Math 951 Lecture Notes
Chapter 4 – Introduction to Fixed Point Methods

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1 Introduction

To this point, we have been using linear functional analytic tools (eg. Riesz Representation Theorem, etc.) to study the existence and properties of solutions to linear PDE. This has largely followed a well developed general theory which proceeded quite methodoligically and has been widely applicable. As we transition to nonlinear PDE theory, it is important to understand that there is essentially no widely developed, overaching theory that applies to all such equations. The closest thing I would way that exists is the famous Cauchy-Kovalyeskaya Theorem, which asserts quite generally the local existence of solutions to systems of partial differential equations equipped with initial conditions on a "noncharacteristic" surface. However, this theorem requires the coefficients of the given PDE system, the initial data, and the surface where the IC is described to all be real analytic. While this is a very severe restriction, it turns out that it can not be removed. For this reason, and many more, the Cauchy-Kovalyeskaya Theorem is of little practical importance (although, it is obviously important from a historical context... hence why it is usually studied in Math 950).

Throughout the remainder of the class, we will be studying a variety of analytical techniques to approach the existence problem in nonlinear PDE theory, each of which is applicable in some cases and not in others. In that sense, the rest of the class will likely not feel as cohesive as the first part, and this is true and, unfortunatley, simply the nature of the subject.

In this chapter, we provide a brief introduction to the use of fixed point methods in the study of nonlinear PDE theory. I will only provide some simple applications of one of the most basic fixed point arguments (the Contraction Mapping Principle). There are, of course, many other more sophisticated fixed point methods available and such methods could be the topic of a final project in the class.

2 Contraction Mapping Principle

We begin our study of nonlinear PDE with the following abstract, yet basic, result.

Theorem 1 (Contraction Mapping Principle)**.** *Let (X,d) be a complete metric space, and assume*

$$A : X \to X$$

is a contraction on X, i.e. there exists a constant $\gamma \in (0,1)$ such that

$$d(A(x), A(y)) \leq \gamma d(x,y)$$

for all $x, y \in X$. Then there exists a unique $x_0 \in X$ such that $A(x_0) = x_0$, i.e. A has a unique fixed point in X.

Proof. The proof is based on a simple iteration scheme. Let $x_1 \in X$ be arbitrary, and inductively define

$$x_{k+1} = A(x_k), \quad k \in \mathbb{N}$$

We claim that the sequence $\{x_k\}$ is Cauchy in X. Indeed, notice if $k \in \mathbb{N}$ then

$$d(A(x_{k+1}), A(x_k)) \leq \gamma d(A(x_k), A(x_{k-1}) \leq \ldots \leq \gamma^k d(x_2, x_1)$$

and hence for arbitrary $k, \ell \in \mathbb{N}$ with $k \geq \ell$ we have

$$d(x_k, x_\ell) = d(A(x_{k-1}), A(x_{\ell-1})) \leq \sum_{j=\ell-1}^{k-2} d(A(x_{j+1}), A(x_j)) \leq d(x_2, x_1) \sum_{j=\ell-1}^{k-2} \gamma^j.$$

Since $\gamma \in (0,1)$ we have $\sum_{j=1}^{\infty} \gamma^j < \infty$ and hence the sequence $\{x_k\}$ is Cauchy in X, as claimed. Since X is complete, it follows three exists a $x_0 \in X$ such that $x_k \to x_0$ in X as $k \to \infty$. Since

$$x_{k+1} = A(x_k)$$

and since A is continuous, taking $k \to \infty$ gives $x_0 = A(x_0)$ so that x_0 is a fixed point of A.

Moreover, if $\tilde{x}$ is any other fixed point of A, note that

$$d(x_0, \tilde{x}) = d(A(x_0), A(\tilde{x})) \leq \gamma d(x_0, \tilde{x}).$$

Since $\gamma \in (0, 1)$, it follows that $\tilde{x} = x_0$, establishing uniqueness of the fixed point. $\square$

The above theorem, sometimes called the Banach Fixed Point Theorem, is incredibly simple yet powerful. It is especially powerful in the context of linear problems, as the next example illustrates.

Example: Let $[a, b] \subset \mathbb{R}$ be a bounded interval and suppose $k : [a, b] \times [a, b] \to \mathbb{R}$ is a continuous function. Given a continuous function $g : [a, b] \to \mathbb{R}$ consider the following *Fredholm integral equation*: find $f \in C([a, b])$ such that

$$f(x) = g(x) + \int_a^b k(x, y) f(y) dy. \tag{1}$$

Such an integral equation is ideally setup for the use of the Contraction Mapping Theorem[2]. Indeed, note if we fix $g \in C([a, b])$ and define the map

$$C([a, b]) \ni f \mapsto T(f)(x) := g(x) + \int_a^b k(x, y) f(y) dy$$

then a continuous solution to (1) is simply a fixed point of the map T. To apply the contraction mapping theorem here, we first note that if $f \in C([a, b])$ then given $x_1, x_2 \in [a, b]$ we have

$$T(f)(x_1) - T(f)(x_2) = \int_a^b \left(k(x_1, y) - k(x_2, y) \right) f(y) dy$$

so that, using the fact that continuous functions on compact domains are uniformly continuous, we see that $T(f)$ is continuous on $[a, b]$. This verifies that $C([a, b])$ is an invariant set for T, i.e. that

$$T : C([a, b]) \to C([a, b]).$$

To see further that T is a contraction on all of $C([a, b])$, simply observe that if $f_1, f_2 \in C([a, b])$ then the triangle inequality gives for all $x \in [a, b]$

$$\begin{aligned} |T(f_1)(x) - T(f_2)(x)| &\leq \int_a^b |k(x, y)||f_1(y) - f_2(y)| dy \\ &\leq \|f_1 - f_2\|_{L^\infty([a,b])} \sup_{x \in [a,b]} \int_a^b |k(x, y)| dy \end{aligned}$$

