Math 951 – Advanced PDE II
Homework 4 – Solutions!

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. Let $U \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary and consider the following generalized linear diffusion equation:

$$\begin{cases} u_t = -\Delta^2 u, & \text{in } U \\ u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial U \\ u = u_0, & \text{on } U \times \{t = 0\} \end{cases}$$

where $u_0 \in L^2(U)$ is given. Using methods analogous to those of Problem # 3 of HW2, it is possible to show that the differential operator[1] $L = \Delta^2$ has a countably infinite number of positive eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of finite multiplicity that, when listed with respect to multiplicity, can be listed as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty.$$

Furthermore, the associated eigenfunctions $\{\phi_j\}_{j=1}^\infty$ in form a orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^2(U)$ with respect to the inner product $\langle v_1, v_2 \rangle_* := \int_U \Delta v_1 \Delta v_2 dx$. With these preparations in mind, the goal of this exercise is to show that

$$u(t) = \sum_{j=1}^{\infty} \langle u_0, \phi_j \rangle_{L^2(U)} e^{-\lambda_j t} \phi_j, \tag{0.1}$$

is the unique weak solution of the above IVBVP.

(a) We say $u : [0, \infty) \to L^2(U)$ is a weak solution of the above IVBVP if

(A) $u \in C([0, \infty); L^2(U))$ with $u(0) = u_0$.

(B) $u \in C((0, \infty); H_0^2(U))$.

(C) We have

$$\frac{d}{dt} \langle u(t), v \rangle_{L^2(U)} = - \langle u(t), v \rangle_* \tag{0.2}$$

for all $t \in (0, \infty)$ and for all $v \in H_0^2(U)$.

Prove that the function defined in equation (0.1) is indeed a weak solution of the given problem.

[1] Considered here as a closed densely defined operator on $L^2(U)$ with form domain $H_0^2(U)$.

(b) Prove that weak solutions of the given IVBVP are unique.

(c) (Suggested) Verify the claims about the structure of the eigenvalues and eigenfunctions of the operator L discussed above.

Solution: **(a)** For each $k \in \mathbb{N}$, define the function $u_k : [0, \infty) \to L^2(U)$ by

$$u_k(t) := \sum_{j=1}^{k} a_j e^{-\lambda_j t} \phi_j, \quad a_j := \langle u_0, \phi_j \rangle_{L^2(U)},$$

in particular noticing that $u_k \in C([0, \infty); L^2(U))$ for each $k \in \mathbb{N}$. We want to show that the sequence u_k converges to a weak solution of the given problem. To this end, notice for $K, L \in \mathbb{N}$ with $K > L$ and all $t > 0$ we have

$$\begin{aligned}
\|u_K(t) - u_L(t)\|_{L^2(U)}^2 &= \left\| \sum_{j=L+1}^{K} a_j e^{-\lambda_j t} \phi_j \right\|_{L^2(U)}^2 \\
&= \sum_{j=L+1}^{K} a_j^2 e^{-2\lambda_j t} \\
&\leq \sum_{j=L+1}^{\infty} a_j^2,
\end{aligned}$$

where the above is justified by the facts that $\{\phi_j\}_{j=1}^\infty$ is an ONB of $L^2(U)$ and that $\lambda_j > 0$ for all $j \in \mathbb{N}$. Since $u_0 \in L^2(U)$, it follows that $\sum_{j=1}^\infty a_j^2 < \infty$ so that the above calculation shows that $\|u_K(t) - u_L(t)\|_{L^2(U)} \to 0$ as $K, L \to \infty$ uniformly for $t > 0$. Hence, $\{u_k(t)\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(U)$ for each $t > 0$ and hence, defining the function $u : [0, \infty) \to L^2(U)$ by

$$u(t) := \lim_{k \to \infty} u_k(t),$$

it follows that $u \in C([0, \infty); L^2(U))$ with $u(0) = u_0$, verifying part (A).

Next, we check that $u(t) \in H_0^2(U)$ for each $t > 0$. To this end, notice that since $\phi_j \in H_0^2(U)$ for each $j \in \mathbb{N}$, we have $u_k(t) \in H_0^2(U)$ for each $j \in \mathbb{N}$. Let $S : L^2(U) \to H_0^2(U)$ denote the weak solution operator associated with the bilinear form $B : H_0^2(U) \times H_0^2(U) \to \mathbb{R}$ defined by

$$B[v, w] := \int_U \Delta v \Delta w \, dx,$$

i.e. for each $f \in L^2(U)$ we have

$$B[S(f), v] = \int_U f v \, dx \quad \forall v \in H_0^2(U).$$

Further, $B[\cdot,\cdot]$ induces an inner product on $H_0^2(U)$ equivalent to the standard one (Check!). By the definition of the ϕ_j we have that

