Note on Density of $C_c^\infty(U)$ in $W^{k,p}(U)$

As stated in class, although $C_c^\infty(U)$ is dense in $L^p(U)$ for any $1 \leq p < \infty$, it is not in general dense in $W^{k,p}(U)$ for $k > 1$. Indeed, we have the following fact.

Theorem 0.1. *Let $U \subset \mathbb{R}^n$ be a open and bounded, $k \geq 1$, and $1 \leq p \leq \infty$. Then $C_c^\infty(U)$ is not dense in $W^{k,p}(U)$.*

Proof. We give the proof first for $n = 1$. The big idea here is to come up with a continuous linear functional on $W^{k,p}(U)$ that vanishes identically on $C_c^\infty(U)$. To this end, notice for each fixed $u \in C^\infty(\bar{U})$ the mapping

$$L(\phi) = \int_U \sum_{j=0}^{k} \frac{d^j u}{dx^j} \cdot \frac{d^j \phi}{dx^j}\, dx, \quad \phi \in W^{k,p}(U) \tag{0.1}$$

is a continuous linear functional on $W^{k,p}(U)$. Indeed, continuity follows from Hölder's inequality via the estimate

$$|L(\phi)| \leq \sum_{j=0}^{k} \int_U \left|\frac{d^j u}{dx^j}\right| \cdot \left|\frac{d^j \phi}{dx^j}\right| dx \leq \sum_{j=0}^{k} \left\|\frac{d^j u}{dx^j}\right\|_{L^q(U)} \left\|\frac{d^j \phi}{dx^j}\right\|_{L^p(U)} \leq \|u\|_{W^{k,q}(U)} \|\phi\|_{W^{k,p}(U)}$$

where we note the smoothness of u implies that $\|u\|_{W^{k,q}(U)} < \infty$. Moreover, if u in (0.1) is chosen to solve the ODE

$$\sum_{j=0}^{k} (-1)^j \frac{d^{2j} u}{dx^{2j}} = 0 \tag{0.2}$$

on an interval containing $\bar{U}$, then by integration by parts we find that $L(\phi) = 0$ for all $\phi \in C_c^\infty(U)$. Note that the ODE (0.2) has a smooth solution of the form $u(x) = e^{rx}$ provided r be a root of the polynomial equation

$$\sum_{j=0}^{k} (-1)^j r^{2j} = 0; \tag{0.3}$$

recall that complex roots of (0.3) can be associated to real valued solutions of (0.2) via Euler's formula $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

Now, let u be any non-trivial solution of the ODE (0.2). Clearly $u \in W^{k,p}(U) \cup C^\infty(\bar{U})$ and, using this function u in (0.1), we clearly have have

$$L(u) = \sum_{j=0}^{k} \int_U \left(\frac{d^j u}{dx^j}\right)^2 dx > 0.$$

Thus, if $C_c^\infty(U)$ were dense in $W^{k,p}(U)$, there would exist a sequence $\{\phi_j\}_{j=1}^\infty$ of functions in $C_c^\infty(U)$ such that $\phi_j \to u$ in $W^{k,p}(U)$. But then the continuity of L on $W^{k,p}(U)$ would imply that

$$0 = \lim_{j\to\infty} L(\phi_j) = L\left(\lim_{j\to\infty} \phi_j\right) = L(u) > 0,$$

a contradiction.

In higher dimensions, L should be replaced by

$$L(\phi) = \int_U \sum_{|\alpha|\leq k} D^\alpha u \cdot D^\alpha \phi dx, \quad \phi \in W^{k,p}(U) \tag{0.4}$$

where $u \in C^\infty(\bar{U})$ and (0.2) by the PDE equivalent

$$\sum_{|\alpha|\leq k} (-1)^{|\alpha|} D^{2\alpha} u = 0. \tag{0.5}$$

As above, it is readily checked that L is a continuous linear functional on $W^{k,p}(U)$. By first enclosing $\bar{U}$ in a parallelepiped, one can find a non-trivial solution u of (0.5) using the method of separation of variables. Using this u in (0.4), one finds $L(u) > 0$ but that $L(\phi) = 0$ for all $\phi \in C_c^\infty(U)$. It follows that $C_c^\infty(U)$ can not be dense in $W^{k,p}(U)$, as claimed. $\square$