[2]Here, recall that since the uniform limit of continuous functions is continuous, the space $C([a, b])$ equipped with the $L^\infty([a, b])$ norm is a Banach space.

so that, in particular,

$$\|T(f_1) - T(f_2)\|_{L^\infty([a,b])} \le \left(\sup_{x\in[a,b]} \int_a^b |k(x,y)|dy \right) \|f_1 - f_2\|_{L^\infty([a,b])}.$$

It follows that if k is such that

$$\sup_{x\in[a,b]} \int_a^b |k(x,y)|dy < 1$$

then T is a contraction on all of $C([a,b])$ and hence, by the Contraction Mapping Theorem, there exists a unique $f \in C([a,b])$ such that $T(f) = f$, as desired.

Note in the above example that the fact that the map T was affine in f is what led to the mapping T to be a contraction on the entire space $C([a,b])$. For nonlinear equations, however, it is often the case that the mapping A is only a contraction on a particular subset of the Banach space X. For this reason, the following variant of Theorem 1 is often useful.

Theorem 2. *Let (X,d) be a complete metric space, and assume $A : X \to X$. Furthermore, assume there exists an $a \in X$ and an $r > 0$ such that*

(i) The ball $B(a,r) := \{x \in X : d(x,a) < r\}$ is an invariant set for A, i.e.

$$A : B(a,r) \to B(a,r).$$

(ii) the map A is a contraction on $B(a,r)$, i.e. there exists a constant $\gamma \in (0,1)$ such that

$$d(A(x), A(y)) \le \gamma d(x,y)$$

for all $x, y \in B(a,r)$.

Then there exists a $x_0 \in B(a,r)$ such that $A(x_0) = x_0$ and, furthermore, this is the unique fixed point of A inside the ball $B(a,r)$.

The proof of the above similar is similar to that for the Contraction Mapping Principle, and is hence omitted[3]. We now employ the above result in a number of examples.

3 Example: Nonlinear Elliptic PDE

For our first example, let $\Omega \subset \mathbb{R}^3$ be open and bounded with smooth boundary, and consider the nonlinear elliptic BVP

$$\begin{cases} \Delta u + \lambda u + u^2 = g, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

[3]Of course, students should provide a full proof themselves!

where $\lambda \in \mathbb{R}$ is a constant and $g \in L^2(\Omega)$ is given. Note that due to the nonlinear term, our first job is to find a function space where the PDE makes sense. Specifically, since we require $g \in L^2(\Omega)$ we need to work in a function space where the left hand side of (2) is an $L^2(\Omega)$ function and, ideally, where the boundary condition is enforced. I claim a natural candidate for this space is

$$X = H^2(\Omega) \cap H_0^1(\Omega)$$

equipped with the $H^2(\Omega)$ norm since, in particular, if $u \in X$ then $\Delta u + \lambda u \in L^2(\Omega)$ and $u = 0$ (in the trace sense) on $\partial\Omega$. As for the nonlinear term, recall by Sobolev embedding that if $k \in \mathbb{N}$ with $k > \frac{n}{2}$, n being the spatial dimension, then $H^k(\Omega)$ is continuously embedded in $C^0(\Omega) \cap L^\infty(\Omega)$. Since here $n = 3$ and $2 > \frac{3}{2}$ this implies in the present context that

$$H^2(\Omega) \cap H_0^1(\Omega) \subset C^0(\Omega) \cap L^\infty(\Omega)$$

with the continuity estimate

$$\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{H^2(\Omega)} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

In particular, if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ then we know $u^2 \in L^2(\Omega)$ with

$$\|u^2\|_{L^2(\Omega)} \leq C\|u\|_{H^2(\Omega)}^2$$

for some constant $C > 0$ (independent of u). It follows that the equation is at least makes sense in $L^2(\Omega)$ for all $u \in X$.

As for the solvability of (2), we begin by considering the nonlinear term as a nonhomogeneous term. In particular, for each $u \in X$ we set

$$h(u) = g - u^2 \in L^2(\Omega)$$

and consider the linear elliptic BVP

$$\begin{cases} \Delta w + \lambda w = h(u), & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

By our elliptic existence and (boundary) regularity theory[4], it follows that if $\lambda \notin \sigma_p(-\Delta)$ then there exists a unique weak solution $\tilde{w} \in H^2(\Omega) \cap H_0^1(\Omega)$ of (3). Since the weak solution will depends on the nonhomogeneous term, this naturally defines a map

$$A : X \to X, \quad A(u) = \tilde{w},$$

where $\tilde{w}$ is the unique weak solution of (3). The main observation here is that, by construction, fixed points of A are weak solutions of (2).

