I especially recommend reading sections 1.1-1.3, 2.2, 3.1-3.2.
Please be sure to read this handout by the end of the first week of classes.

– Mat

NOTES ON FUNCTIONAL ANALYSIS FOR MATH 951

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1. Banach Spaces

Summary: A Banach space is a (typically) infinite-dimensional vector space equipped with a norm. For PDE purposes, most of the interesting examples are spaces of functions. Also, most of the spaces of functions we encounter in studying PDE are Banach spaces.

Some examples are given, and basic properties are discussed.

1.1. **loose definition.** A Banach space X is a complete normed linear space. This means

- **" linear"** If $x, y \in X$ and a, b are scalars then $ax + by \in X$. In other words, X is a vector space. We will usually work with real Banach spaces, ie spaces for which the scalars are real numbers.
- **"normed"** There is a function which assigns to an element $x \in X$ a non-negative number $||x||$, called "the norm of x". This function satisfies certain familiar axioms such as the triangle inequality, $||x + y|| \le ||x|| + ||y||$. (The axioms are recalled in Section 1.2)
- **"complete"** Since the space has a norm, it makes sense to talk about convergence: a sequence of elements $x_n \in X$ converges to a limit $x \in X$ if $||x_n - x|| \to 0$ as $n \to \infty$. The completeness property means that every Cauchy sequence has a limit.

1.2. **examples.** Most interesting Banach spaces are infinite dimensional, but there are also finite-dimensional examples, and we include some of them.

Note that in order to describe a Banach space, we have to state not only what the elements of the space are, but also what the norm is.

1. Fix some $1 \le p < \infty$. A finite-dimensional Banach space is given by $X \approx \mathbb{R}^n$, where the norm is defined by

$$||x||_p := (\sum_{i=1}^n |x_i|^p)^{1/p},$$

where $x = (x_1, ..., x_n) \in X$.

Remark: We write $X \approx R^n$ instead of $X = \mathbb{R}^n$ to emphasize that X is determined by not only the set of points that compise it, but also the norm. We usually think of $\mathbb{R}^n$ as having the standard Eucidean norm, and our space X does not have this norm, except when $p = 2$.

2. Again taking $X \approx \mathbb{R}^n$, we get a different Banach space by defining the norm

$$||x|| := \max_i |x_i|.$$

We write this norm as $\|x\|_\infty$, since

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p.$$

Prove this!

Remark: For any finite n, all n-dimensional Banach spaces are topologically equivalent.

3. Take X to be the set of continuous functions on a bounded open subset Ω of some $\mathbb{R}^n$, with the norm $||f|| := \max_{x\in\Omega} |f(x)|$. This space is called $C(\Omega)$.
4. Take X to be the set of all continuously differentiable functions on a bounded open subset Ω of some $\mathbb{R}^n$, with the norm $||f|| := \max_{x\in\Omega} |f(x)| + \max_{x\in\Omega} |Df(x)|$. This space is called $C^1(\Omega)$.
5. Fix some $1 \le p < \infty$, and take X to be the set of all measurable functions $f : \Omega \to \mathbb{R}$, for some open subset $\Omega \subset \mathbb{R}^n$, such that

$$\left(\int |f|^p dx\right)^{1/p} < \infty.$$

The norm of f in X is then defined to be the quantity on the left-hand side of the above inequality. This space is $L^p(\Omega)$.

6. Fix some $1 \le p < \infty$, and take X to be the set of all measurable functions $f : \Omega \to \mathbb{R}$, for some subset $\Omega \subset \mathbb{R}^n$, such that f has a "weak derivative" Df (in some sense that can be made precise) and

$$\left(\int |f|^p + |Df|^p dx\right)^{1/p} < \infty.$$

The norm of f is then defined to be the quantity on the left-hand side of the above inequality. This space is $W^{1,p}(\Omega)$.

This is an example of a Sobolev space. These provide the natural functional analytic setting for many PDE questions, so we will study their properties in great detail.

1.3. **PDE considerations.** In PDE applications, it is important to select the "correct" space X in which to work; what this space is will depend on the details of the problem under consideration. For example, suppose that Ω is a bounded open set, and I want to find a function $u : \Omega \to \mathbb{R}$ satisfying

$$u - \Delta u = f \in L^2(\Omega)$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

If I multiply the equation by u and integrate, integrating by parts in the process, I find that a solution u must satisfy

$$\begin{aligned}\int_\Omega (u^2 + |Du|^2)dx &= \int_\Omega u^2 - u\Delta u dx \\ &= \int_\Omega f u dx \\ &\le \frac{1}{2}\int_\Omega (f^2 + u^2)dx,\end{aligned}$$

using the fact that $2ab \le a^2 + b^2$. By subtracting, we find that

$$\int_\Omega (u^2 + |Du|^2)dx \le C \int f^2 dx$$

for our solution u. This suggests that, if I want to use techniques from functional analysis to solve the equation, I should look for a solution in the space $W^{1,2}(\Omega)$, defined in Example 6 above, since the quantity on the left is just the $W^{1,2}$ norm squared.