In one dimension, even more can be said: by a straight forward ODE analysis, it can be shown that

$$\left(H_0^1(0,1)\right)^\perp = \text{span}\{e^x, e^{-x}\},$$

where here $\left(H_0^1(0,1)\right)^\perp$ denotes the orthogonal complement of $H_0^1(0,1)$ within $H^1(0,1)$. Thus, every function $f \in H^1(0,1)$ can be written in the form

$$f = f_0 + c_1 e^x + c_2 e^{-x}$$

for some $f_0 \in H_0^1(0,1)$ and $c_1, c_2 \in \mathbb{R}$. (Check!)

The above ideas should be able to be extneded to show that if $U \subset \mathbb{R}^n$ is bounded in ANY direction, then $C_c^\infty(U)$ is not dense in $W^{k,p}(U)$. As a complementary result, we next show that if $U = \mathbb{R}^n$, then such a density result DOES hold.

Theorem 0.2. *For any $1 \leq p < \infty$ and $k \geq 0$, $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. That is, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.*

Proof. Clearly, we have $W_0^{1,p}(\mathbb{R}^n) \subset W^{1,p}(\mathbb{R}^n)$, so that we only need to prove the reverse containment. To this end, let $f \in W^{1,p}(\mathbb{R}^n)$. Let $\eta^\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be the standard mollifier and notice then that $\eta^\varepsilon * f \in C^\infty \cap W^{k,p}(\mathbb{R}^n)$ and, for $|\alpha| \leq k$ we have

$$\partial^\alpha(\eta^\varepsilon * f) = \eta^\varepsilon * (\partial^\alpha f) \to \partial^\alpha f$$

in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. Hence, we have $\eta^\varepsilon * f \to f$ in $W^{k,p}(\mathbb{R}^n)$ as $\varepsilon \to 0$. In particular, given any $\delta > 0$ we can find a function $\psi \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ such that

$$\|f - \psi\|_{W^{k,p}(\mathbb{R}^n)} < \delta.$$

We now show that the smooth function ψ can be approximated by functions in $C_0^\infty(\mathbb{R}^n)$. To this end, let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a smooth cut-off function such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}$$

and define $\phi^R(x) = \phi(x/R)$ and $\psi^R = \phi^R \psi$ for each $R > 0$. Then clearly $\psi^R \in C_c^\infty(\mathbb{R}^n)$ for each $R > 0$ and, by the Leibniz rule,

$$\partial^\alpha \psi^R = \phi^R \partial^\alpha \psi + \frac{1}{R} h^R,$$

for each $|\alpha| \leq k$, where h^R is bounded in L^p uniformly in R. Therefore, by the dominated convergence theorem we have

$$\partial^\alpha \psi^R \to \partial^\alpha \psi$$

in L^p as $R \to \infty$ for each $|\alpha| \leq k$, i.e. we have that $\psi^R \to \psi$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. In particular, we can find an $R^* > 0$ sufficiently large such that

$$\|\psi - \psi^{R^*}\|_{W^{k,p}(\mathbb{R}^n)} < \delta$$

from which it follows that

$$\|f - \psi^{R^*}\|_{W^{k,p}(\mathbb{R}^n)} < 2\delta$$

and hence the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$, which completes the proof. $\square$

Notice that Theorem 0.2 fails in the endpoint $p = \infty$. Indeed, $1 \in W^{k,\infty}(\mathbb{R}^n)$ but $1 \notin W_0^{k,\infty}(\mathbb{R}^n)$ so that $W_0^{k,\infty}(\mathbb{R}^n) \subsetneq W^{k,\infty}(\mathbb{R}^n)$. Can you characterize the closure of $C_c^\infty(\mathbb{R}^n)$ in $W^{k,\infty}(\mathbb{R}^n)$?