To prove A has a fixed point in X, we employ the Contraction Mapping Theorem. To this end,, given $u, \tilde{u} \in X$ notice the difference $A(u) - A(\tilde{u})$ satisfies (3) with the nonhomogeneous

[4]Specifically, see Theorem 4 in Section 6.4 of Evans.

term replaced by $u^2 - \tilde{u}^2$. Using the continuity of the weak solution operator associated to the linear problem (3) it follows that

$$\begin{aligned}\|A(u) - A(\tilde{u})\|_{H^2(\Omega)} &\le C\left\|u^2 - \tilde{u}^2\right\|_{L^2(\Omega)} \\ &\le C\left(\|u\|_{H^2(\Omega)} + \|\tilde{u}\|_{H^2(\Omega)}\right)\|u - \tilde{u}\|_{H^2(\Omega)}.\end{aligned}$$

In particular, we see that the quadratic nature of the nonlinearity implies that A is not a contraction on all of X and hence the classical Contraction Mapping Theorem given by Theorem 1 does not apply.

Nevertheless, we can attempt to apply the variant of the Contraction Mapping Theorem given by Theorem 2. To this end, observe that given $u \in X$ we have, again by the continuity of the weak solution operator associated to (3),

$$\begin{aligned}\|A(u)\|_{H^2(\Omega)} &\le C\|h(u)\|_{L^2(\Omega)} \\ &\le C\left(\|g\|_{L^2(\Omega)} + \|u^2\|_{L^2(\Omega)}\right) \\ &\le C\left(\|g\|_{L^2(\Omega)} + \|u\|^2_{H^2(\Omega)}\right).\end{aligned}$$

Consequently, given $r > 0$ we see that if $u, \tilde{u} \in B_X(0, r)$ then

$$\begin{cases}\|A(u)\|_{H^2(\Omega)} \le C_1\left(\|g\|_{L^2(\Omega)} + r^2\right) \\ \|A(u) - A(\tilde{u})\|_{H^2(\Omega)} \le C_2(2r)\|u - \tilde{u}\|_{H^2(\Omega)}\end{cases}$$

for some constants $C_1, C_2 > 0$. In order for $B_X(0, r)$ to be an invariant set for A we need to choose r such that

$$C_1\left(\|g\|_{L^2(\Omega)} + r^2\right) < r, \tag{4}$$

which is possible provided that $\|g\|_{L^2(\Omega)}$ is sufficiently small. Indeed, given $\|g\|_{L^2(\Omega)}$ sufficiently small (depending on the size of C_1) there exists an $\tilde{r} > 0$ such that (4) holds for all $r \in (0, \tilde{r})$. Furthermore, in order for A to be a contraction on $B_X(0, r)$ we clearly need $r < \frac{1}{2C_2}$. It follows that if $\|g\|_{L^2(\Omega)}$ is sufficiently small and

$$0 < r_0 < \min\left\{\tilde{r}, \frac{1}{2C_2}\right\}$$

then

$$A : B_X(0, r_0) \to B_X(0, r_0)$$

is a contraction.

By Theorem 2 it follows that if $\lambda \notin \sigma_p(-\Delta)$ then for $\|g\|_{L^2(\Omega)}$ sufficiently small there exists a unique $u_0 \in B_X(0, r_0)$ such that $A(u_0) = u_0$, and hence u_0 is a weak solution of (2). Moreover, this is the unique weak solution of (2) of small norm.

Before continuing to more examples, there are several remarks to make about the above result. First, note it was assumed throughout that $\lambda \notin \sigma_p(-\Delta)$. In particular, for such λ the unique weak solution (with small norm) of the problem

$$\begin{cases} \Delta u + \lambda u + u^2 = 0, & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

is clearly $u = 0$. An interesting question to ask is what happens if $\lambda \in \sigma_p(-\Delta)$. At such a point, multiple non-trivial solutions may "bifurcate" from the trivial branch of solutions[5]. The construction of such non-trivial solutions of (5) is the subject of "local bifurcation theory" and is typically completed via a procedure known as a Lyapunov-Schmidt reduction. Interested students are encouraged to consider this topic for a final project.

Finally, as the above example demonstrates, when applying the Contraction Mapping Theorem to nonlinear problems one must typically take certain parameters to be small to ensure the contraction property holds (e.g. r_0 and $\|g\|_{L^2(\Omega)}$ in the above example). Sometimes, however, one can eliminate teh need for such smallness assumptions through iteration, as the next example demonstrates.

4 Example: Nonlinear Reaction Diffusion

For this example, let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, and consider the IVBVP

$$\begin{cases} u_t - \Delta u = f(u), & \text{in } \Omega \times [0, T] \\ \qquad u = 0, & \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = g(x), & \text{for } x \in \Omega, \end{cases} \tag{6}$$

where here $g \in H_0^1(\Omega)$ and $T > 0$ is finite and fixed (but arbitrary). Further, we suppose the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is *globally Lipschitz*[6], i.e. there exists a constant $C > 0$ such that

$$|f(z) - f(\tilde{z})| \leq C|z - \tilde{z}| \quad \forall z, \tilde{z} \in \mathbb{R}. \tag{7}$$

In particular, observe that for all $z \in \mathbb{R}$ we have

$$|f(z)| = |f(0) + (f(z) - f(0))| \leq C(1 + |z|) \tag{8}$$

for some constant $C > 0$. It follows that $f : L^2(\Omega) \to L^2(\Omega)$. The goal of this exercise is to prove there exists a unique weak solution of (6), i.e. there exists a unique

$$u \in C((0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

such that $u(0) = g$ and

$$\langle u_t, v \rangle_{L^2(\Omega)} + B[u, v] = \langle f(u), v \rangle_{L^2(\Omega)}$$

[5] This is akin to asking in Math 766 what happens at a point where the implicit function theorem fails.

