Math 951 – Advanced PDE II
Homework 6 – Solutions!

Spring 2020

Turn in solutions to all problems. Working together in groups is HIGHLY suggested, although each person from the group must submit their own solutions.

4. Let $V : \mathbb{R}^n \to \mathbb{R}$ be such that $V \in L^\infty(\mathbb{R}^n)$ and $\lim_{|x|\to\infty} V(x) = 0$. The goal of this problem is to analyze the "Schrödinger" eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}, \tag{0.1}$$

posed on $H^1(\mathbb{R}^n)$. For this purpose, define the admissible set

$$\mathcal{A} := \left\{ u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |u|^2 dx = 1 \right\}$$

and consider the minimization problem

$$\mu = \inf_{u\in\mathcal{A}} \int_{\mathbb{R}^n} \left(|Du|^2 + V(x)|u|^2\right) dx \tag{0.2}$$

(a) Show that if $u \in \mathcal{A}$ is a minimizer for (0.2), then u is a weak solution of (0.1) for $\lambda = \mu$.

(b) Suppose $\{u_k\}_{k=1}^\infty$ is a sequence in $H^1(\mathbb{R}^n)$ such that u_k converges weakly to a function $u \in H^1(\mathbb{R}^n)$ weakly in $H^1(\mathbb{R}^n)$. Establish the following "compactness" result: under the above hypothesis on V, there exists a subsequence[1] $\{u_{k_j}\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} V(x)|u_{k_j}(x)|^2 dx \to \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx.$$

(*Hint: A fundamental results of functional analysis, known as the Banach-Steinhaus theorem, or principle of uniform boundedness, implies that all weakly convergent sequences in a Banach space are bounded*).

(c) Let V be as above and suppose there exists a function $w \in \mathcal{A}$ such that

$$\int_{\mathbb{R}^n} \left(|Dw|^2 + V(x)|w|^2\right) dx < 0.$$

Show that the minimization problem (0.2) has a minimizer $u \in \mathcal{A}$.

Solution: (a) Define the functionals $F, G : H^1(\mathbb{R}^n) \to \mathbb{R}$ by

$$F(u) = \int_{\mathbb{R}^n} \left(|Du|^2 + V(x)|u|^2\right) dx, \quad G(u) = \int_{\mathbb{R}^n} |u|^2 dx,$$

[1] Actually, one can verify this result *with out* passing to a subsequence. While this is not necessary for the problem, I suggest trying to understand why this is so.

noting in particular that F is well defined since $V \in L^\infty(\mathbb{R}^n)$. By direct calculation, we find that F and G are both Gâteaux differentiable on $H^1(\mathbb{R}^n)$ with

$$dF[u]v = \int_{\mathbb{R}^n} (Du \cdot Dv + V(x)uv)\, dx, \quad dG[u]v = 2\int_{\mathbb{R}^n} uv\ dx$$

for all $v \in H^1(\mathbb{R}^n)$. Furthermore, by Problem #2 in the notes we know that $G : H^1(\mathbb{R}^n) \to \mathbb{R}$ is C^1 and hence, by the Lagrange multiplier theorem, any minimizer of $F|_A$ must satisfy

$$\int_{\mathbb{R}^n} (Du \cdot Dv + V(x)uv)\, dx = \lambda \int_{\mathbb{R}^n} uv\ dx, \quad \forall v \in H^1(\mathbb{R}^n) \tag{0.3}$$

for some $\lambda \in \mathbb{R}$, i.e. u must be a weak solution of

$$-\Delta u + V(x)u = \lambda u$$

on $H^1(\mathbb{R}^n)$. Taking $u = v$ in (0.3), it follows from the definition of μ that

$$\mu = F(u) = \lambda \int_{\mathbb{R}^n} |u|^2 dx = \lambda$$

so that u is a weak solution of (0.1) with $\lambda = \mu$.

