^3$ for all $t \in \mathbb{R}$ and let $U \subset \mathbb{R}^3$ be an open and bounded set. Prove that if $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE[1]

$$-\Delta u = f(u) \quad \text{ in } U$$

then in fact we have $u \in H^2_{\mathrm{loc}}(U)$. In particular, prove that for every $V \Subset U$ we have $u \in H^2(V)$ and that there exists a constant $C > 0$ independent of u such that

$$\|u\|_{H^2(V)} \leq C\|u\|_{H^1(U)} \left(1 + \|u\|^2_{H^1(U)}\right).$$

Hint: Here, you will need to use a Sobolev embedding theorem known as the Gagliardo-Nirenberg-Sobolev (GNS) inequality. This states that if $U \subset \mathbb{R}^n$ is open and bounded and if $k < \frac{n}{p}$, we have the inequality

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}, \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

for all $u \in W^{k,p}(U)$. In particular, GNS implies the embedding $W^{k,p}(U) \subset L^q(U)$ is continuous. The main point of this problem is to use the GNS inequality to show that show that $u \in H^1(U)$ implies the "inhomogeneous" term $f(u)$ belongs to $L^2(U)$. Now, try to apply the H^2 interior regularity theorem from class.

4. Let $U \subset \mathbb{R}^3$ be open and bounded with smooth boundary and let $f \in C^\infty(\mathbb{R})$ be given. Suppose that $u \in H^1(U)$ is a weak solution of the semilinear elliptic PDE

$$-\Delta u = f(u) \quad \text{ in } U$$

and that, moreover, there exists a constant $M > 0$ such that $\|u\|_{L^\infty(U)} \leq M$. Prove that in fact one has $u \in C^2(U)$ and hence u is a classical solution of the PDE in the domain U.

Hint: Notice that for $U \in \mathbb{R}^3$ open and bounded with smooth boundary, $H^2(U)$ is contained in $C(U) \cap L^\infty(U)$. First then, figure out how large k needs to be to ensure that $u \in H^k(U)$ implies $u \in C^2(U)$, then use a strategy similar to that of the previous problem to prove that u in fact has this many derivatives in $L^2(U)$.

Extended Hint: First show that $f(u) \in L^2(U)$, so that $u \in H^2(V)$ for all $V \Subset U$. Then, for a given $V \Subset U$ and for each $i = 1, 2, \ldots, n$, find a PDE that the function u_{x_i} satisfies weakly in V, and use the interior regularity result again to conclude that $u_{x_i} \in H^2(W)$ for every $W \Subset V$, and then conclude that $u \in H^3(\Omega)$ for every $\Omega \Subset U$. Continue this process to a point that you can conclude that $u \in C^2(U)$.

[1] By a weak solution, we mean a function $u \in H^1(U)$ such that $\int_U Du \cdot Dv \; dx = \int_U f(u)v \; dx$ for every $v \in H^1(U)$.

6. (EXTRA CREDIT!!!) Consider the operator $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by $K := (-\Delta + I)^{-1}$. The aim of this exercise is to prove that although K is a bounded linear operator on $L^2(\mathbb{R}^n)$ (as guaranteed by the first existence theorem), it is *not* a compact operator from $L^2(\mathbb{R}^n)$ into itself.

   (a) First, fix $\phi \in C_c^\infty(\mathbb{R}^n)$ and consider a sequence $\{a_j\} \subset \mathbb{R}^n$ such that $|a_j| \to \infty$ as $j \to \infty$. Prove that the sequence of functions $\{\phi_j\}$ defined by $\phi_j(x) := \phi(x - a_j)$ is bounded in $L^2(\mathbb{R}^n)$ and converges weakly[2] to zero as $j \to \infty$.

   (b) Next, prove the following general theorem: if H is a Hilbert space and if M is a bounded, linear, compact operator then given any sequence $\{u_k\}$ which converges weakly to u in H, we have $Mu_k \to Mu$ strongly in H. That is, compact operators on Hilbert spaces upgrade weak convergence to strong (norm) convergence. (*Hint: A fundamental result of functional analysis, known as the Banach-Steinhaus Theorem, or principle of uniform boundedness, implies that all weakly convergent sequences in a Banach space are bounded.*)

   (c) Prove that, if $K \in \mathcal{L}(L^2(\mathbb{R}^n))$ were compact, then $K\phi_j \to 0$ in $H^1(\mathbb{R}^n)$. (*Hint: Here, use the fact that, by construction, the range of K is actually contained in $H^1(\mathbb{R}^n)$.*)

   (d) Still assuming that K is compact, use the fact that the bilinear form $B[u,v] := \int_U (Du \cdot Dv + uv)dx$ defined on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ is bounded and the result of part (c) to derive a contradiction.

   Hint: (a) First, show that $\int_U \phi_j v \, dx \to 0$ as $j \to \infty$ for every $v \in C_c^\infty(\mathbb{R}^n)$. Then, use a density argument to show that $\int_U \phi_j v \, dx \to 0$ as $j \to \infty$ for every $v \in L^2(\mathbb{R}^n)$.

   (b) Use the fact that a sequence $\{x_k\}$ converges to x in a metric space X if and only if every subsequence of $\{x_k\}$ has a subsequence that converges to x in X. To apply this in the current situation, start with an arbitrary subsequence of $\{u_k\}$, say $\{u_{k_j}\}$, and prove there exists a further subsequence $\{u_{k_{j_l}}\}$ such that $Mu_{k_{j_l}} \to Mu$ in H. Since this holds for any arbitrary subsequence of $\{u_k\}$, it follows that $Mu_k \to Mu$ in H. Also, at some point, you will need to use the fact that weak limits are unique. Note: You do NOT need to verify either of the above claims (regarding convergent subsequences and uniqueness of weak limits), although doing so will earn you extra kudos points!

   (c) Show first that $\psi_j := K\phi_j$ converges strongly to zero in $L^2(\mathbb{R}^n)$. It remains to show that the sequence $D\psi_j$ converge strongly to zero in $L^2(\mathbb{R}^n$ as well. To this end,

[2] Recall, a sequence $\{f_n\}$ in a Hilbert space H converges weakly to $f \in H$, denoted $f_n \rightharpoonup f$, if $F(f_n) \to F(f)$ as $n \to \infty$ for every $F \in H^*$. In the setting of problem #6(a) then, we have $\phi_j \rightharpoonup 0$ if $\int_U \phi_j v \, dx \to 0$ for every $v \in L^2(\mathbb{R}^n)$.

show that the functions ψ_j satisfy

$$\int_{\mathbb{R}^n} (D\psi_j \cdot Dv + \psi_j v)\, dx = \int_{\mathbb{R}^n} \phi_j v \ dx$$

for every $v \in H^1(\mathbb{R}^n)$. Show that choosing $v = \psi_j$ and taking limits as $j \to \infty$ above implies that $D\psi_j \to 0$ in $L^2(\mathbb{R}^n)$ as $j \to \infty$.

(d) Show that since $\psi_j = K\phi_j$ we have

$$B[\psi_j, w] = \int_{\mathbb{R}^n} \phi_j w \ dx$$

for every $w \in H^1(\mathbb{R}^n)$, where B is the bilinaer form on $H^1(\mathbb{R}^n)$ associated to the operator $-\Delta + I$. Use the boundedness of B on $H^1(\mathbb{R}^n)$ to contradict that $\psi_j \to 0$ in $H^1(\mathbb{R}^n)$ as $j \to \infty$.