[6] As we will discuss in Section 5 later, this assumption is not practical in most applications...

for a.e. $t \in [0,T]$ and for all $v \in H_0^1(\Omega)$. As we will see, this is possible via the Contraction Mapping Theorem *provided the time of existence is sufficiently small.*

To this end, we aim to apply the Contraction Mapping Theorem on the space

$$X_\tau := C([0,\tau]; L^2(\Omega))$$

where here $\tau \in (0,T]$ will be chosen later, and where we equip X_τ with the natural norm

$$\|v\|_\tau := \max_{0\le t\le\tau} \|v(t)\|_{L^2(\Omega)}.$$

As with the previous example, we begin by considering the nonlinearity as an inhomogeneous term. Given $u \in X_\tau$ and note that by (8) we have

$$\begin{aligned}\int_0^\tau \|f(u(t))\|^2_{L^2(\Omega)} dt &= \int_0^\tau \left(\int_\Omega |f(u(x,t))|^2 dx\right) dt \\ &\le C \int_0^\tau \left(\int_\Omega (1+|u(x,t)|)^2\, dx\right) dt \\ &\le C\tau \left\|1+|u|\right\|_\tau^2\end{aligned}$$

for all $t \in [0,\tau]$ so that, in particular

$$u \in X_\tau \quad \Rightarrow \quad f(u) \in L^2([0,\tau]; L^2(\Omega)).$$

For $u \in X_\tau$ fixed, we now consider the linear parabolic IVBVP

$$\begin{cases} w_t - \Delta w = f(u), & \text{in } \Omega \times [0,T] \\ w = 0, & \text{on } \partial\Omega \times [0,T] \\ w(x,0) = g(x), & \text{for } x \in \Omega, \end{cases} \tag{9}$$

and note by our parabolic existence theory, since $f(u) \in L^2([0,\tau]; L^2(\Omega))$ there exits a unique weak solution

$$\tilde{w} \in C((0,\tau]; H_0^1(\Omega)) \cap C^1([0,\tau]; L^2(\Omega))$$

such that for all $v \in H_0^1(\Omega)$ we have

$$\langle \tilde{w}_t, v\rangle_{L^2(\Omega)} + B[\tilde{w}, v] = \langle f(u), v\rangle_{L^2(\Omega)}$$

for a.e. $t \in [0,\tau]$. Since the weak solution will depend on the nonhomogeneous term, this naturally defines a map

$$A : X_\tau \to X_\tau, \quad A(u) = \tilde{w}$$

where $\tilde{w}$ is the unique weak solution of (9). The main observation is that, by construction, fixed points of A are weak solutions of (6).

To prove A has a fixed point in X_τ, we attempt to use the Contraction Mapping Theorem. To this end, given $u_1, u_2 \in X_\tau$ set $w_j = A(u_j)$ and note that the difference $v := w_1 - w_2$ weakly satisfies the IVBVP

$$\begin{cases} (w_1 - w_2)_t - \Delta(w_1 - w_2) = f(u_1) - f(u_2), & \text{in } \Omega \times [0, T] \\ w_1 - w_2 = 0, & \text{on } \partial\Omega \times [0, T] \\ (w_1 - w_2)(x, 0) = 0, & \text{for } x \in \Omega, \end{cases}$$

which, by the weak formulation, implies that

$$\frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|^2_{L^2(\Omega)} + \int_\Omega |D(w_1 - w_2)|^2 dx = \langle f(u_1) - f(u_2), w_1 - w_2 \rangle_{L^2(\Omega)}.$$

Now, observe by the Cauchy-with-ϵ inequality we have

$$\begin{aligned} \left|\langle f(u_1) - f(u_2), w_1 - w_2 \rangle_{L^2(\Omega)}\right| &\leq \frac{1}{4\epsilon}\|f(u_1) - f(u_2)\|^2_{L^2(\Omega)} + \epsilon\|w_1 - w_2\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{4\epsilon}\|f(u_1) - f(u_2)\|^2_{L^2(\Omega)} + C\epsilon \int_\Omega |D(w_1 - w_2)|^2 dx, \end{aligned}$$

where the last inequality follows by Poincaré. By choosing $\epsilon >$ above so that $C\epsilon < 1$ we have

$$\frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|^2_{L^2(\Omega)} \leq C\|f(u_1) - f(u_2)\|^2_{L^2(\Omega)}$$

which, using again that f is globally Lipshitz, gives

$$\frac{1}{2}\frac{d}{dt}\|w_1 - w_2\|^2_{L^2(\Omega)} \leq C\|u_1 - u_2\|^2_{L^2(\Omega)}.$$

Integrating and using the initial condition for $w_1 - w_2$, we find for all $t \in [0, \tau]$ we have

$$\|w_1 - w_2\|^2_{L^2(\Omega)}(t) \leq C \int_0^t \|u_1 - u_2\|^2_{L^2(\Omega)}(s) ds \leq C\tau\|u_1 - u_w\|_\tau. \tag{10}$$

which gives (taking max over all $t \in [0, \tau]$ yields the estimate

$$\|A(u_1) - A(u_2)\|^2_\tau \leq C\tau\|u_1 - u_2\|^2_\tau \tag{11}$$

valid for some constant $C > 0$ which depends only on the geometry of Ω (via Poincaré) and the Lipschitz constant associated to f. In particular, it does not depend on the initial condition.