(b) By the Banach-Steinhaus theorem, the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $H^1(\mathbb{R}^n)$ so that, in particular, there exists a constant $M > 0$ such that $\|u_k\|_{H^1(\mathbb{R}^n)} \leq M$ for all $k \in \mathbb{N}$ and, consequently, $\|u\|_{H^1(\mathbb{R}^n)} \leq M$. Now, for each $k \in \mathbb{N}$ let

$$\widetilde{V}_k := \int_{\mathbb{R}^n} V(x)|u_k(x)|^2 dx$$

and set $\widetilde{V} := \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx$. We want to show $\lim_{j\to\infty} \widetilde{V}_{k_j} = \widetilde{V}$ for some subsequence $\{\widetilde{V}_{k_j}\}_{j=1}^\infty$ of $\{\widetilde{V}_k\}_{k=1}^\infty$. To this end, let $\varepsilon > 0$ be arbitrary. Since $V(x) \to 0$ as $|x| \to \infty$, there exists a $R = R(\varepsilon) > 0$ such that

$$|V(x)| \leq \frac{\varepsilon}{4M^2}$$

for all $|x| > R$. By Rellich-Kondrachov, $H^1(B(0,R)) \Subset L^2(B(0,R))$ and hence there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ and a function $u_0 \in L^2(B(0,R))$ such that $u_{k_j} \to u_0$ strongly in $L^2(B(0,R))$. Since strong convergence implies weak convergence, and since weak limits are unique, it follows that $u_0 = u$. Thus, there exists a constant $K > 0$ such that

$$\left\| |u_{k_j}|^2 - |u|^2 \right\|_{L^1(B(0,R))} < \frac{\varepsilon}{2\|V\|_{L^\infty}} \quad \forall j > K.$$

For $j > K$ then, we have

$$
\begin{aligned}
\left|\widetilde{V}_{k_j} - \widetilde{V}\right| &\le \int_{\mathbb{R}^n} |V(x)| \left||u_{k_j}|^2 - |u|^2\right| dx \\
&\le \int_{B(0,R)} |V(x)| \left||u_{k_j}|^2 - |u|^2\right| dx \\
&\quad + \int_{|x|>R} |V(x)| \left(|u_{k_j}|^2 + |u|^2\right) \\
&\le \|V\|_{L^\infty(\mathbb{R}^n)} \left\||u_{k_j}|^2 - |u|^2\right\|_{L^1(B(0,R))} \\
&\quad + \frac{\varepsilon}{4M^2}\left(\|u_{k_j}\|^2_{L^2(\mathbb{R}^n)} + \|u\|^2_{L^2(\mathbb{R}^n)}\right) \\
&< \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\widetilde{V}_{k_j} \to \widetilde{V}$ as claimed.

(c) First, notice that for all $u \in \mathcal{A}$ we have

$$F(u) \ge \int_{\mathbb{R}^n} |Du|^2 dx - \|V\|_{L^\infty(\mathbb{R}^n)}$$

so that, in particular, F is bounded below on $\mathcal{A}$. Moreover, the above inequality implies that F is coercive on $\mathcal{A}$ by the following reasoning: if a sequence $\{\phi_k\}_{k=1}^\infty$ in $\mathcal{A}$ is unbounded in $H^1(\mathbb{R}^n)$, it must be that $\|D\phi_k\|_{L^2(\mathbb{R}^n)} \to \infty$ as $k \to \infty$ and hence the above inequality implies that $F(\phi_k) \to \infty$. That is,

$$\lim_{\substack{\|u\|_{H^1(\mathbb{R}^n)} \to \infty \\ u \in \mathcal{A}}} F(u) = \infty,$$

i.e. F is coercive on $\mathcal{A}$.

Now, the fact that F is bounded below on $\mathcal{A}$ allows us to take a minimizing sequence $\{u_k\}_{k=1}^\infty$ in $\mathcal{A}$ such that $F(u_k) \to \mu$. Since F is coercive on $\mathcal{A}$, it follows that $\{u_k\}_{k=1}^\infty$ must be bounded in $H^1(\mathbb{R}^n)$. Since $H^1(\mathbb{R}^n)$ is reflexive, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ and a function $u_0 \in H^1(\mathbb{R}^n)$ such that $u_{k_j} \rightharpoonup u^*$ in $H^1(\mathbb{R}^n)$. By the "compactness" result in part (b) above, upon passing to a further subsequence we can assume that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} V(x)|u_{k_j}|^2 dx = \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx$$

and hence, by the weak lower semicontinuity of the functional

$$H^1(\mathbb{R}^n) \ni u \mapsto \int_{\mathbb{R}^n} |Du|^2 dx \in \mathbb{R},$$

it follows that

$$F(u_0) \le \liminf_{j\to\infty} F(u_{k_j}).$$

In particular, by the definition of μ and the fact that $\{u_{k_j}\}_{j=1}^{\infty}$ is a minimizing sequence for $F|_{\mathcal{A}}$, we find that $F(u_0) \leq \mu$. To conclude that u_0 is a minimizer for $F|_{\mathcal{A}}$, it remains to show that $u_0 \in \mathcal{A}$.

