Notes on Poincaré Type Inequalities

As discussed in class, the development of Poincaré inequalities will prove to be essential throughout our analysis of linear PDE theory. In these notes, when $U \subset \mathbb{R}^n$ is bounded, I will provide an integration by parts based proof of the Poincar'e inequality on $W_0^{1,p}(U)$. I will then show while such a Poincaré inequality can not hold on the unbounded domain $\mathbb{R}$, a varient of it CAN hold when $U = \mathbb{R}^n$ for $n \geq 2$.

1 Poincaré Inequality on $W_0^{1,p}(U)$

First, let me give an easy proof of the inequality

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}, \quad \forall u \in W_0^{1,p}(U) \tag{1.1}$$

when $1 \leq p < \infty$ and U is an open, bounded subset of $\mathbb{R}^n$. By density, it is clearly enough to prove the inequality for all $u \in C_c^\infty(U)$, which we now do. To this end, simply notice[1] that since U is bounded we have for any $j \in \{1, 2, \ldots, n\}$ and $u \in C_c^\infty(U)$

$$\begin{aligned} \int_U |u|^p dx = \int_U \frac{\partial}{\partial x_j}(x_j)|u|^p dx &= -p \int_U x_j |u|^{p-1} \mathrm{sgn}(u) u_{x_j} dx \\ &\leq C \int_U |u|^{p-1}|Du| dx \end{aligned}$$

for some constant $C > 0$. Now, noticing that

$$\frac{1}{p} + \frac{1}{\left(\frac{p}{p-1}\right)} = 1$$

it follows by Hölder's inequality that

$$\|u\|_{L^p(U)}^p \leq C \left(\int_U \left(|u|^{p-1}\right)^{p/(p-1)} dx \right)^{(p-1)/p} \left(\int_U |Du|^p dx \right)^{1/p} = \|u\|_{L^p(U)}^{p-1} \|Du\|_{L^p(U)}$$

from which the desired inequality follows. By density of $C_c^\infty(U)$ in $W_0^{1,p}(U)$ then, we obtain the inequality for all $u \in W_0^{1,p}(U)$ also.

2 Poincaré Inequality on $\mathbb{R}$

I claim that Poincaré's inequality can not hold on the unbounded domain $\mathbb{R}$. Indeed, consider the sequence of smooth functions

$$\phi_k(x) = \begin{cases} 0 & |x| > k + \frac{1}{10} \\ -\mathrm{sign}(x), & |x| \in (k, k+1) \\ 0 & |x| < k - \frac{1}{10} \end{cases}$$

[1] We consider here only the case $1 \leq p < \infty$, since the case $p = \infty$ is trivial.

where the function is smooth and monotone (say) where I give no definition. Then for all $1 \le p < \infty$ we have $\|\phi_k\|_{L^p(U)} \approx 2$ for all k while the smooth functions $\psi_k(x) := \int_{-\infty}^x \phi_k(s)ds$ satisfy $\|\psi_k\|_{L^p(U)} \to \infty$ as $k \to \infty$. Thus, it is not possible to find a constant $C > 0$ such that $\|\psi_k\|_{L^p(U)} \le C\|\psi_k'\|_{L^p(U)}$ for all k, and hence the Poincaré inequality must fail in $\mathbb{R}$.

3 Poincaré Inequality in $\mathbb{R}^n$ for $n \ge 2$

Even though the Poincaré inequality can not hold on $W^{1,p}(\mathbb{R})$, a variant of it can hold on the space $W^{1,p}(\mathbb{R}^n)$ when $n \ge 2$. To see why this might be true, let me first explain why the above example does not serve as a counterexample on $\mathbb{R}^n$.

If you wanted to extend the counter example from the previous section to the present case, you could define a sequence of radial functions $f_k(x) := \phi_k(|x|)$ and $g_k(x) := \psi_k(|x|)$, where ϕ_k and ψ_k are as above. Then as before you can convince yourself that $\|g_k\|_{L^q(\mathbb{R}^n)} \approx k^{n/q} \to \infty$ as $k \to \infty$ for all $1 \le q < \infty$. However, when $n \ge 2$ the sequence $\|f_k\|_{L^p(\mathbb{R}^n)}$ is no longer bounded for any $1 \le p < \infty$ since

$$\|f_k\|_{L^p(\mathbb{R}^n)} = C\left(\int_0^\infty |\phi_k(r)|^p r^{n-1} dr\right)^{1/p} \approx \left(\int_k^{k+1} r^{n-1} dr\right)^{1/p} \approx k^{(n-1)/p}$$

for all $k \in \mathbb{N}$. Thus, if an inequality of the form

$$\|g_k\|_{L^q(\mathbb{R}^n)} \le C\|f_k\|_{L^p(\mathbb{R}^n)}$$

is to hold, it immediately follows that we must have $q > p$ and $n \ge 2$.

To further investigate this issue, suppose we want to prove an inequality of the type

$$\|f\|_{L^q(\mathbb{R}^n)} \le C\|Df\|_{L^p(\mathbb{R}^n)}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

where p and q are appropriately chosen and $C = C(n, p, q) > 0$. To see for which indices p and q such an inequality can hold, fix a $f \in C_c^\infty(\mathbb{R}^n)$ and for $\lambda > 0$ let f_λ denote the rescaled function

$$f_\lambda(x) := f\left(\frac{x}{\lambda}\right).$$

Then, performing the change of variables $x \mapsto \lambda x$ in the integrals that define the L^p and L^q norms, with $1 \le p, q < \infty$, and using the fact that

$$Df_\lambda = \frac{1}{\lambda}(Df)_\lambda,$$

we find that

$$\left(\int_{\mathbb{R}^n} |Df_\lambda|^p dx\right)^{1/p} = \lambda^{n/p-1}\left(\int_{\mathbb{R}^n} |Df|^p dx\right)^{1/p}$$

and

$$\left(\int_{\mathbb{R}^n} |f_\lambda|^q dx\right)^{1/q} = \lambda^{n/q}\left(\int_{\mathbb{R}^n} |f|^q dx\right)^{1/q}.$$

These norms must scale according to the same exponent if we are to have an inequality of the desired form, otherwise we can violate the inequality by taking $\lambda \to 0$ or $\lambda \to \infty$. The equality of the exponents implies that we must choose $q = p^*$, where p^* satisfies

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

In particular, notice that we need $n \geq 2$ and $1 \leq p < n$ in order to ensure that $p^* > 0$, in which case $p < p^* < \infty$. Given a $p \in [1, n)$, the number p^* is known as the Sobolev conjugate of p. The fact that such a inequality is true when $q = p^*$ was obtained by Sobolev in 1938, and is now usually referred to as the Gagliardo-Nirenberg inequality and is an example of a Sobolev embedding theorem. Specifically, we have the following theorem.

Theorem 1. *Assume that $n \geq 2$ and $1 \leq p < n$. Then there exists a constant $C = C(n, p) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

A proof of this theorem can be found in Evans. Using an extension operator then, it immediately follows that we can obtain versions of this theorem valid on open, bounded domains as well.