From above, it follows that A is a contraction on all of X_τ provided

$$0 < \tau < \frac{1}{C}, \tag{12}$$

where the constant $C > 0$ is from (11). Consequently, given any $\tau_1 > 0$ such that (12) holds for $\tau = \tau_1$, the Contraction Mapping Theorem implies there exists a unique $u \in X_{\tau_1}$ such that

$$A(u) = u$$

so that u is a weak solution of the IVBVP (6) on the time interval $[0,\tau_1]$.

In this example, it turns out we can extend the solution beyond the time $\tau=\tau_1$. Indeed, notice that since $u(t)\in H^1_0(\Omega)$ for a.e. $t\in[0,\tau_1]$, we can assume WLOG (redefining τ_1 if necessary) that $u(\tau_1)\in H^1_0(\Omega)$, and hence we can use the "final data" $u(\tau_1)$ as an initial condition from which to evolve from. In particular, since the constant C in (11) depends only on the geometry of Ω and $\mathrm{Lip}(f)$, we can repeat the above argument with initial data $g=u(\tau_1)$ to extend our weak solution to the time interval $[0,2\tau_1]$. Continuing, after finitely many steps we obtain a weak solution of (6) on the entire time interval $[0,T]$.

Finally, to verify uniqueness over the interval $[0,T]$, suppose u_1 and u_2 are two weak solutions of (6) and note, using previous notation,

$$w_j=A(u_j)=u_j$$

so that (10) implies

$$\|u_1-u_2\|^2_{L^2(\Omega)}(t)\leq C\int_0^t\|u_1-u_2\|^2_{L^2(\Omega)}(s)ds$$

for all $t\in[0,T]$. Using Gronwall's inequality, it follows that $\|u_1-u_2\|_{L^2(\Omega)}(t)=0$ for all $t\in[0,T]$ as desired.

In the above example, we avoided a smallness assumption on our weak solutions or on $\mathrm{Lip}(f)$ by iteration. This was possible because the constant C in (11) did not depend on the initial condition nor on τ itself. We will see this type of trick again later when considering well-posedness of the nonlinear Schrödinger equation.

5 The Chaffee-Infante Problem & Finite Time Blowup

It is important to note that the assumption in Section 4 that f be globally Lipschitz is unrealistic. In applications it often happens that the nonlinearity f is a polynomial in u, giving rise to nonlinearities which are locally, but certainly not globally, Lipschitz continuous. In these situations solutions may not exist globally in time, as this next example illustrates.

Consider the one-dimensional IVBVP

$$\begin{cases} u_t=u_{xx}+u^3, \quad x\in(0,1),\ t>0\\ u(0,t)=u(1,t)=0, \quad t>0\\ u(x,0)=g(x), \quad x\in(0,1)\end{cases}\tag{13}$$

Note that (13) combines two interesting and competing effects. First, if one ignores the nonlinearity then we have the classical diffusion equation, which we know is smoothing and solutions exist for all time. On the other hand, if we ignore the diffusion term then the PDE reduces to the ODE

$$u_t=u^3$$

which has solutions

$$u(t) = \frac{u(0)}{\sqrt{1-2u(0)^2 t}},$$

which become singular (i.e. blow up) in finite time. To analyze (13), one must untable these these competing effects.

Here, we aim at proving that if the initial data is "sufficiently large", in a sense to be determined later, then there do not exist smooth solutions of (13) defined for all $t > 0$, i.e. smooth solutions will not be "global". We begin by recalling the eigenvalue-eigenvector pairs of uniformly elliptic operator $-\partial_x^2$ on $H_0^1(0,1)$ are given explicitly by

$$\mu_j = (j\pi)^2, \quad \phi_j(x) = \sin(j\pi x), \quad j \in \mathbb{N}.$$

In particular, the principle (i.e. lowest) eigenvalue is $\mu_1 = \pi^2$ with eigenfunction $\phi_1(x) = \sin(\pi x)$ satisfying

$$\phi_1(x) > 0 \ \text{ on } (0,1) \quad \text{and} \quad \int_0^1 \phi_1(x)^2 dx = \frac{1}{2}.$$

Now, assume that $u(x,t)$ is a smooth solution of (13) and, for so long the solution u is smooth, let

$$\eta(t) := \int_0^1 u(x,t)\sin(\pi x)dx$$

denote the first Fourier sine coefficient of u. Our goal is to prove that if $\eta(0)$ is sufficiently large, then $\eta(t)$ blows up in finite time.

