Math 951 Lecture Notes
The Principle Eigenvalue Theorem

Mathew A. Johnson [1]
Department of Mathematics, University of Kansas
matjohn@ku.edu

1 Introduction

There has been several questions concerning the simplicity of the principle eigenvalue for uniformly elliptic operators on bounded domains. In these notes, I aim to present the big idea in at least the symmetric case. For those interested in the non-symmetric case, see Theorem 3 in Section 6.5 in Evans.

To this end, let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, and consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

As we know from class, non-trivial solutions of (1) exist only when λ is an $H_0^1(\Omega)$ eigenvalue of the operator $-\Delta$. Such eigenvalues form an increasing sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \nearrow +\infty$$

and each eigenvalue λ_j has corresponding eigenfunction $\phi_j \in H_0^1(\Omega)$. Above, each eigenvalue is listed with respect to its multiplicity. That is, if λ_j has, say, m linearly independent eigenfunctions, then it is listed exactly m times in the above list.

Theorem 1 (Principle Eigenvalue Theorem)**.** *The principle eigenvalue λ_1 for the operator $-\Delta$ on $H_0^1(\Omega)$ is simple, that is, there exists $\phi_1 \in H_0^1(\Omega)$ such that*

$$\operatorname{Ker}(-\Delta - \lambda_1 I) = \operatorname{span}\{\phi_1\}.$$

Furthermore, the principle eigenfunction ϕ_1 may be chosen to be strictly positive on Ω.

The above result is very powerful, and applies in various other contexts[2]. Its proof is primarily composed of 2 main components:

(1) A variational characterization of the principle eigenvalue λ_1.

[2] In particular, this result continues to be true even for non-symmetric uniformly elliptic differential operators on Ω. See Theorem 3 in Section 6.5.2 in Evans. The proof, however, is considerably more complicated...

(2) The Strong Maximum Principle for uniformly elliptic operators on Ω

Once these are established, the proof proceeds by using the variational formulation to prove that ϕ_1 may be chosen to be either non-negative or non-positive on Ω. Once this is established, the Strong Maximum Principle is then used to conclude that ϕ_1 can not vanish on the interior of Ω. Finally, the simplicity of λ_1 follows by another application of the Strong Maximum Principe.

2 Proof of the Principle Eigenvalue Theorem

Our first step in the proof of Theorem 1 is to establish the following fundamental result.

Theorem 2 (Variational Characterization of λ_1)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, and let λ_1 be the principle eigenvlaue of $-\Delta$ on $H_0^1(\Omega)$ with eigenfunction ϕ_1. Then*

$$\lambda_1 = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |Du|^2 dx}{\|u\|_{L^2(\Omega)}^2}$$

and, further, this minimum is achieved if and only if $u \in \operatorname{span}\{\phi_1\}$.

The proof of Theorem 2 is an exercise listed at the end of these notes. The above is sometimes referred to as the Rayleigh quotient characterization of the principle eigenvalue λ_1 and, in particular, it shows that $\lambda_1^{-1/2}$ is the *sharp* Poincaré constant for the domain Ω. Specifically, we have

$$\int_\Omega |u|^2 dx \leq \frac{1}{\lambda_1} \int_\Omega |Du|^2 dx \ \text{ for all } u \in H_0^1(\Omega)$$

with equality if and only if $u = \gamma\phi_1$ for some constant $\gamma \in \mathbb{R}$.

We now use Theorem 2 to prove that the principle eigenfunction ϕ_1 may be chosen to be either non-negative or non-positive. To this end, we need the following two technical lemmas.

Lemma 1 (The Chain Rule)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and suppose $F \in C^1(\mathbb{R})$ is such that $F' \in L^\infty(\mathbb{R})$. If $u \in H^1(\Omega)$ then $F(u) \in H^1(\Omega)$ with*

$$\partial_{x_j} F(u) = F'(u) u_{x_j}$$

for each $j = 1, 2, \ldots, n$.