Suppose $u_0 \notin \mathcal{A}$ and note since the functional $H^1(\mathbb{R}^n) \ni u \mapsto \|u\|_{L^2(\mathbb{R}^n)}^2$ is weakly lower semicontinuous we have

$$\|u_0\|_{L^2(\mathbb{R}^n)}^2 \leq \liminf_{j\to\infty} \|u_{k_j}\|_{L^2(\mathbb{R}^n)}^2 = 1,$$

and hence we must have $\|u_0\|_{L^2(\mathbb{R}^n)} < 1$. Further, since $F(u) < 0$ for some $u \in \mathcal{A}$, it follows that $F(u_0) \leq \mu < 0$ so that, in particular, u_0 is not identically zero. Thus, $0 < \|u_0\|_{L^2(\mathbb{R}^n)}^2 < 1$ and hence

$$\tilde{u}_0 := \frac{u_0}{\|u_0\|_{L^2(\mathbb{R}^n)}} \in \mathcal{A}$$

and, recalling that $F(u_0) < 0$,

$$F(\tilde{u}_0) = \frac{F(u_0)}{\|u_0\|_{L^2(\mathbb{R}^n)}^2} < F(u_0) \leq \mu. \tag{0.4}$$

which contradicts the definition of μ. Notice in obtaining the second inequality in (0.4), we are very heavily using the fact that $\mu < 0$. It follows that u_0 must indeed belong to the admissible set $\mathcal{A}$ and hence u_0 is a minimizer of $F|_{\mathcal{A}}$.

5. Continuing the above problem, set $F(u) := \int_{\mathbb{R}^n} \left(|Du|^2 + V(x)|u|^2\right) dx$ and let $\mathcal{A}$ be defined as above. A major question in mathematical quantum mechanics is to identify a class of potentials V for which it is true that $F(w) < 0$ for some $w \in \mathcal{A}$. Clearly, for this to be true the potential must be "negative enough": this intentially vague notion is typically quantified by requiring that $\int_{\mathbb{R}^n} V(x)dx < 0$, in which case we say V is an "attractive" potential. However, in high dimensions, it is not the case that $F(w) < 0$ for some $w \in H^1(\mathbb{R}^n)$ for every attractive potential V. This exercise explores this idea.

   (a) Show that if $n = 1, 2$ and the potential $V \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is attractive, then there exists a $u \in \mathcal{A}$ such that $F(u) < 0$.

   (b) Prove Hardy's inequality: if $n \geq 3$, then

   $$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |Du|^2 dx$$

   for all $u \in H^1(\mathbb{R}^n)$.

   (c) Let $n \geq 3$ and define the potential $V(x) := -\frac{(n-2)^2}{4|x|^2}$ on $\mathbb{R}^n$. Conclude that even though V is an attractive potential, we have $F(u) \geq 0$ for every $u \in H^1(\mathbb{R}^n)$.

Note: This part explicitly shows that for the time-dependent Schrödinger equation

$$iu_t = -\Delta u + V(x)u, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

attractive potentials in $\mathbb{R}^n$ for $n \geq 3$ may not support "bound states", i.e. solutions of the form $u(x,t) = e^{-i\lambda t}\phi(x)$ for some $\lambda \in \mathbb{R}$ and non-trivial $\phi \in H^1(\mathbb{R})$.