$$S(\phi_j) = \frac{1}{\lambda_j}\phi_j$$

for each $j \in \mathbb{N}$ and, moreover, we have

$$\|S(f)\|_{H^2(U)} \leq C\|f\|_{L^2(U)}, \quad \forall f \in L^2(U)$$

for some constant $C > 0$. Thus, for all $K > L$ and $t > 0$ we have

$$\begin{aligned}
\|u_K(t) - u_L(t)\|_{H^2(U)}^2 &= \left\| \sum_{j=L+1}^{K} a_j e^{-\lambda_j t}\phi_j \right\|_{H^2(U)}^2 \\
&= \left\| S\left(\sum_{j=L+1}^{K} a_j \lambda_j e^{-\lambda_j t}\phi_j \right) \right\|_{H^2(U)}^2 \\
&\leq C \left\| \sum_{j=L+1}^{K} a_j \lambda_j e^{-\lambda_j t}\phi_j \right\|_{H^2(U)}^2 \\
&\leq C \sum_{j=L+1}^{K} a_j^2 \left(\lambda_j e^{-\lambda_j t}\right)^2.
\end{aligned}$$

Notice for all $\lambda > 0$ we have the inequality

$$\left|\lambda e^{-\lambda t}\right| \leq t^{-1} \sup_{a>0} a e^{-a} \leq C t^{-1}$$

for some constant $C > 0$. For any given $\tau > 0$, it follows that for $t > \tau$ we have

$$\|u_K(t) - u_L(t)\|_{H^2(U)}^2 \leq \frac{C}{\tau^2} \sum_{j=L+1}^{\infty} a_j^2,$$

which goes to zero uniformly in t for $t > \tau$ as $K, L \to \infty$. It follows then that

$$u \in C([\tau,\infty); H_0^2(U)), \quad \forall \tau > 0$$

which, in particular, verifies (B).

Finally, to verify (C), notice that $\sum_{j=1}^{\infty} a_j e^{-\lambda_j t}\phi_j$ converge in $H_0^2(U)$ for each fixed $t > 0$ and that the bilinear form B introduced above is continuous in the first slot.

For a fixed $\tau > 0$ and $t_2 \geq t_1 \geq \tau$, $v \in H_0^2(U)$, we have

$$\begin{aligned}\int_{t_1}^{t_2} B\left[u(t), v\right] dt &= \int_{t_1}^{t_2} \left(\sum_{j=1}^{\infty} a_j e^{-\lambda_j t} B\left[\phi_j, v\right] \right) dt \\ &= \int_{t_1}^{t_2} \left(\sum_{j=1}^{\infty} a_j \lambda_j e^{-\lambda_j t} \left\langle \phi_j, v \right\rangle_{L^2(U)} \right) dt.\end{aligned}$$

Now, for each $k \in \mathbb{N}$ we define the function $f_k : [\tau, \infty) \to \mathbb{R}$ by

$$f_k(t) := \sum_{j=1}^{k} a_j \lambda_j e^{-\lambda_j t} \left\langle \phi_j, v \right\rangle_{L^2(U)}.$$

Then f_k is continuous for each $k \in \mathbb{N}$ and, furthermore, for each $K > L$ we have

$$\begin{aligned}|f_K(t) - f_L(t)| &\leq \sup_{\lambda > 0} \left| \lambda e^{-\lambda t} \right| \sum_{j=L+1}^{K} |a_j| \cdot |\langle \phi_j, v \rangle| \\ &\leq \frac{C}{\tau} \left(\sum_{j=L+1}^{\infty} a_j^2 \right)^{1/2} \left(\sum_{j=L+1}^{\infty} |\langle \phi_j, v \rangle|^2 \right)^{1/2},\end{aligned}$$

which goes to zero as $K, L \to \infty$ uniformly in t for $t > \tau$. Thus, for each $v \in H_0^2(U)$ we have

$$\begin{aligned}\int_{t_1}^{t_2} B\left[u(t), v\right] dt &= \sum_{j=1}^{\infty} a_j \left\langle \phi_j, v \right\rangle_{L^2(U)} \int_{t_1}^{t_2} \lambda_j e^{-\lambda_j t} dt \\ &= -\sum_{j=1}^{\infty} a_j \left(e^{-\lambda_j t_2} - e^{-\lambda_j t_1} \right) \left\langle \phi_j, v \right\rangle_{L^2(U)} \\ &= -\left(\left\langle u(t_2), v \right\rangle_{L^2(U)} - \left\langle u(t_1), v \right\rangle_{L^2(U)} \right).\end{aligned}$$

Since $u \in C([\tau, \infty); H_0^2(U))$, it follows that the map $t \mapsto \langle u(t), v \rangle$ is differentiable on (τ, ∞) with

$$\frac{d}{dt} \left\langle u(t), v \right\rangle_{L^2(U)} = -B\left[u(t), v\right].$$

Since $\tau > 0$ was arbitrary, this verifies the existence of a weak solution of the given IVBVP.