To this end, note from (13) that

$$\begin{aligned}\eta'(t) &= \int_0^1 u_t \sin(\pi x)dx = \int_0^1 \left(u_{xx} + u^3\right)\sin(\pi x)dx \\ &= -\pi^2\eta(t) + \int_0^1 u^3 \sin(\pi x)dx.\end{aligned}$$

where the last equality follows from integration by parts. Now, suppose that $g(x) \geq 0$ for all $x \in (0,1)$ and note that the maximum principle then implies $u(x,t) \geq 0$ for all $0 < x < 1$ and for all $t > 0$ for which the solution is defined. Using Hölder's inequality it follows that

$$\begin{aligned}\int_0^1 u(x,t)\sin(\pi x)dx &= \int_0^1 \left(u^3\sin(\pi x)\right)^{1/3}(\sin(\pi x))^{2/3}dx \\ &\leq \left(\int_0^1 u^3\sin(\pi x)dx\right)^{1/3}\left(\int_0^1 \sin(\pi x)dx\right)^{2/3} \\ &= \left(\frac{2}{\pi}\right)^{2/3}\left(\int_0^1 u^3\sin(\pi x)dx\right)^{1/3}\end{aligned}$$

so that

$$\int_0^1 u^3\sin(\pi x)dx \geq \frac{\pi^2}{4}\eta(t)^3.$$

It follows that for all $t > 0$ for which u exists and is smooth we have

$$\eta'(t) \geq \pi^2 \left(\frac{1}{4}\eta(t)^3 - \eta(t) \right) = \pi^2 \eta(t) \left(\frac{1}{4}\eta(t)^2 - 1 \right).$$

Consequently, if[7] $\eta(0) > 2$ then Gronwall's inequality implies that $\eta(t) \geq y(t)$ where $y(t)$ solves the ODE

$$\frac{dy}{dt} = \pi^2 \left(\frac{1}{4} y(t)^3 - y(t) \right)$$

which has explicit solution[8] given by

$$y(t) = \frac{2y(0)}{\sqrt{y(0)^2 - (y(0)^2 - 4)e^{2\pi^2 t}}}.$$

Clearly if $y(0) > 2$ then $y(t) \to \infty$ as $t \to t_*^-$, where, explicitly, we have

$$t_* := \frac{1}{\pi^2} \log \left(\frac{\eta(0)}{\sqrt{\eta(0)^2 - 4}} \right).$$

It immediately follows that no smooth solution of (13) can exist beyond $t = t_*$.

Note that the above argument does not actually prove that $\eta(t)$ blows up at $t = t_*$, since it may happen that the solution u loses smoothness at time t_*, since it may happen that u loses smoothness at an earlier time (for example, another Fourier coefficient may blow up before $t = t_*$), which would invalidate the above argument. One can only get a sharp result if the quantity blowing up is a "controlling norm", in the sense that all local solutions remain smooth if and only if the controlling norm is finite.

The above proof shows that if the initial data g is "sufficiently large" then there exists a finite $\tau > 0$ such that

$$\lim_{t \to \tau^-} \|u(t)\|_{H^1(0,1)} = \infty.$$

A natural question is to ask what happens if g is *not* large, but is instead "small" in some sense. It turns out that using techniques from Dynamical Systems one can show that if $\|g\|_{H^1(0,1)}$ is sufficiently small, then the unique solution of (13) exists for all time $t \geq 0$ and, in particular, one has

$$\lim_{t \to \infty} \|u(t)\|_{H^1(0,1)} = 0$$

at an exponential rate. In the language of Dynamical Systems, this says that the equilibrium solution $u \equiv 0$ is an exponentially, asymptotically stable solution to the nonlinear reaction

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[7] Note the fixed points of the above ODE are $\eta = 0, \pm 2$, and hence the flow generated by the above ODE is monotone increasing for $\eta > 2$.

[8] This can be easily found by substituting $\xi(t) = e^{\pi^2 t} y(t)$ and noting that ξ satisfies the separable ODE $\xi'(t) = \frac{\pi^2}{4} e^{-2\pi^2 t} \xi(t)^3$.

diffusion equation (13). For details, take Math 851 with me sometime (for example, in Spring 2021).

Furthermore, observe in the above example that the sign of the nonlinearity was crucial. Indeed, it is possible to show (via Galerkin's method) that there exists a unique weak solution (appropriately defined) of the IVBVP

$$\begin{cases} u_t = u_{xx} - u^3, \quad x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = g(x), \quad x \in (0,1) \end{cases} \tag{14}$$

regardless of the "size" of the initial condition g. This is not unexpected, since solutions of the ODE

$$u_t = -u^3$$

exist globally in time (i.e. they have no finite time blow up). Interested students could consider this as a topic for a final project.

Finally, I want to emphasize that similar non-existence results are possible for other polynomial nonlinearities. For example, see Section 9.4.1 in Evans for an example with quadratic nonlineraity posed in $\mathbb{R}^n$.

6 Some Final Thoughts

In the above, we explored some applications of one of the most basic fixed point arguments: the Contraction Mapping Theorem. Clearly there are several other fixed point methods exist. The next result is an example of a fixed point result in finite-dimensions which is purely topological in nature.