1.4. **Some details.** Here we record the precise definition of a norm.

A function $x \mapsto \| x \|$ is a **norm** if it has the following properties:

$$||x|| \geq 0 \quad \forall x \in X, \qquad \text{and } ||x|| = 0 \iff x = 0.$$

$$|| x + y || \leq ||x|| + ||y||;$$

$$||\alpha x|| = |\alpha| \, ||x||;$$

for all scalars α and $x \in X$.

It is not obvious that all of the norms given in the examples above in fact have these properties. Clearly, the triangle inequality is the hardest to check. The statement that this holds for $L^p(\Omega)$ is known as Minkowski's ineqality.

It is also not obvious that the above normed linear spaces are complete, and hence are Banach spaces.

Strictly speaking, an element of the space $L^p(\Omega)$ is not a function, but rather an equivalence class of functions, where we set $f \sim g$ if $f = g$ a.e. That is, we identify all functions which agree up to a set of measure zero.

The same is true for Sobolev spaces.

We will discuss at some length what it means for a function to have weak derivatives.

2. Hilbert Spaces

Summary: A Hilbert space is a particularly well-behaved kind of Banach space, in which the norm is derived from an inner product.

Some examples are given, and basic properties are discussed.

2.1. **definition.** A Hilbert space H is a Banach space in which the norm is derived from an inner product. For concreteness I will define *real* Hilbert spaces; there is a corresponding definition for *complex* Hilbert spaces that is similar in spirit but differs in one or two important details.

An inner product is a function that takes a pair of elements x, y of the space and produces a scalar. The inner product of x and y is usually written

$$(x, y)$$

It should satisfy a number of familiar axioms, for example,

$$(x, y) = (y, x) \qquad (ax + by, z) = a(x, z) + b(y, z)$$

for $a, b \in \mathbb{R}$ and $x, y, z \in H$. Moreover, it should have the property that $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$. Hence

$$||x|| := \sqrt{(x, x)}$$

defines a norm that makes H into a Banach space.

2.2. **examples.** Most interesting Hilbert spaces are infinite dimensional, but there are also finite-dimensional examples, and we include some of them.

Note that in order to describe a Hilbert space, we have to state not only what the elements of the space are, but also what the norm is.

1. A finite-dimensional Hilbert space is given by $\mathbb{R}^n$ with the standard inner product,
$$(v, w) = \sum_{i=1}^{n} v_i w_i, \qquad v = (v_1, ..., v_n), \quad w = (w_1, ..., w_n).$$
This is essentially the only finite-dimensional Hilbert spaces. (Why?)
2. Suppose Ω is a bounded open subset of some Euclidean space $\mathbb{R}^n$. Then $L^2(\Omega)$ is a Hilbert space with the inner product
$$(u, v) = \int_\Omega uv \, dx.$$
3. The Sobolev space $W^{1,2}$ (a special case of the spaces defined in Example 6 in Section 1.2) is a Hilbert space, with the inner product
$$(u, v) = \int_\Omega uv + Du \cdot Dv \, dx.$$

2.3. **basic properties.** Two vectors $x, y \in H$ are said to be *orthogonal* if $(x, y) = 0$. Similarly, a vector $x \in H$ is orthogonal to a subspace $Y \subset H$ if $(x, y) = 0 \forall y \in Y$.

Note that the notion of orthogonality does not make sense in a general Banach space.

3. Dual spaces

Summary: Every Banach space has a dual space, which is again a Banach space. The definition of a dual space is given. An easy example is presented to show how the dual of a Banach space can be concretely represented, and a number of harder and more important examples are stated.

3.1. **Definition.** Whenever X is a Banach space we can define the dual space X^* to be the collection of all bounded linear functionals on X, endowed with the dual norm. This means:

- For our purposes, a **functional** is a function $l : X \to \mathbb{R}$. (If X is a complex Banach space, then we would instead consider $l : X \to \mathbb{C}$.) Such a functional is **linear** if the obvious identity holds:
$$l(ax + by) = al(x) + bl(y), \qquad x, y \in X, \quad a, b \in \mathbb{R}.$$
It is **bounded** if there exists some constant C such that
$$|l(x)| \le C\,||x|| \qquad \forall x \in X. \tag{3.1}$$
- We define the **dual norm** on X^* by
$$||\, l \,||_{X^*} := \sup\{l(x) \mid x \in X, ||x||_X \le 1\}$$
for a linear functional $l \in X^*$. Equivalently, $||l||$ is the smallest value that can be taken for the constant C on the right-hand side of (3.1).

·Def: A Banach space is reflexive if $\underset{\overset{\shortparallel}{(X^*)^*}}{X^{**}} = X$

It is a fact that, for any Banach space X, the dual space is again a Banach space. That is, it is a complete linear space, and moreover the dual norm defined above has all the properties that we require of a norm.