(b) To verify uniqueness, let u be a weak solution (guaranteed to exist by part (a)) and notice that since $\{\phi_j\}_{j=1}^{\infty}$ is complete in $L^2(U)$ we can represent for each $t \geq 0$ the function $u(t)$ by

$$u(t) = \sum_{k=1}^{\infty} b_k(t) \phi_k,$$

where $b_k(t) := \langle u(t), \phi_k \rangle_{L^2(U)}$ for each $k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, b_k is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, and satisfies $b_k(0) = a_k$. Moreover, for each $t > 0$ and $k \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{d}{dt} b_k(t) &= -B[u(t), \phi_k] \\
&= -B[\phi_k, u(t)] \\
&= -\lambda_k B[S(\phi_k), u(t)] \\
&= -\lambda_k \int_U \phi_k u(t) dx \\
&= -\lambda_k b_k(t),
\end{aligned}$$

so that

$$b_k(t) = b_k(0) e^{-\lambda_k t} = a_k e^{-\lambda_k t}.$$

This verifies the uniqueness of the weak solution of the given IVBVP.

(c) I think I'll skip this part.... follows from similar arguments done in a previous HW.

2. (Galerkin's Method for Elliptic BVP[2]) Suppose $U \subset \mathbb{R}^n$ is open and bounded and consider the Poisson equation

$$\begin{cases} -\Delta u = f \text{ in } U \\ \quad u = 0 \text{ on } U, \end{cases}$$

where $f \in L^2(U)$. Furthermore, let $\{\phi_j\}_{j=1}^{\infty} \subset C^{\infty}(\bar{U})$ be the eigenfunctions of $-\Delta$ taken with Dirichlet boundary conditions, chosen to be an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^1(U)$.

(a) Prove that for each $m \in \mathbb{N}$ there exists constants d_m^k such that the function $u_m := \sum_{k=1}^{m} d_m^k \phi_k$ satisfies

$$\int_U Du_m \cdot D\phi_j \; dx = \int_U f \phi_j \; dx, \quad \forall j = 1, 2, \ldots, m.$$

(b) Now, show there exists a subsequence of $\{u_m\}$ which converges weakly[3] in $H_0^1(U)$ to a weak solution u of the above Poisson problem.

[2] Based on Problem 7.4 from Evans

[3] Here, we say a sequence $\{f_j\}$ converges weakly to f in $H_0^1(U)$ if $\lim_{j \to \infty} F(f_j) \to F(f)$ for every bounded linear functional F on $H_0^1(U)$.

Solution: **(a)** Letting $u_m = \sum_{k=1}^m d_m^k \phi_k$, we easily see that the above condition is equivalent to

$$\int_U f\phi_j \; dx = \sum_{k=1}^m d_m^k \int_U D\phi_k \cdot D\phi_j \; dx = d_m^j \int_U |D\phi_j|^2 dx = \lambda_j d_m^j$$

where λ_j is the Dirichlet eigenvalue of $-\Delta$ corresponding to the function ϕ_j. Since the Dirichlet eigenvalues of $-\Delta$ are positive, it follows that we can take

$$d_m^j := \lambda_j^{-1} \int_U f\phi_j \; dx$$

for all $j = 1, 2, \ldots, m$.

(b) First, notice that the sequence $\{Du_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(U)$. Indeed, by definition we see that for each $m \in \mathbb{N}$ we have

$$\begin{aligned}
\|Du_m\|_{L^2(U)}^2 &= \sum_{j=1}^m d_m^j \int_U Du_m \cdot D\phi_j \; dx = \sum_{j=1}^m d_m^j \int_U f\phi_j dx \\
&= \int_U f u_m \; dx \leq \|f\|_{L^2(U)} \|u_m\|_{L^2(U)} \\
&\leq C\|f\|_{L^2(U)} \|Du_m\|_{L^2(U)}
\end{aligned}$$

where in the last inequality we used the Poincaré inequality. Thus, again by Poincaré, the sequence $\{u_m\}$ is bounded in the reflexive Banach space $H_0^1(U)$. By Banach-Alaoglu then, there exists a subsequence $\{u_{m_j}\}$ which converges weakly to a function u in $H_0^1(U)$. I claim that the weak limit u is actually a weak solution of the stated Poisson problem.

To see this, notice that since the map $H_0^1(U) \ni \phi \mapsto \int_U \phi g \; dx$ for a given $g \in H_0^1(U)$ is a bounded linear functionals on $H_0^1(U)$, it follows by weak convergence that for all $k \in \mathbb{N}$ we have

$$\int_U Du \cdot D\phi_k \; dx = \lim_{j\to\infty} \int_U Du_{m_j} \cdot D\phi_k \; dx = \int_U f\phi_k \; dx.$$

Since $\{\phi_j\}$ is a basis for $H_0^1(U)$, it follows by a similar argument then that

$$\int_U Du \cdot Dv \; dx = \int_U fv \; dx$$

for all $v \in H_0^1(U)$. It follows that u is a weak solution of the Poisson problem, as claimed.