Theorem 3 (Brouwer's Fixed Point Theorem)**.** *If $B \subset \mathbb{R}^n$ is a closed ball and if $f : B \to B$ is continuous, then f has a fixed point in B.*

The above result follows directly by the Intermediate Value Theorem when $n = 1$, but is surprisingly difficult to prove[9] for $n \geq 2$. The key points used in the proof are the convexity and the compactness of the closed unit ball in $\mathbb{R}^n$. With that said, it is natural to ask if a generalization to infinite dimensional Banach spaces may be possible. This question has been addressed by numerous researchers, and two well-known extensions to the infinite-dimensional context are given as follows.

Theorem 4 (Schauder's Fixed Point Theorem)**.** *Let X be a real Banach space and suppose $K \subset X$ is compact and convex. Also, suppose that $A : K \to K$ is continuous. Then A has a fixed point in K.*

[9]Interestingly, one proof for the higher-dimensional case of Brouwer's Fixed Point Theorem can be done using the Calculus of Variations, which is the next topic for the class!

Theorem 5 (Schaefer's Fixed Point Theorem)**.** *Let X be a real Banach space and suppose that $A : X \to X$ is a continuous. Assume also that $A : X \to X$ is compact, i.e. for each bounded sequence $\{u_k\} \subset X$ there exists a subsequnece $\{u_{k_j}\}$ such that the sequence $\{A(u_{k_j})\}$ converges in X. Furthermore, assume that the set*

$$\{u \in X : u = \lambda A(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then A has a fixed point in X.

Note that in Schaefer's Fixed Point Theorem, the final hypotheses essentially just guarantees that the set of all possible fixed points of the family of operators λA with $\lambda \in [0, 1]$, then the mapping A itself has a fixed point. As described in Evans, this is an example of the informal principle that "if we can prove appropriate estimates for solutions of a nonlinear PDE, under the assumption that such solutions exist, then in fact these solutions do exist". This is an example of the method of a-priori estimates, which is used throughout the study of nonlinear PDE.

As you might expect, applications of these fixed point theorems would be an interesting topic for final projects in the class!

7 Exercises

Complete the following exercises.

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the Lipshitz condition

$$|f(x, z_1) - f(x, z_2)| \leq L_1|z_1 - z_2|$$

for all $(x, z_1), (x, z_2) \in \Omega \times \mathbb{R}$. Assume also that $f(\cdot, 0) \in L^2(\Omega)$ and consider the nonlinear elliptic BVP

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Prove there exists a unique weak solution $u \in H_0^1(\Omega)$ of the above BVP provided that

$$L_1 < \lambda_1,$$

where λ_1 is the principle eigenvalue of $-\Delta$ with respect to $H_0^1(\Omega)$. Here, we say $u \in H_0^1(\Omega)$ is a weak solution of the given BVP if

$$\int_U Du \cdot Dv \; dx = \int_U f(x, u)v \; dx$$

for all $v \in H_0^1(\Omega)$.

Hint: For this problem, let $X = H_0^1(\Omega)$ and note by the Poincaré inequality that X is a Banach space with the norm

$$\|u\|_X := \|Du\|_{L^2(\Omega)}.$$

First, show that for each $u \in X$ the linear elliptic

$$\begin{cases} -\Delta w = f(x,u), & \text{in } \Omega \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{15}$$

has a unique weak solution $w \in X$. This naturally defines a map

$$A : X \to X, \quad A(u) = w$$

where w is the unique weak solution of (15). Now, prove that A is a strict contraction on X. For this, the variational characterization of λ_1 (see Theorem 2 in the "Principle Eigenvalue Theorem" notes) will be very helpful.

2. (Based on #4, Section 9.7 from Evans) Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary and consider a parabolic IVBVP of the form

$$\begin{cases} u_t - \Delta u = f \text{ in } U \times (0,\infty) \\ u = 0 \text{ on } \partial U \times [0,\infty) \\ u = g \text{ on } U \times \{t = 0\}, \end{cases}$$

where $g \in L^2(U)$ and $f \in L^\infty(U \times [0,\infty))$.

(a) Suppose $f = 0$ above. Using the eigenfunction expansion of the solution derived in class, show directly that if u is the weak solution of the above IVBVP then

$$\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t}\|g\|_{L^2(U)} \quad \forall\, t \ge 0,$$

where $\lambda_1 > 0$ is the principle eigenvalue of $-\Delta$ with Dirichlet boundary conditions on U.

(b) Now, suppose there exists a $\tau > 0$ such that f is τ-periodic in t, i.e. $f(x,t+\tau) = f(x,t)$ for all $(x,t) \in U \times (0,\infty)$. Prove there exists a unique function $g \in L^2(U)$ for which the corresponding weak solution u is τ-periodic in t as well.

Hint: Let $f \in L^\infty(U \times [0,\infty))$ be fixed as in the statement of the problem. Parabolic existence theory guarantees that for each $g \in L^2(U)$ there exists a unique weak solution $u \in C((0,\infty); L^2(U)) \cap L^2((0,\infty); H_0^1(U))$ of the given parabolic IVBVP; see Section 11.2(b) of McOwen for details. Use this result together with the Banach Fixed Point Theorem on an appropriate map between Banach spaces. Also, you may find the exponential bound derived in #4 above helpful in verifying the contraction property.

