# Math 951 – Advanced PDE II
# Homework 5 – Solutions!

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the Lipshitz condition

$$|f(x, z_1) - f(x, z_2)| \leq L_1 |z_1 - z_2|$$

for all $(x, z_1), (x, z_2) \in \Omega \times \mathbb{R}$. Assume also that $f(\cdot, 0) \in L^2(\Omega)$ and consider the nonlinear elliptic BVP

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Prove there exists a unique weak solution $u \in H_0^1(\Omega)$ of the above BVP provided that

$$L_1 < \lambda_1,$$

where $\lambda_1$ is the principle eigenvalue of $-\Delta$ with respect to $H_0^1(\Omega)$. Here, we say $u \in H_0^1(\Omega)$ is a weak solution of the given BVP if

$$\int_U Du \cdot Dv \, dx = \int_U f(x, u) v \, dx$$

for all $v \in H_0^1(\Omega)$.

**Solution:** Throughout this problem, we consider $H_0^1(U)$ as a Banach space equipped with the norm

$$\|u\|_{H_0^1(U)} := \|Du\|_{L^2(U)}.$$

For each fixed $u \in H_0^1(U)$, set $g(x) := f(x, u)$ and consider the linear elliptic problem

$$\begin{cases} -\Delta w = g(x), & \text{in } U \\ w = 0, & \text{on } \partial U. \end{cases} \tag{0.1}$$

By the Lipshitz condition on $f$,

$$|g(x)| \leq |f(x, u) - f(x, 0)| + |f(x, 0)| \leq L_1 |u| + |f(x, 0)|,$$

it follows that $g \in L^2(U)$ so that there exists a unique weak solution $w \in H_0^1(U)$ of (0.1). Define the operator $A : H_0^1(U) \to H_0^1(U)$ by $w = A(u)$ and note that a weak solution of the original nonlinear elliptic BVP corresponds to a fixed point of $A$ on

$H_0^1(U)$. To prove the existence of such a fixed point, first notice that if $w \in H_0^1(U)$ is a weak solution of (0.1) then, by definition, we have by Poincaré's inequality and the variational characterization of $\lambda_1$ that

$$\int_U |Dw|^2 dx = \int_U gw dx \leq \|w\|_{L^2(U)} \|g\|_{L^2(U)} \leq \lambda_1^{-1/2} \|Dw\|_{L^2(U)} \|g\|_{L^2(U)}.$$

Thus,

$$\|A(u)\|_{H_0^1(U)} \leq \lambda_1^{-1/2} \|f(\cdot, u)\|_{L^2(U)}$$

for all $u \in H_0^1(U)$. It follows that for any $u_1, u_2 \in H_0^1(U)$ we have

$$\begin{aligned} \|A(u_1) - A(u_2)\|_{H_0^1(U)} &\leq \lambda_1^{-1/2} \|f(\cdot, u_1) - f(\cdot, u_2)\|_{L^2(U)} \\ &\leq L_1 \lambda_1^{-1/2} \|u_1 - u_2\|_{L^2(U)} \\ &\leq L_1 \lambda_1^{-1} \|u_1 - u_2\|_{H_0^1(U)}. \end{aligned}$$

Thus, if $L_1 < \lambda_1$ the map $A$ is a strict contraction on the space $H_0^1(U)$ so that, by the Banach Fixed Point Theorem, $A$ has a unique fixed point on $H_0^1(U)$.

2. (Based on #4, Section 9.7 from Evans) Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary and consider a parabolic IVBVP of the form

$$\begin{cases} u_t - \Delta u = f \text{ in } U \times (0, \infty) \\ \quad\quad u = 0 \text{ on } \partial U \times [0, \infty) \\ \quad\quad u = g \text{ on } U \times \{t = 0\}, \end{cases}$$

where $g \in L^2(U)$ and $f \in L^\infty(U \times [0, \infty))$.

(a) Suppose $f = 0$ above. Using the eigenfunction expansion of the solution derived in class, show directly that if $u$ is the weak solution of the above IVBVP then

$$\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\lambda_1 t} \|g\|_{L^2(U)} \quad \forall\ t \geq 0,$$

where $\lambda_1 > 0$ is the principle eigenvalue of $-\Delta$ with Dirichlet boundary conditions on $U$.

(b) Now, suppose there exists a $\tau > 0$ such that $f$ is $\tau$-periodic in $t$, i.e. $f(x, t+\tau) = f(x, t)$ for all $(x, t) \in U \times (0, \infty)$. Prove there exists a unique function $g \in L^2(U)$ for which the corresponding weak solution $u$ is $\tau$-periodic in $t$ as well.

**Solution: (a)** Let $\{(\lambda_j, \phi_j)\}$ be the eigenvalue/eigenfunction pairs of the operator $-\Delta$ with Dirichlet boundary conditions and recall from class that we can write the unique weak solution $u$ of the above heat equation as

$$u(t) := \sum_{j=1}^{\infty} d_k(t) e^{-\lambda_j t} \phi_j$$

with the series converging in $L^2(U)$, where $d_k := \langle u(t), \phi_j \rangle$. Then since $0 < \lambda_1 \leq \lambda_j$ for all $j \geq 2$ we have

$$\|u(\cdot,t)\|_{L^2(U)}^2 = \sum_{k=1}^{\infty} e^{-2\lambda_k t} d_k^2 \leq e^{-2\lambda_1 t} \sum_{k=1}^{\infty} d_k^2 = e^{-2\lambda_1 t} \|g\|_{L^2(U)}^2$$

as claimed.

**(b)** For each $g \in L^2(U)$, there exists a unique weak solution $u \in C(0,\infty; L^2(U)) \cap L^2(0,\infty; H_0^1(U))$ of the above parabolic IVBVP. Define the mapping $S : L^2(U) \times (0,\infty) \to L^2(U)$ by $S(g)(t) = u(t)$. Now, set $w = S(g)(t) - S(g)(t+\tau)$ and notice that $w$ satisfies

$$\begin{cases} w_t - \Delta w = 0, & \text{in } U \times (0,\infty) \\ w = 0, & \text{on } \partial U \times (0,\infty) \\ w = g - S(g)(\tau), & \text{on } U \times \{t = 0\}, \end{cases}$$

If $g = S(g)(\tau)$, then by uniqueness we would have $w = 0$ so that $u$ would be $\tau$-periodic in $t$. To prove the existence of such a function $g$, define the map $A : L^2(U) \to L^2(U)$ by

$$A(v) = S(v)(\tau)$$

and note it is enough to prove that $A$ has a unique fixed point in $L^2(U)$. Using the exponential decay bound in the previous exercise, we find for any $v_1, v_2 \in L^2(U)$ that

$$\|A(v_1) - A(v_2)\|_{L^2(U)} \leq e^{-\lambda_1 \tau} \|v_1 - v_2\|_{L^2(U)}$$

so that $A$ is a strict contraction on $L^2(U)$. By the Banach Fixed Point Theorem, there exists a unique $g_0 \in L^2(U)$ such that $A(g_0) = g_0$. Thus, by uniqueness, the function $g_0$ is the unique initial data in $L^2(U)$ such that the weak solution of the given IVBVP is $\tau$-periodic in time.