Proof. This is an exercise. I suggest either using the global approximation of $H_0^1(\Omega)$ by $H_0^1(\Omega) \cap C^\infty(\Omega)$, or else using molifiers. ☐

Lemma 2 (See Exercise #18 in Chapter 5 of Evans)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is bounded and that $u \in H^1(\Omega)$. Then $|u| \in H^1(\Omega)$ with $D|u| = \operatorname{sign}(u) Du$ a.e. in Ω.*

Proof. Given $u \in H_0^1(\Omega)$ we decompose $|u| = u^+ + u^-$ where

$$u^+ = \max\{u, 0\}, \quad u^- = -\min(u, 0)$$

and note to show $|u| \in H^1(\Omega)$ it is enough to prove that $u^+, u^- \in H^1(\Omega)$. Here, we concentrate on the function u^+. To get in a context where we can use the Chain Rule in Lemma 1, observe that

$$u^+ = \lim_{\epsilon \to 0^+} F_\epsilon(u) \text{ a.e.},$$

where

$$F_\epsilon(z) := \begin{cases} (z^2 + \epsilon^2)^{1/2} - \epsilon, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0 \end{cases}$$

can be easily seen to satisfy $F_\epsilon \in C^1(\mathbb{R})$ with

$$F_\epsilon'(z) \begin{cases} \dfrac{z}{\sqrt{z^2 + \epsilon^2}}, & \text{if } z \geq 0 \\ 0, & \text{if } z < 0. \end{cases}$$

In particular, note $\|F_\epsilon'\|_{L^\infty(\mathbb{R})} = 1$. By the Chain Rule, it follows that for each $\epsilon > 0$ we have $F_\epsilon(u) \in H^1(\Omega)$ with $DF_\epsilon(u) = F_\epsilon'(u)Du$, i.e. for every test function $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega F_\epsilon(u) D\phi \, dx = -\int_{u>0} \left(\frac{uDu}{\sqrt{u^2 + \epsilon^2}} \right) \phi \, dx.$$

The result for u^+ now follows by taking $\epsilon \to 0^+$ and using Dominated Convergence. A similar argument applies to u^-, which completes the proof. □

With the above results in hand, let ϕ_1 be a weak eigenfunction associated to λ_1. By Lemma 2 it follows that $|\phi_1| \in H_0^1(\Omega)$ with $|D|\phi_1|| = |D\phi_1|$. In particular, using Theorem 2 we have

$$\frac{\int_\Omega |D|\phi_1||^2 \, dx}{\||\phi_1|\|_{L^2(\Omega)}^2} = \frac{\int_\Omega |D\phi_1|^2 \, dx}{\|\phi_1\|_{L^2(\Omega)}^2} = \lambda_1,$$

which, by again by Theorem 2, implies that $\phi_1 = \alpha|\phi_1|$ for some constant $\alpha = \pm 1$. In particular, it follows that the principle eigenfunction may be chosen to be either non-negative on Ω or non-positive on Ω.

Our next step in the proof of Theorem 1 is to prove that, in fact, ϕ_1 must be non-zero on the interior of Ω. The main ingredient here is the following.

Theorem 3 (Strong Maximum Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, and suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$-\Delta u \geq 0 \text{ in } \Omega.$$

If u attains its minimum over $\overline{\Omega}$ at an interior point of Ω, then u must be a constant in Ω.

To see this, recall by the Boundary Regularity Theory for uniformly elliptic operators on Ω that any $H_0^1(\Omega)$-eigenfunction associated to $-\Delta$ must belong to $C^\infty(\overline{\Omega})$. Indeed, the Boundary Regularity Theory guarantees that if $f \in H^m(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $u \in H^{m+2}(\Omega)$. If λ_j is an $H_0^1(\Omega)$ eigenvalue of $-\Delta$ with eigenfunction ϕ_j, it then follows from the identity $-\Delta\phi_j = \lambda_j\phi_j$ and Sobolev embedding that $\phi_j \in C^\infty(\overline{\Omega})$, as claimed.

In particular, it follows by the above observation that $|\phi_1| \in C^\infty(\overline{\Omega})$ must be a smooth solution of the PDE

$$-\Delta|\phi_1| = \lambda_1|\phi_1|.$$

Since $\lambda_1 > 0$, the Strong Maximum Principle implies that if $|\phi_1|$ attains its minimum value in Ω then it must be constant: specifically $|\phi_1|$ would have to be identically zero since $|\phi_1| \in H_0^1(\Omega)$. Since $|\phi_1|$ is non-trivial by virtue of being an eigenfunction, it follows that $|\phi_1$ can not achieve its minimum value on the interior of Ω. Since clearly $|\phi_1| \geq 0$ on $\overline{\Omega}$ and since $|\phi_1| = 0$ on $\partial\Omega$, it follows that we must have

$$|\phi_1(x)| > 0 \quad \text{for all } x \in \Omega$$

which implies that the principle eigenfunction must be sign definite on Ω.