<u>Extended Hint</u>: Continuing to have f fixed, the above "weak solution operator" produces a map from the initial data $g \in L^2(U)$ to the weak solution $u = S[g] \in C((0,\infty); L^2(U)) \cap L^2((0,\infty); H_0^1(U))$. In order for given initial data g to produce a solution u that is τ-periodic in time, it must be that $u(t) = u(t+\tau)$ for a.e. $t \geq 0$, i.e. we must have that $S[g](t) - S[g](t+\tau) = 0$ for a.e. $t \geq 0$. Set $w(t) := S[g](t) - S[g](t+\tau)$ and prove there exists a unique $g \in L^2(U)$ such that $w(t) = 0$ for a.e. $t \geq 0$.

3. This exercise introduces you to "Duhamel's Principle" for nonhomogeneous linear PDE, and demonstrates how it can be used in the nonlinear PDE context.

   (a) Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, and for $f \in L^2(\Omega)$ and $g \in H_0^1(\Omega)$ consider the IVBVP

$$\begin{cases} u_t = \Delta u + f, & x \in \Omega, t > 0 \\ u(x,0) = g, & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{16}$$

   Recall from Section 2 in Chapter 3 (of our notes[10]) that when $f = 0$ and $g \in H_0^1(\Omega)$ the above problem has a unique weak solution

$$u \in C([0,\infty); H_0^1(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$$

   which is given explicitly by

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-\mu_k t} \phi_k(x)$$

   where the $\{(\mu_k, \phi_k)\}$ are the Dirichlet eigenvalues and eigenfunctions associated to $-\Delta$ on $H_0^1(\Omega)$, with eigenfunctions normalized to be an ONB for $L^2(\Omega)$. Use this explicit form to prove that weak solutions of the above IVBVP with $f = 0$ satisfy the parabolic smoothing estimate

$$\|u(\cdot,t)\|_{H^1(\Omega)} \leq \frac{C}{t^{1/2}} \|g\|_{L^2(\Omega)} \tag{17}$$

   for all $t > 0$, where here $C > 0$ is some constant.

   (b) For each $t \geq 0$, define the weak solution operator

$$S(t) : H_0^1(\Omega) \to C([0,\infty); H_0^1(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$$

[10] This is not *exactly* the result we proved, but this is proved in the same way... here, just take this reformulation for granted.

by $u(x,t) = S(t)g(x)$. It can be shown for each $f \in L^2(\Omega)$ the unique weak solution associated to the nonhomogeneous IVBVP (16) is given by

$$u(x,t) = S(t)g(x) + \int_0^t S(t-s)f(x)ds.$$

Taking uniqueness for granted, formally show by differentiating the above in t that this solves (16). This is called Duhamel's Principle, and essentially says you can solve non-homogeneous problems whenever you can solve homogeneous problems.

(c) Now, lets see how the above can be used to solve nonlinear PDE. Let $f \in C^1(\mathbb{R})$, and for each $g \in H_0^1(\Omega)$ consider the one-dimensional IVBVP

$$\begin{cases} u_t + uu_x = u_{xx} + f(u), & x \in (0,1),\ t > 0 \\ u(x,0) = g(x), & x \in (0,1) \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$$

Prove that for $\tau > 0$ sufficiently small, there exits a unique weak solution

$$u \in C([0,\tau]; H_0^1(\Omega)) \cap C^1((0,\tau); L^2(\Omega))$$

to the above IVBVP.

Hint: By part (b) above, we see that *if* a weak solution exists then it satisfies

$$u(x,t) = S(t)g(x) + \int_0^t S(t-s)F(u(x,s))ds$$

where here $F(u) = f(u) - uu_x$, i.e. u would have to be a fixed point of the mapping

$$\mathcal{T}(u) = S(t)g + \int_0^t S(t-s)F(u(s))ds$$

on the space

$$C([0,\tau]; H_0^1(\Omega)) \cap C^1((0,\tau); L^2(\Omega)).$$

Show that $\mathcal{T}$ is a strict contraction on this space, provided that $\tau > 0$ is sufficiently small. The estimate (17) will be crucial here.

4. (Based on #2, Section 13.3 from McOwen[11]) Let $U \subset \mathbb{R}^n$ be open and bounded with smooth domain, and consider the following nonlinear Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0, & \text{in } U \\ u = 0 & \text{on } \partial U, \end{cases} \tag{18}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

[11] Note: This exercise will be important in future discussions...

(a) If $F(u) = \int_0^u f(t)dt$, use integration by parts to show that

$$n \int_U F(u)dx + \int_U f(u) \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i} \, dx = 0$$

for any $u \in C^1(\bar{U})$ with $u = 0$ on ∂U.

(b) If $u \in C^2(U) \cap C(\bar{U})$ satisfies (18), then prove *Pohozaev's identity*

$$\frac{n-2}{2} \int_U |Du|^2 dx - n \int_U F(u)dx + \frac{1}{2} \int_{\partial U} \left(\frac{\partial u}{\partial \nu} \right)^2 (x \cdot \nu) dS = 0,$$

where ν is the exterior unit normal vector to ∂U.

(c) When U is a ball in $\mathbb{R}^n$, show that (18) admits no non-trivial solution $u \in C^2(U) \cap C(\bar{U})$ when $f(u) = |u|^{p-1}u$ with $p > \frac{n+2}{n-2}$.