·Def: x_n cvgs. to $x \in X$ "weakly", denoted $x_n \rightharpoonup x$, if $l(x_n) \to l(x)$ $\forall l \in X^*$.

·Thm (Banach-Alaoglu): Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.

3.2. **Representation theorems.** Given any Banach space X, the dual space X^* has the abstract description that we have given above, as a space of linear functionals endowed with a certain norm. It is often of interest to find some more concrete description of X^*. Results of this sort are called representation theorems. We first indicate how this is done by looking at

3.2.1. *an extremely simple example.* Let X be a n-dimensional vector space, which we can identify with $\mathbb{R}^n$. We make X into a Banach space by giving it the norm

$$||(x_1, ..., x_n)|| = |x_1| + ... + |x_n|.$$

We will write $e_1, ..., e_n$ for the standard basis vectors in $X \approx \mathbb{R}^n$.

We claim that linear functionals on X can be identified with vectors $v \in \mathbb{R}^n$.

To see this, note first that, given any $v \in \mathbb{R}^n$, we can use it to define a linear functional l on X as follows:

$$l_v(x) = x \cdot v. \tag{3.2}$$

Conversely, given any linear functional l on X, we claim that there is a v in $\mathbb{R}^n$ such that $l = l_v$ as defined above. In fact v can be written down explicitly: $v := (l(e_1), ..., l(e_n))$.

We check that this works. For any $x = (x_1, ..., x_n) \in X$ we have by linearity

$$\begin{aligned} l(x) &= l(x_1e_1 + ... + x_ne_n) \\ &= x_1l(e_1) + ... + x_nl(e_n) \\ &= x \cdot (l(e_1), ..., l(e_n)) = x \cdot v \end{aligned} \tag{3.3}$$

as required.

Thus we can identify linear functionals on X with vectors $v \in \mathbb{R}^n$.

Once we have done this, we can ask, what is the dual norm on $X^* \approx \mathbb{R}^n$? (Remember, $\mathbb{R}^n$ can be given any number of different norms that make it into a Banach space.)

Given $v \in X^* \approx \mathbb{R}^n$, the dual norm by definition is

$$||v||_{X^*} := \sup\{v \cdot x \mid |x_1| + ... + |x_n| \leq 1\}.$$

It is not hard to see that

$$||v||_{X^*} = \max\{|v_1|, ..., |v_n|\}, \qquad v = (v_1, ..., v_n). \tag{3.4}$$

Thus, starting from the abstract description of X^* as the dual space of a certain Banach space X, we have found a very concrete representation of X^* as an n-dimensional vector space with the norm given by (3.4), and the correspondence between the concrete and the abstract realizations given by (3.3) and (3.2).

3.2.2. *harder examples.* A lot of effort in the early part of this century was devoted to finding concrete and natural ways of representing the dual spaces of certain Banach spaces that frequently appear in analysis. Here are some of the most important results:

- Suppose that $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $\mathbb{R}^n$ with the $p-$norm is precisely $\mathbb{R}^n$ with the q-norm. In the simple example above we have verified this statement for the case $p = 1$.

- Suppose that $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let U be any bounded open subset of $\mathbb{R}^n$. Then the dual of $L^p(U)$ can be identified with $L^q(U)$. In particular, given any bounded linear functional $l : L^p(U) \to \mathbb{R}$, there is a unique $v \in L^q(U)$ such that

$$l(u) = \int_U uv\, dx$$

for all $u \in L^p(U)$. (This fails for $p = \infty, q = 1$. In fact, $L^1 \subset (L^\infty)^*$ by below, but the reverse inclusion fails.) In the other direction, given any $v \in L^q(U)$ the mapping

$$u \mapsto \int_U uv\, dx$$

defines a bounded linear functional on $u \in L^p(U)$ (by Hölder's inequality), and moreover

$$\|v\|_{L^q} = \sup\{\int_U uv\, dx \ : \ u \in L^p(U), \|u\|_{L^p} \leq 1\}.$$

(This also holds for $p = \infty, q = 1$.)
- If H is any Hilbert space, then the dual of H can be identified with H itself. In other words, given any bounded linear functional $l : H \to \mathbb{R}$, there exists a unique element x of H such that $l(y) = (x, y)$, for all $y \in H$. (This is the Riesz Representation Theorem, also known as the Riesz–Fischer Theorem.)
- Note that the dual of $L^2(U)$ is $L^2(U)$. This is a special case of both the above examples.
- The dual space of $L^\infty(U)$ is **not** $L^1(U)$, as mentioned above. In fact, $L^1(U)$ is not the dual space of any Banach space.
- Let $C(U)$ be the Banach space of continuous functions on U, as in example 3 in Section 1.2. Its dual space is the space of signed Radon measures on U. This is a large space which includes not only all L^1 functions but also things like "delta functions". Thus, although L^1 is not a dual space, it is contained in a larger space which is a dual space.