To complete the proof of Theorem 1, it remains to show that the simplicity of λ_1. To this end, let ϕ_1 be as above and suppose that $\psi \in H_0^1(\Omega)$ is some other eigenfunction associated to λ_1. By our previous work, we know that $\psi \in C^\infty(\overline{\Omega})$ and $|\psi(x)| > 0$ for all $x \in \Omega$. Fixing $x_0 \in \Omega$ it follows that the function

$$\Phi(x) := \phi_1(x_1)\psi(x) - \psi(x_0)\phi_1(x)$$

is also a smooth eigenfunction associated to λ_1. However, since $\Phi(x_0) = 0$ it follows by the Strong Maximum Principle that $\Phi(x) = 0$ for all $x \in \Omega$, which implies that

$$\psi(x) = \left(\frac{\psi(x_0)}{\phi_1(x_0)}\right)\phi_1(x),$$

i.e. ψ must be a multiple of ϕ_1. This completes the proof of Theorem 1.

3 Exercises

Complete the following exercises.

1. Let $U \subset \mathbb{R}^n$ be open and bounded and consider the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ be the sequence of Dirichlet eigenvalues and $\{\phi_j\}_{j=1}^{\infty}$ the corresponding eigenfunctions in $H_0^1(U)$, i.e. suppose $-\Delta\phi_j = \lambda_j \phi_j$.

(a) Prove the variational characterization of λ_1 provided in Theorem 2.

(b) Continuing, prove that for all $k \in \mathbb{N}$, $k \geq 2$, we have the following variational characterization of the eigenvalue λ_k:

$$\lambda_k = \min_{u \in H_0^1(U) \setminus \{0\} : u \in \Sigma_{k-1}^{\perp}} \frac{\int_U |Du|^2 dx}{\int_U |u|^2 dx}$$

where $\Sigma_{k-1} := \overline{\text{span}\{\phi_1, \phi_2, \ldots, \phi_{k-1}\}}$ and $\perp$ means orthogonal in $H_0^1(U)$. (Note: the above two results can be useful for finding upper bounds on the λ_k, since it gives λ_k as the minimum of something.)

(c) Use the above two results to prove the Dirichlet eigenvalues are "monotone with respect to domain". That is, given two open and bounded sets $U, V \in \mathbb{R}^n$, let $\lambda_k(V)$ and $\lambda_k(U)$ denote the k-th Dirichlet eigenvalues of $-\Delta$ on the domains V and U respectively. Prove that if $V \subset U$, then $\lambda_k(V) \geq \lambda_k(U)$. (Note: This result if FALSE for the Neumann eigenvalues...)

(d) Prove the *Caurant minimax principle* for this problem: that is, prove that for all $k \in \mathbb{N}$ the eigenvalue λ_k can be expressed variationally as

$$\lambda_k = \max_{S \in \Omega_{k-1}} \left(\min_{u \in S^{\perp}} \frac{\int_U |Du|^2 dx}{\int_U |u|^2 dx} \right).$$

Here, Ω_{k-1} denotes the collection of all $(k-1)$-dimensional subspaces of $H_0^1(U)$. (Note: The Courant principle can be useful for finding a *lower* bound on λ_k, since it gives λ_k as the maximum of something.)

2. Prove Lemma 1.

3. Fill in the details concerning the regularity of the eigenfunctions of $-\Delta$. That is, prove that if ϕ_j is a $H_0^1(\Omega)$-eigenfunction associated to $-\Delta$, then $\phi_j \in C^{\infty}(\overline{\Omega})$. For this, it will be helpful to see the General Sobolev Embedding Theorem stated in Theorem 6 in Section 5.7 of Evans, along with the higher-order boundary regularity result in Theorem 5 in Section 6.3 of Evans.

4. Look up and study the proof of the Strong Maximum Principle. Note it relies on two previous results: the Weak Maximum Principle, and Hopf's Lemma... for completeness, you should study these results as well.