Solution: (a) Fix $u \in C_c^\infty(\mathbb{R}^n) \cap \mathcal{A}$ with $u(0) \neq 0$ and define for each $L > 0$ the dilation $u_L(x) := L^{-n/2}u(x/L)$. Then for each $L > 0$ we have $u_L \in C_c^\infty(\mathbb{R}^n) \cap \mathcal{A}$ and

$$F(u_L) = L^{-2}\int_{\mathbb{R}^n} |Du|^2 dx + L^{-n}\int_{\mathbb{R}^n} V(x)|u(x/L)|^2 dx.$$

The key observation here is that, by our assumptions on V, we have

$$\lim_{L\to\infty}\int_{\mathbb{R}^n} V(x)|u(x/L)|^2 dx = |u(0)|^2\int_{\mathbb{R}^n} V(x)dx.$$

Indeed, assuming $\text{spt}(u) \subset B(0,R)$ for some $R > 0$, there exists a constant $C_R > 0$ such that for all $L > 0$ we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} V(x)|u(x/L)|^2 dx - |u(0)|^2\int_{\mathbb{R}^n} V(x)dx\right| &\leq \|V\|_{L^\infty(\mathbb{R}^n)}\int_{B(0,R)} |u(x/L) - u(0)|^2\, dx \\
&= \|V\|_{L^\infty(\mathbb{R}^n)}\left(L^n\int_{B(0,R/L)} |u(z) - u(0)|^2 dz\right) \\
&= C_R\|V\|_{L^\infty(\mathbb{R}^n)}\left(\frac{1}{|B(0,R/L)|}\int_{B(0,R/L)} |u(z) - u(0)|^2 dz\right),
\end{aligned}$$

which clearly converges to zero as $L \to \infty$ by the continuity of u.

With the above identity in mind, notice that if $n = 1$ we have

$$F(u_L) = \frac{1}{L}\left(\frac{1}{L}\int_{\mathbb{R}^n} |Du|^2 dx + \int_{\mathbb{R}^n} V(x)|u(x/L)|^2 dx\right),$$

which is negative for L sufficientyl large by our assumption that V is an attractive potential. Similarly, if $n = 2$ we have

$$F(u_L) = \frac{1}{L^2}\left(\int_{\mathbb{R}^n} |Du|^2 dx + \int_{\mathbb{R}^n} V(x)|u(x/L)|^2 dx\right)$$

so that, by choosing $u(0)$ such that

$$\int_{\mathbb{R}^n} |Du|^2 dx + |u(0)|^2\int_{\mathbb{R}^2} V(x)dx < 0,$$

then $F(u_L) < 0$ for L sufficiently large in this case as well.

(b) Notice that for all $\lambda \in \mathbb{R}$ we have

$$0 \leq \int_{\mathbb{R}^n} \left[|Du|^2 + 2\lambda u Du \cdot \frac{x}{|x|^2} + \lambda^2 \frac{u^2}{|x|^2} \right] dx.$$

Moreover, integrating by parts yields the identity

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{u}{|x|^2} Du \cdot x dx &= - \int_{\mathbb{R}^n} u \nabla \cdot \left(\frac{u}{|x|^2} x \right) dx \\ &= - \int_{\mathbb{R}^n} \left[u^2 \left(D|x|^{-2} \right) \cdot x + \frac{u}{|x|^2} Du \cdot x + n \frac{u^2}{|x|^2} \right] dx. \end{aligned}$$

This may be further simplified by noticing

$$x \cdot D|x|^{-2} = x \cdot \left(\frac{-2x}{|x|^4} \right) = - \frac{2}{|x|^2}$$

and hence, after rearranging,

$$2 \int_{\mathbb{R}^n} \frac{u}{|x|^2} Du \cdot x dx = (2-n) \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx.$$

It follows that for any $\lambda \in \mathbb{R}$ we have the inequality

$$0 \leq \int_{\mathbb{R}^n} |Du|^2 dx + (\lambda(2-n) + \lambda^2) \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx.$$

Finally, notice that the function $f(\lambda) = (\lambda(2-n) + \lambda^2)$ has a negative absolute min value of $-\frac{(n-2)^2}{4}$ corresponding to $\lambda = \frac{n-2}{2}$. Using this distinguished value of λ above yields the inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{R}^n} |Du|^2 dx$$

as claimed. (Note: This bound is sharp!)

(c) Clearly V is an attractive potential since $V(x) < 0$ for all $x \in \mathbb{R}$. Nevertheless, Hardy's inequality implies $F(u) \geq 0$ for all $u \in H^1(\mathbb{R}^n)$), as claimed.